Pre-Algebra Lecture 2. Factors

This lesson will review the following concepts:

- Factors and multiples
- Prime numbers, composite numbers, and prime factorization
- Tests for divisibility by 2, 3, and 5.
- Greatest common factor
- Least common multiple

(1) Definitions of "factor" and "multiple"

Definition: The *factors* of an integer are those integers by which it is divisible.

(Remember that "integers" are whole numbers, whether positive, negative, or 0.) Here "divisible" means that the division leaves zero remainder. So 3 is a factor of 6 because $6 = 3 \times 2$. But 5 is not a factor of 17 because if we divide 17 by 5, we get 3 with remainder 2: that is to say, $17 = (5 \times 3) + 2$.

The factors of any nonzero integer include the integer itself. So 6 is a factor of 6 because $6 = 6 \times 1$. Clearly every integer also has 1 as a factor.

Here are some numbers together with all of their positive factors:

- 5: 1,5
- 6: 1,2,3,6
- 8: 1,2,4,8
- 12: 1,2,3,4,6,12
- 8357: 1,61,137,8357

Factors of negative integers, and of zero

Negative integers also have factors. For example, 3 is a factor of -15 because $-15 = 3 \times (-5)$. This shows that -5 is also a factor of -15. So a negative integer has both positive and negative factors. This is also true of positive integers. For example, -2 is a factor of 6 because $6 = (-2) \times (-3)$. The *complete* list of all integer factors of 6 is

$$-6, -3, -2, -1, 1, 2, 3, 6$$

Normally however, when we ask for the factors of a positive integer, we are interested in the positive factors, not the negative ones.

Here are two tricky points:
First, 0 has every other integer as a factor.
Second, 0 is not a factor of any integer. The first statement is true because dividing 0 by any nonzero integer leaves zero remainder. For example, 0 has 2 as a factor because $0 = 0 \times 2$. The second statement above is true because one cannot divide any number by zero. We cannot even divide 0 by 0. So 0 cannot be a factor of anything. Also, 0 is the *only* integer that does not have itself as a factor.

Multiples

Closely related to the notion of factor is the notion of multiple.

> **Definition:** An integer is a *multiple* of another integer if the first is divisible by the second. In other words, an integer a is a multiple of another integer b if and only if b is a factor of a.

For example, the nonnegative multiples of 3 are

$$0, 3, 6, 9, 12, 15, 18, 21, \ldots$$

where "…" means "and so on." Let us check this. 21 is listed as a multiple of 3. According to the definition, 3 should then be a factor of 21. Is that true? Yes: $21 = 3 \times 7$. From this we see that 21 is also a multiple of 7.

(2) Prime and composite numbers; Prime factorization

> **Definition:** A *prime number* is a positive integer greater than 1 whose only positive factors are 1 and itself.

For example, 2 is a prime number because its only positive factors are 1 and 2; also 3 is a prime number because its only positive factors are 1 and 3. But 1 is not a prime number, for reasons explained later. Also 4 is not a prime number because it has 2 as well as 1 and 4 as factors. 4 is an example of a *composite number*:

> **Definition:** A *composite* integer is one that has positive factors other than itself and 1.

The importance of primes stems from the fact that every positive integer can be factored into a product of primes, and the factorization is unique apart from changes in order. This is called **prime factorization**. For example,

$$\begin{aligned} 6 &= 2 \times 3 \\ 12 &= 2 \times 2 \times 3 \\ 21 &= 3 \times 7 \\ 51 &= 3 \times 17. \end{aligned}$$

Notice that we could have written 3×2 instead of 2×3 as the prime factorization of 6, but these are the same factorization because the order of the factors doesn't matter. Notice also that 2 occurs twice in the prime factorization of 12.

The word "prime" has the same origin as "primary," meaning "first" or "most important". Primes come first because all other integers can be built up as products of them.

The reason that 1 is not considered to be prime is that prime factorization would then not be unique: we could write, for example, $6 = 1 \times 2 \times 3 = 1 \times 1 \times 2 \times 3 = \ldots$, so 6 could be said to have any number of different factorizations. To prevent this, we declare that 1 is not a prime, even though it has only itself as a factor.

