Pre-Algebra
Class 4 - Fractions II

In this lecture we are going to go over the most basic operations involving fractions, namely addition and subtraction. We will then use those operations to solve algebraic equations involving fractions.

1 Adding and Subtracting Fractions

We will start by going over how to add and subtract fractions. There are three possible cases we'll have to deal with: fractions with like denominators, fractions with unlike denominators, and mixed numbers. In all cases before we can do the addition or subtraction of the fractions we must initially transform the fractions so that they have like denominators (ideally the lowest common denominator). The addition and subtraction then becomes straightforward, as explained below.

1.1 Fractions with like denominators

For fractions with like denominators one should add or subtract the numerators while leaving the denominator unchanged:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

If you prefer thinking about real life objects, imagine that you have a pie which you have cut into 12 slices. Each slice is then $\frac{1}{12}$ of the total pie. If you are given 7 of the slices you have $7 \cdot \frac{1}{12} = \frac{7}{12}$ of the pie. If you now eat three slices you have four slices left $(7 - 3 = 4)$, i.e. you have $\frac{4}{12} = \frac{1}{3}$ of the total pie left. Putting this into a mathematical expression, we have:

$$\frac{7}{12} - \frac{3}{12} = \frac{7-3}{12} = \frac{4}{12} = \frac{1}{3}.$$

Note: You only add or subtract the numerator ('upstairs') - the denominator ('downstairs') is left unchanged.

Examples: A few examples involving subtraction and addition:

(a) $\frac{1}{2} + \frac{1}{2} = \frac{1+1}{2} = \frac{2}{2} = 1$

(b) $\frac{1}{2} - \frac{1}{2} = \frac{1-1}{2} = \frac{0}{2} = 0$

(c) $\frac{7}{12} + \frac{1}{12} = \frac{7+1}{12} = \frac{8}{12} = \frac{2}{3}$

(d) $\frac{7}{12} - \frac{1}{12} = \frac{7-1}{12} = \frac{6}{12} = \frac{1}{2}$

(e) $\frac{1}{12} - \frac{7}{12} = \frac{1-7}{12} = \frac{-6}{12} = -\frac{1}{2}$

(f) $-\frac{1}{12} + \frac{7}{12} = \frac{-1+7}{12} = \frac{6}{12} = \frac{1}{2}$

Example: Imaginee that you have a cake which you have sliced in 6 pieces. If you have two guests who both eat 1 piece each, what fraction of the cake is left?

Solution: Each slice corresponds to $\frac{1}{6}$ of the cake. The full cake is $1 = \frac{6}{6}$ and your guests are eating two slices $\frac{1}{6} + \frac{1}{6} = \frac{1+1}{6} = \frac{2}{6}$. Therefore you have

$$\frac{6}{6} - \frac{2}{6} = \frac{6-2}{6} = \frac{4}{6} = \frac{2}{3}$$

left of the cake.

1.2 Fractions with unlike denominators

To add or subtract fractions with unlike denominators you must first change them to have the same denominator. Ideally you should find the least common denominator, but any common denominator will do. Do not try to add or subtract fractions before having a common denominator i- attempting to do so is asking for a (mathematical) disaster.

Example:

$$\frac{1}{20} + \frac{2}{5}$$

We can factorize $20 = 2 \cdot 2 \cdot 5$ but 5 can't be factorized further. Therefore the least common denominator is $lcd = 2 \cdot 2 \cdot 5 = 20$. We then get:

$$\frac{1}{20} + \frac{2}{5} = \frac{1}{20} + \frac{4 \cdot 2}{4 \cdot 5} = \frac{1}{20} + \frac{8}{20} = \frac{1+8}{20} = \frac{9}{20}$$

Example:

$$\frac{3}{20} - \frac{2}{35}$$

We can factorize $20 = 2 \cdot 2 \cdot 5$ and $35 = 5 \cdot 7$. Therefore the least common denominator is $lcd = 2 \cdot 2 \cdot 5 \cdot 7 = 140$. We then get:

$$\frac{3}{20} - \frac{2}{35} = \frac{7 \cdot 3}{7 \cdot 20} - \frac{4 \cdot 2}{4 \cdot 35} = \frac{21}{140} - \frac{8}{140} = \frac{21 - 8}{140} = \frac{13}{140}$$

which can't be simplified further.

