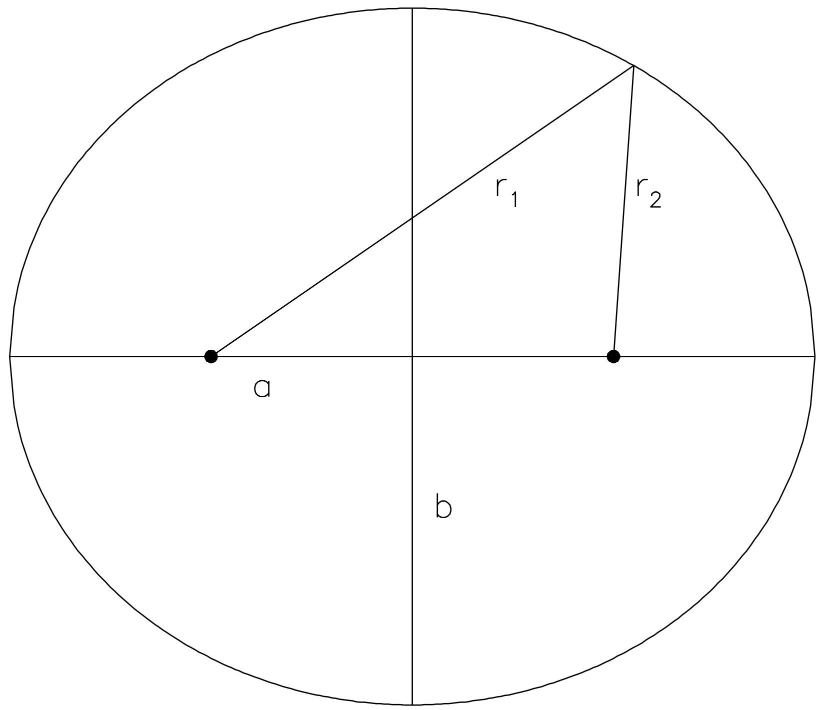
Astro 205. Lecture 2 September 17, 2003

Gravity, Kepler's Laws and Orbits



Ellipse with e = 0.5. a and b are the semi-major and semi-minor axes, and the foci are marked by dots

To first order, the planets orbit the Sun in a plane on an ellipse, with the Sun at one focus. An ellipse has 2 foci, and its definition is:

$$r_1 + r_2 = 2a$$

2a is the largest diameter, or *major axis*, and 2b the smallest, or *minor axis*. The equation of an ellipse in Cartesian (x,y) coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and

$$b^2 = a^2(1 - e^2)$$

where the quantity e is the *orbital eccentricity.* The larger the eccentricity, the more elongated is the orbit. If $e = 0$, then $a = b$ and the ellipse becomes a circle with both foci at the same place, the center of a circle. The point of closest approach of a planet to the Sun is called the *perihelion*: the perihelion distance is

$$r = a(1 - e)$$

The point of furthest distance is the aphelion.

Gravity

Kepler's analysis of the observed planetary motions led to the formulation of his three empirical laws, obeyed by all of the planets in the solar system:
1. The orbits of the planets are ellipses with the Sun at one focus.
2. The radius vector between the planet and the Sun sweeps out equal areas in equal time.
3. The period ("year") P of the planet and its mean distance from the Sun a are related by:

$$\frac{P^2}{a^3} = \text{constant}$$

Table 1. P^2/a^3 for the Solar System Planets

Body	D(A.U.)	Period(yrs)	P^2/a^3
Mercury	0.39	0.24	0.99
Venus	0.72	0.61	1.01
Earth	1.00	1.00	1.00
Mars	1.52	1.88	1.01
Jupiter	5.20	11.9	1.01

Let's check Kepler's third law for the Solar System. We'll use the periods and mean distances from Table 1 in Lecture 1. Let's express the orbital periods in years (a year being the time it takes the Earth to orbit the Sun) and the mean distance in A.U. (the mean distance between the Earth and the Sun, 1.5×10^{13} cm), then calculate P^2/a^3. (I've done it only out to Jupiter, but you can check the remaining planets). Note the units of P^2/a^3: years^2/AU^3.