Most integers are composite, especially large integers. Nevertheless, **there are infinitely many primes**. Thus, it is not possible to make a complete list of all primes; **there is no such thing as the largest prime number**.[1] How do we know this? The reason is that, given any set of primes, we can find a prime that does not appear in the list in the following way: multiply all of the primes in the list by one another to make a number—call it N—and then add one to make $N + 1$. Clearly N has every prime on the list as a factor. For example, if our list is $\{2, 3, 5\}$, then $N = 2 \times 3 \times 5 = 30$. Now think about $N + 1$. This is *not* divisible by any prime on the list, because it leaves 1 as remainder when divided by any of them. In the example just given, $N + 1 = 31$, which leaves remainder 1 when divided by 2, 3, or

[1] You may skip the rest of this paragraph if you are willing to take this fact on faith.

5. Therefore, the prime factorization of $N + 1$ consists of primes that are not on the list. In fact, $N + 1$ may actually be prime itself, as 31 is. But not necessarily. For example, if we had started with the list $\{3, 5, 7\}$ then by the same method we would have got

$$3 \times 5 \times 7 + 1 = 106 = 2 \times 53.$$

So in this case $N + 1 = 106$ isn't prime, but its prime factorization 2×53 involves primes not on the original list. The point of all this is that no matter how long a list of primes we might have, there must always be primes that aren't on that list. Hence there are infinitely many primes.

(3)Tests for divisibility by 2, 3, and 5

Prime factorizations can be found by trial and error. It is usually best to start by trying small factors, because (i) these are most likely to succeed (small primes have frequent multiples); (ii) the divisions are easy; and (iii) if a small factor is found, then the *quotient* (the result of the division) is smaller and more easily factored than the original number. Tests for divisibility by 2, 3, and 5 are particularly easy.

To test whether an integer is divisible by 2, look at the least significant digit—meaning, the last digit on the right when the integer is written out in base 10. If this digit is even, then the whole number is even, which is the same as saying that it is divisible by 2. Thus one can see at a glance that 345678912 is divisible by 2 but 987654321 is not.

Testing for divisibility by 5 is again a matter of looking at the last digit. If this is 0 or 5, then the number is divisible by 5, otherwise not. Thus 105 and 110 are divisible by 5, but 107 is not.

Notice that if the last digit is 0, then according to these rules, the number is divisible by both 2 and 5. This means that it is also divisible by $2 \times 5 = 10$. But you knew that already!

The trick for testing for divisibility by 3 is only slightly more complicated. Add up all of the digits of the number; if the result has more than one digit, then add those digits to one another. Keep going like this until you get a single-digit number. If this result is divisible by 3—meaning that it is 0, 3, 6, or 9—then the original number is divisible by 3. For example, the sum of the digits of 87 is $8+7 = 15$, and $1+5 = 6$, so 87 must be divisible by 3. Indeed, $87 = 3 \times 19$. A quick check shows that 19 is prime, so the prime factorization is $87 = 3 \times 19$. How did we determine that 19 is prime? By the rules above, none of 2, 3, and 5 is a factor of 19. We never have to look for factors other than primes. And in this case, we don't need to look for factors larger than 5 because $5 \times 5 = 25 > 19$: if 19 had a factor greater than 5, then it would have to have a second factor less than 5, which we would already have found. This trick works generally: when looking for factors of some integer n, if we have some smaller integer m such that $m \times m > n$, then we need only look for prime factors $< m$. Any larger factors of n, if they exist, will be found as quotients.

Since 11, 31, 41, 61, and 71 are prime, people sometimes mistake 51 for a prime. But since $5 + 1 = 6$, the rule says that 51 is divisible by 3.