Example:

$$\frac{3}{15} + \frac{2}{25}$$

We can factorize $15 = 3 \cdot 5$ and $25 = 5 \cdot 5$. Therefore the least common denominator is $lcd = 3 \cdot 5 \cdot 5 = 75$. We then get:

$$\frac{3}{15} + \frac{2}{25} = \frac{5 \cdot 3}{5 \cdot 15} + \frac{3 \cdot 2}{3 \cdot 25} = \frac{15}{75} + \frac{6}{75} = \frac{21}{75} = \frac{7}{25}$$

Example: You have a pizza which has been cut into 8 slices. If you eat half the pizza and your friend eats one slice, what fraction of the pizza has been eaten?

Solution: You have eaten half, i.e. $\frac{1}{2}$, of the pizza and your friend has had 1 slice, i.e. $\frac{1}{8}$. So in total you've had:

$$\frac{1}{2} + \frac{1}{8} = \frac{4 \cdot 1}{4 \cdot 2} + \frac{1}{8} = \frac{4 + 1}{8} = \frac{5}{8}.$$

If you are having troubles finding the least common denominator you can always find a common denominator by multiplying the two denominators of the fractions you are trying to add or subtract, i.e.

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d}{b \cdot d} + \frac{c \cdot b}{d \cdot b} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d}{b \cdot d} - \frac{c \cdot b}{d \cdot b} = \frac{ad - bc}{bd}$$

The only downside is that the numbers may become rather large - and you may have to do additional simplifying of the final result.

Example: Let's again look at:

$$\frac{1}{20} + \frac{2}{5}$$

We found previously that $lcd = 20$. But assuming we were having troubles finding it we could also take as the denominator $20 \cdot 5 = 100$:

$$\frac{1}{20} + \frac{2}{5} = \frac{5 \cdot 1}{5 \cdot 20} + \frac{20 \cdot 2}{20 \cdot 5} = \frac{5}{100} + \frac{40}{100} = \frac{5 + 40}{100} = \frac{45}{100} = \frac{9}{20}$$

This is the same result as before, but involved one extra step to simplify the final expression.

Remember: Do NOT try to add or subtract fractions which have unlike denominators - before adding or subtracting you MUST first change them to have the same denominator.

1.3 Mixed numbers

Lecturer: It is important that we keep the notation for $A \cdot \frac{b}{c}$ separate from $A\frac{b}{c}$.

There are two equally valid methods for adding and subtracting mixed numbers. The first one is based on the fact that a mixed number $A\frac{b}{c} = A + \frac{b}{c}$. Therefore, when adding or subtracting mixed numbers one can add or subtract them separately:

$$A\frac{b}{c} + D\frac{e}{f} = A + \frac{b}{c} + D + \frac{e}{f} = (A+D) + \left(\frac{b}{c} + \frac{e}{f}\right) = (A+D) + \frac{bf+ce}{cf}$$

$$A\frac{b}{c} - D\frac{e}{f} = A + \frac{b}{c} - \left(D + \frac{e}{f}\right) = (A-D) + \left(\frac{b}{c} - \frac{e}{f}\right) = (A-D) + \frac{bf-ce}{cf}$$

Let's look at a few examples:

Example:

$$4\frac{1}{20} + 12\frac{2}{5}$$

We have that $lcd = 20$.

$$4\frac{1}{20} + 12\frac{2}{5} = 4 + \frac{1}{20} + 12 + \frac{2}{5} = (4+12) + \left(\frac{1}{20} + \frac{4\cdot 2}{4\cdot 5}\right) = 16 + \frac{9}{20} = 16\frac{9}{20}$$