Table 2. P^2/a^3 for the Galilean satellites of Jupiter

Body	D(A.U.)	Period(yrs)	P^2/a^3
Io	2.8×10^{-3}	4.8×10^{-3}	1050
Europa	4.5×10^{-3}	9.7×10^{-3}	1032
Ganymede	7.1×10^{-3}	2.0×10^{-2}	1117
Callisto	1.26×10^{-2}	4.6×10^{-2}	1058

Within the accuracy of the calculations, then, the value of P^2/a^3 is indeed constant.
Let's use another example: the moons of Jupiter. We'll consider the four largest moons, the *Galilean* satellites (so-called because they were first seen by Galileo through his telescope - and their evident orbital path around Jupiter was one of the strong pieces of evidence showing that not everything in the Universe orbits the Earth, a conclusion that got various scientists into big trouble a few hundred years ago. And as another aside, it's very likely the case that Galileo was not the first person to see Jupiter's moons, but was the first to observe that they orbit Jupiter. They are actually bright enough that you can see them with the naked eye under very favorable conditions - not in New Jersey, though). In the same units (AU, years):
So P^2/a^3 is also constant for the Jovian moons (within numerical accuracy) but this "constant" is different from that for the solar system. The "constant" in Kepler's third law is thus not a physical constant like the speed of light, for example, but depends on the system; it is different for the planets orbiting the Sun than for the satellites orbiting Jupiter. What's the difference? It's the *mass* of the central body in the system. Now let's derive Kepler's laws.

Gravity

Newton's enormous intellectual leap of realizing that the force which causes objects to fall is the same as that which keeps the planets orbiting the Sun, led to his formulation of the *laws of gravity.*

Newton's laws of gravity say:
1. for every action there is an equal and opposite reaction
2. the force of gravity between any two bodies is

$$\underline{\mathrm{F}} \quad = \quad \mathrm{m}\underline{\mathrm{a}} \quad [= \quad \mathrm{m}_1 \frac{\mathrm{d}^2\underline{\mathrm{r}}}{\mathrm{dt}^2}] \quad = \quad \frac{\mathrm{Gm}_1\mathrm{m}_2\underline{\mathrm{r}}}{\mathrm{r}^3}$$

where m_1 and m_2 are the masses of the bodies, r is the distance between them and $\underline{r}$ is the *vector* distance, i.e. the force of gravity is in the direction of the line joining the two bodies. [d $\underline{r}$/dt is the velocity $\underline{v}$, and $d^2\underline{r}/dt^2 = d\underline{v}/dt$ is the *acceleration* $\underline{a}$]. Gravity is a *central force* (not all forces are; think of friction, which is strictly local and doesn't operate unless two bodies are touching), and this is the basic reason why the strength of the force goes as $1/r^2$ - the area of a sphere of radius r is $4\pi r^2$.

The derivation of orbits in a central potential is given in Shu, pp 463-466, the handout attached to the end of these lecture notes, in CO Ch2, pp 25-39 and in Zeilik and Gregory, P1-8. In the following discussion, we'll deal with the simplified situation of circular orbits.

All of the objects in the Solar System attract each other and move about a common center of mass, the *barycenter* of the Solar System. The formal definition of the center of mass is that it is the weighted mean position of the positions of a group of objects. The "weights" are the masses of the individual objects. This definition can be extended to a continuous object. The center of mass of a human being is approximately located in the abdomen. In some athletic events, like the pole vault or the high jump, the trajectory of the center of mass passes well under the bar, but the athlete moves her body during the jump so that all parts of the body pass above the bar - the athlete's body is curved as it passes over the bar. The center of mass of some SUVs is high above the wheelbase, which is why these vehicles can sometimes roll over too easily).

Back to a simple situation. The center of mass of two bodies of masses m_1 and m_2 is at

$$\underline{\mathrm{r}} \quad = \quad \frac{\mathrm{m}_1\underline{\mathrm{r}}_1 \ + \ \mathrm{m}_2\underline{\mathrm{r}}_2}{\mathrm{m}_1 \ + \ \mathrm{m}_2}$$

where $\underline{r}_1$ and $\underline{r}_2$ are the positions of the two bodies. We can estimate the approximate location of the Solar System barycenter by considering just Jupiter, which has more mass than all the other planets put together. If $\underline{r}_1 = 0$, $\underline{r}_2$ is the Sun-Jupiter distance, m_1 is the Sun's mass and m_2 is Jupiter's mass, then

$$r = \frac{0.001}{1.001} r_2 = 7.8 \times 10^{10} \text{ cm}$$

Since the Sun's radius is about 7×10^{10} cm, the barycenter of the Solar System lies just above the solar surface.