When looking for a prime factorization, we don't stop after finding the smallest factor. We keep going until we are left with nothing but primes. For example,

$$52 = 2 \times 26 = 2 \times 2 \times 13$$

After the first division by 2, we got 26, which isn't prime, so we we factored it as $26 = 2 \times 13$. Finally 13 *is* prime, so we knew that we were done, and we collected together all the primes we found. The principle here is that if a and b are two integers, then the prime factorization of $a \times b$ is the prime factorization of a times the prime factorization of b. So, the prime factorization of 52 is the prime factorization of 2 times the prime factorization of 26: $52 = (2) \times (2 \times 13)$. The prime factorization of 24 is the prime factorization of $(4) \times (6) = (2 \times 2) \times (2 \times 3) = 2 \times 2 \times 2 \times 3$.

(4) Greatest common factor

> The *greatest common factor* of two positive integers is the largest integer that is a factor of both.

For example, the greatest common factor of 18 and 24 is 6. Both 18 and 24 have larger factors than 6—for example 9 divides 18 and 12 divides 24—but 6 is the largest factor that divides both 18 and 24. As another example, the GCF (greatest common factor) of 5 and 15 is 5—this is true because 5 is a factor of itself.

How do we find the GCF?

Method #1: By prime factorization. This is best explained by example. The prime factorizations of 18 and 24 are

$$\begin{aligned} 18 &= 2 \times 3 \times 3 \\ 24 &= 2 \times 2 \times 2 \times 3 \end{aligned}$$

Looking at the two sets of prime factors $\{2, 3, 3\}$ and $\{2, 2, 2, 3\}$ (notice that repeated factors are listed as often as they appear), we see that the largest subset common to both is $\{2, 3\}$. Thus the GCF is $2 \times 3 = 6$.

Method #2: By Euclid's Algorithm. This works by repeated division with remainder, starting with the two numbers in question. At each step, we divide the smaller of the two numbers from the previous step by the remainder from that earlier division. We keep going until we get a remainder of 0. The last divisor is the GCF.

Let's do some examples of Euclid's Algorithm, starting with 18 and 24.

$$\begin{aligned} 24 \div 18 &= 1 \text{ with remainder } 6 \\ 18 \div 6 &= 3 \text{ with remainder } 0 \end{aligned}$$

Thus the GCF of 24 and 18 is 6. Here's a longer example: Find the GCF of 1848 and 1547:

$$\begin{aligned} 1848 \div 1547 &= 1 \text{ with remainder } 301 \\ 1547 \div 301 &= 5 \text{ with remainder } 42 \\ 301 \div 42 &= 7 \text{ with remainder } 7 \\ 7 \div 7 &= 1 \text{ with remainder } 0 \end{aligned}$$

The GCF is the remainder in the second-to-last step (i.e., the last step that yielded a nonzero remainder), in this case 7. *Study the pattern above carefully:* in each line, the first number (the dividend) is the same as the second number (the divisor) in the previous row, and the divisor in each line is the remainder from the previous line.[2] Method #2 usually requires less work than method #1.

(5) Least common multiple

> **Definition:** The *least common multiple* of two integers is the smallest positive integer that has both as factors.

Thus the least common multiple of 6 and 4 is 12; the least common multiple of 6 and 12 is also 12. Again, there are two methods for finding LCMs.

Method #1. Find the prime factorizations of both numbers. Construct the smallest set of primes that has both factorizations as subsets, and multiply the members of this set.

For example $6 = 2 \times 3$ and $4 = 2 \times 2$. Since both $\{2, 2\}$ and $\{2, 3\}$ have to be subsets, the set of factors of the LCM must be $\{2, 2, 3\}$. Then the LCM is $2 \times 2 \times 3 = 12$.

[2]Euclid may not have invented this algorithm, but he wrote it down in a textbook more than 2000 years ago, so he gets the credit. He also wrote the proof given earlier that there is an infinite number of primes.

Method #2. First find the GCF of the two integers, for example by Euclid's algorithm. Then multiply the two numbers together and divide the result by the GCF. This will be the LCM. It may be helpful to remember this by the following formula:

$$LCM(a, b) = \frac{a \times b}{GCF(a, b)}\,.$$

Here a and b stand for any two integers, $LCM(a, b)$ stands for their least common multiple, and $GCF(a, b)$ stands for the greatest common factor. For example, $GCF(6, 4) = 2$, so $(6 \times 4)/GCF(6, 4) = 24/2 = 12 = LCM(6, 4)$.