Example:

$$4\frac{7}{9} - 2\frac{1}{5}$$

We have that $lcd = 45$.

$$4\frac{7}{9} - 2\frac{1}{5} = 4 + \frac{7}{9} - \left(2 + \frac{1}{5}\right) = (4-2) + \left(\frac{7}{9} - \frac{1}{5}\right) = 2 + \frac{5\cdot 7 - 9\cdot 1}{45} = 2 + \frac{26}{45} = 2\frac{26}{45}$$

In some cases the fraction you end up with is negative. In that case, to collapse it back to a mixed number notation, you have to 'borrow' a 1 from the whole number to make the fraction be positive. You can do this regardless of whether the whole number is positive, zero or negative.

Example: Let's look at an example where the fraction you end up with is negative.

$$4\frac{1}{9} - 2\frac{1}{5}$$

We have that $lcd = 45$.

$$4\frac{1}{9} - 2\frac{1}{5} = (4-2) + \left(\frac{5\cdot 1}{5\cdot 9} - \frac{9\cdot 1}{9\cdot 5}\right) = 2 + \frac{-4}{45}$$

This can't be collapsed to a mixed number so we need to 'borrow' 1 from the integer: $2 = 1 + 1 = 1 + \frac{45}{45}$, i.e.

$$4\frac{1}{9} - 2\frac{1}{5} = 2 + \frac{-4}{45} = 1 + \frac{45}{45} + \frac{-4}{45} = 1 + \frac{45-4}{45} = 1\frac{41}{45}.$$

Another way for working with mixed numbers is to initially change them to the form of improper fractions, i.e.

$$A\frac{b}{c} = A + \frac{b}{c} = \frac{A \cdot c}{c} + \frac{b}{c} = \frac{A \cdot c + b}{c}$$

We then get for the addition and subtraction that:

$$A\frac{b}{c} + D\frac{e}{f} = \frac{A \cdot c + b}{c} + \frac{D \cdot f + e}{f}$$

$$A\frac{b}{c} - D\frac{e}{f} = \frac{A \cdot c + b}{c} - \frac{D \cdot f + e}{f}$$

To do the addition or subtraction one must then next find the common denominator as for proper fractions. At the end, one should change the final result back to the form of a mixed number.

Example:

$$4\frac{7}{9} + 2\frac{1}{5}$$

We start by changing the mixed numbers to improper fractions.

$$4\frac{7}{9} = \frac{4 \cdot 9 + 7}{9} = \frac{43}{9}$$

$$2\frac{1}{5} = \frac{2 \cdot 5 + 1}{5} = \frac{11}{5}$$

The least common denominator is 45 so we get:

$$4\frac{7}{9} + 2\frac{1}{5} = \frac{43}{9} + \frac{11}{5} = \frac{5 \cdot 43}{5 \cdot 9} + \frac{9 \cdot 11}{9 \cdot 5} = \frac{215 + 99}{45} = \frac{314}{45} = 6\frac{44}{45}$$

Example:

$$4\frac{1}{9} - 2\frac{1}{5}$$

First we change the mixed numbers to improper fractions:

$$4\frac{1}{9} = \frac{4 \cdot 9 + 1}{9} = \frac{37}{9}$$

$$2\frac{1}{5} = \frac{2 \cdot 5 + 1}{5} = \frac{11}{5}$$

We then get:

$$4\frac{1}{9} - 2\frac{1}{5} = \frac{37}{9} - \frac{11}{5} = \frac{5 \cdot 37}{5 \cdot 9} - \frac{9 \cdot 11}{9 \cdot 5} = \frac{185 - 99}{45} = \frac{86}{45} = 1\frac{41}{45}$$

1.4 A word of warning

2 Solving Algebraic Equations Involving Fractions

We have previously learned how to solve algebraic equations involving integers. Solving algebraic equations involving fractions follows the precise same procedure - the only difference is that we now have to add and subtract fractions.