Since this distance is small compared to the distances between the planets and the Sun, let's make the approximation that the Sun doesn't move (this is equivalent to assuming that the Sun has infinite mass or that the planets have zero mass - we are taking them to be *test particles*). Let's also make the simplifying assumption that all orbits are circular (the general case is discussed in the handout). Let the Sun's mass be M and the planet's m, and let r be the distance between them. Note that even though the Sun (and the planets) are very large the assumption that they are point masses located at their center is a very good one (see CO p 37). So

$$m \frac{d^2 \underline{r}}{dt^2} = -\frac{GMm\underline{r}}{r^3}$$

Note that the force always acts along r. If the planet's orbit is circular, its speed is always directed perpendicular to the radius vector and since there is no force acting in the direction of motion, the planet's speed v is constant. However, the planet's *velocity* $\underline{v}$ is not constant, because the planet is following a circular path and not a straight line. See the diagram on p15 of Shu. The change in velocity $\Delta\underline{v}$ in time interval Δt is $\Delta\underline{v} = \underline{v}\Delta\theta$, where $\Delta\theta$ is the change in angle of the planet centered on the Sun. The *acceleration* $\underline{a}$ is then $\Delta\underline{v}/\Delta t = \underline{v} d\theta/dt$. Since the speed

$$v = r\frac{d\theta}{dt}$$

$$\underline{a} = \frac{v^2}{r}\frac{\underline{r}}{r}$$

This is often called the *centrifugal* or, more correctly, *centripetal* acceleration: it is the inertia of the body to having its velocity changed (equal and opposite reaction). Then

$$\frac{mv^2}{r} = -\frac{GMm}{r^2}$$

The *angular momentum* is mvr. From the above

$$mvr = -\frac{GMm}{v}$$

and is *constant*. This is easy to see, in fact is trivial for a circular orbit, but is true for all orbits in central potentials (see the handout). The angular momentum, usually called $\underline{L}$ (it is a vector), determines the *shape* and *orientation* of the orbit and is a *constant of motion*. This is *Kepler's Second Law*: the radius vector sweeps out equal areas in equal times.

The period P is $2\pi r/v$, so we have

$$\frac{v^2}{r} = -\frac{GM}{r^2}$$

i.e.

$$v^2 = -\frac{GM}{r}$$

giving

$$\frac{r^3}{p^2} = -\frac{GM}{4\pi^2} = \text{constant}$$

This is *Kepler's Third Law.* Now we can see why we got different answers for P^2/a^3 for the Solar System and for the Jupiter satellite system (Tables 1 and 2): the masses of the Sun and Jupiter are different. Indeed, by comparing the value of P^2/a^3 for these two systems with the above formulation of Kepler's third law, we can see that Jupiter is about a thousand times less massive than the Sun. Thus this equation lets you measure M, the mass of *any* body which has other bodies in orbit around it, since you can measure the period of the orbiting object and its distance, G is a universal constant (the gravitation constant, numerical value 6.7×10^{-8} $\mathrm{gm^{-1}cm^3s^{-2}}$ in c.g.s units) and the rest is numbers. This works for any body in the Universe if you can make the appropriate measurements.

A note about angular momentum in the Solar System. The *Sun* rotates on its axis every 27 days. Its angular momentum of rotation is approximately

$$\frac{2}{5}M_\odot R_\odot^2 \omega_\odot$$

The 2/5 is a geometric factor for the spherical Sun, and $\omega_\odot$ is the rotation frequency of the Sun, once per 27 days, or 4.3×10^{-7} Hz ($\omega = 1/P$). The Solar angular momentum is thus 1.7×10^{48} $\mathrm{gm\ cm^2\ s^{-1}}$. The orbital angular momentum of Jupiter around the Sun is $M_J D_J^2 \omega_J$, where M_J is the mass of Jupiter, D_J the distance between Jupiter and the Sun, and ω_J is Jupiter's revolution frequency. The numbers from Table 1 give Jupiter's orbital angular momentum = 3.1×10^{49} $\mathrm{gm\ cm^2s^{-1}}$. Thus, while most of the *mass* of the Solar System is in the Sun, most of the *angular momentum* is in the planets.

Finally, let's look at the *energy.* The kinetic energy is $mv^2/2$ and the potential energy is $-GMm/r$. So

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{r} = \text{constant}$$

E is the second *constant of motion*, and determines the size of the orbit.

Note that E is *negative.* Orbits which have negative energy are *bound orbits* or *closed orbits*: the planet orbits the star and does not wander off into outer space. Circular and elliptical orbits are *closed* or *bound* orbits with *negative total energy.* Can there be orbits with positive total energy? Yes, but these are not bound. The object approaches the star, follows a curved orbit around the star, and moves away from the star and never returns. Orbits with $E = 0$ are *parabolic* and those with $E > 0$ are *hyperbolic.* These curves (circle, ellipse, parabola, hyperbola) are sometimes called *conic sections* because you can make them by intersecting a plane with a cone at various angles. Parabolic and hyperbolic orbits are discussed in the attached handout.