Let's look at a few examples:

Example: Solve $x + \frac{7}{9} = \frac{1}{4}$. We start by subtracting $\frac{7}{9}$ from both sides:

$$\begin{aligned} x + \frac{7}{9} &= \frac{1}{4} \\ \left(x + \frac{7}{9}\right) - \frac{7}{9} &= \left(\frac{1}{4}\right) - \frac{7}{9} \\ x + 0 &= \frac{1}{4} - \frac{7}{9} \end{aligned}$$

Next we have to do the subtraction on the right hand side by finding the *lcd*:

$$\begin{aligned} x &= \frac{9 \cdot 1}{9 \cdot 4} - \frac{4 \cdot 7}{4 \cdot 9} = \frac{9 - 28}{36} = \frac{-19}{36} \\ x &= -\frac{19}{36} \end{aligned}$$

Example: Solve $\frac{1}{2} + x = \frac{7}{13}$. We start by subtracting $\frac{1}{2}$ from both sides:

$$\begin{aligned} \frac{1}{2} + x &= \frac{7}{13} \\ \left(\frac{1}{2} + x\right) - \frac{1}{2} &= \left(\frac{7}{13}\right) - \frac{1}{2} \\ x + 0 &= \frac{7}{13} - \frac{1}{2} \end{aligned}$$

Next we have to do the subtraction on the right hand side by finding the *lcd*:

$$\begin{aligned} x &= \frac{2 \cdot 7}{2 \cdot 13} - \frac{13 \cdot 1}{13 \cdot 2} = \frac{14 - 13}{26} = \frac{1}{26} \\ x &= \frac{1}{26} \end{aligned}$$

Example: *Note to lecturer: This might be too advanced at this stage as it requires multiplying fractions. If you wish to simplify, solve the problem for $y = 2x$.*

Solve $\frac{1}{4} + 2 \cdot x = \frac{1}{2}$. We start by subtracting $\frac{1}{4}$ from both sides:

$$\begin{aligned} \frac{1}{4} + 2 \cdot x &= \frac{1}{2} \\ \left(\frac{1}{4} + 2 \cdot x\right) - \frac{1}{4} &= \left(\frac{1}{2}\right) - \frac{1}{4} \\ 2 \cdot x + 0 &= \frac{1}{2} - \frac{1}{4} \end{aligned}$$

Next we have to do the subtraction on the right hand side by finding the *lcd*:

$$\begin{aligned} 2 \cdot x &= \frac{2 \cdot 1}{2 \cdot 2} - \frac{1}{4} = \frac{2-1}{4} = \frac{1}{4} \\ 2 \cdot x &= \frac{1}{4} \end{aligned}$$

Next we have to divide both sides by two - or equivalently multiply by $\frac{1}{2}$. To be able to do this we need to know that $A \cdot \frac{b}{c} = \frac{A \cdot b}{c}$ and $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$.

$$\begin{aligned} 2 \cdot x &= \frac{1}{4} \\ 2 \cdot x \cdot \frac{1}{2} &= \frac{1}{4} \cdot \frac{1}{2} \\ \frac{2 \cdot x}{2} &= \frac{1 \cdot 1}{4 \cdot 2} \\ x &= \frac{1}{8} \end{aligned}$$

Example: Alice, Bob and Claudine are sharing a bottle of wine. Alice drinks $\frac{1}{2}$ the bottle and Bob drinks $\frac{1}{6}$ of the bottle.

(a) How much does Claudine drink?
Let's call the amount Claudine drinks x. The total amount that Alice, Bob and Claudine drink is the full bottle so we have:

$$x + \frac{1}{2} + \frac{1}{6} = 1$$

Let's start by adding the two fractions:

$$\begin{aligned} x + \frac{1}{2} + \frac{1}{6} &= 1 \\ x + \frac{3 \cdot 1}{3 \cdot 2} + \frac{1}{6} &= 1 \\ x + \frac{3+1}{6} &= 1 \\ x + \frac{2}{3} &= 1 \end{aligned}$$

Next we subtract $\frac{2}{3}$ from BOTH sides:

$$\begin{aligned} x+\frac{2}{3} &= 1 \\ \left(x+\frac{2}{3}\right)-\frac{2}{3} &= 1-\frac{2}{3} \\ x+0 &= \frac{3}{3}-\frac{2}{3} \\ x &= \frac{1}{3} \end{aligned}$$

(b) Who drinks the most?
As $\frac{1}{6}<\frac{1}{3}<\frac{1}{2}$ it is Alice who drinks the most. We could have reached this conclusion without any calculation: if Alice drinks half then the other two friends only have half a bottle to share between them and therefore will each have less than Alice.

You can also transform an algebraic equation involving fractions into a regular algebraic equation involving integers by multiplying through with the least common denominator (*lcd*) of all the fractions in the equation. You then solve the algebraic equation as before. To do this you need to know how to multiply a fraction with a whole number - and note that $A\cdot\frac{b}{c}=\frac{A\cdot b}{c}$ is not the same as the mixed number $A\frac{b}{c}=A+\frac{b}{c}$.

Example: Let's look at the equation:

$$\frac{1}{3}+x=\frac{2}{5}$$

We can first solve it like we did before by subtracting $\frac{1}{3}$ from both sides of the equation:

$$\begin{aligned} \left(\frac{1}{3}+x\right) &= \left(\frac{2}{5}\right) && (1) \\ -\frac{1}{3}+\left(\frac{1}{3}+x\right) &= \left(\frac{2}{5}\right)-\frac{1}{3} && (2) \\ 0+x &= \frac{3\cdot 2}{3\cdot 5}-\frac{5\cdot 1}{5\cdot 3} && (3) \\ x &= \frac{6-5}{15}=\frac{1}{15} && (4) \\ & && (5) \end{aligned}$$

Alternatively we could have multiplied both sides through with the $lcd = 3\cdot 5=15$:

$$\begin{aligned} \left(\frac{1}{3}+x\right) &= \left(\frac{2}{5}\right) && (6) \\ 15\cdot\left(\frac{1}{3}+x\right) &= \left(\frac{2}{5}\right)\cdot 15 && (7) \end{aligned}$$

$$
\begin{aligned}
15 \cdot \frac{1}{3} + 15 \cdot x &= \frac{15 \cdot 2}{5} && (8)\\
\frac{15 \cdot 1}{3} + 15 \cdot x &= \frac{30}{5} && (9)\\
5 + 15 \cdot x &= 6 && (10)\\
& && (11)
\end{aligned}
$$

At this stage we have transformed the equation into a normal equation without fractions which we can solve:

$$
\begin{aligned}
-5 + 5 + 15 \cdot x &= 6 - 5 && (12)\\
15 \cdot x &= 1 && (13)\\
\frac{15 \cdot x}{15} &= \frac{1}{15} && (14)\\
x &= \frac{1}{15} && (15)
\end{aligned}
$$

3 Summary of main concepts

1. Adding and subtracting fractions:

   A. Put them in form where they have the same denominator.

   B. Add and subtract the numerator but leave the denominator unchanged.

   C. When dealing with mixed numbers remember $A\frac{b}{c} = A + \frac{b}{c}$.

   D. Sometimes when subtracting mixed numbers you must 'borrow' a 1 from the integer number.

2. Solving equations:

   A. You can solve them in the same way as for equations with integers - but you must follow the rules for adding and subtracting frations.

   B. You can also change the equations to an integer equation by multiplying both sides with the *lcd* of the fractions involved.

3. Words of warning:

   A. NEVER try to add or subtract fractions with different denominators - it will end in tears.

   B. Keep in mind that $A \cdot \frac{b}{c} = \frac{A \cdot b}{c}$ is not the same as the mixed number $A\frac{b}{c} = A + \frac{b}{c}$.