Physics of the Interstellar and Intergalactic Medium: Odd-Numbered Problems with Solutions

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I have assembled this collection of problems to accompany *Physics of the Interstellar and Intergalactic Medium*. The problems (without solutions) are publicly available as a pdf document at

http://www.astro.princeton.edu/∼draine/book/ .

From time to time the problem collection will be updated with new problems, and with corrections to old problems if needed.

For the benefit of instructors and students using *Physics of the Interstellar and Intergalactic Medium*, the odd-numbered problems with solutions are compiled here.. You may wish to check

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for the latest version of the collected problems.

If you detect errors in the problems or in the solutions provided here, please notify the author via email to draine@astro.princeton.edu .

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Chapter 1. Introduction

1.1 The total mass of neutral gas in the Galaxy is $\sim 4\times10^9\,M_\odot$. Assume that it is uniformly distributed in a disk of radius $R_{\rm disk} = 15\,{\rm kpc}$ and thickness $H = 200\,{\rm pc}$, and that it is a mixture of H and He with He/H=0.1 (by number). Assume ionized hydrogen to be negligible in this problem. [Note: even though the assumptions in this problem are very approximate, please carry out calculations to **two** significant digits.]

(a) What is the average number density of hydrogen nuclei within the disk?

(b) If 0.7% of the interstellar mass is in the form of dust in spherical particles of radius $a = 1000\,\text{Å} = 0.1\,\mu{\rm m}$ and density $2\,{\rm g\,cm^{-3}}$, what is the mean number density of dust grains in interstellar space?

(c) Let $Q_{\rm ext}$ be the ratio of the visual (V band, $\lambda = 0.55\,\mu{\rm m}$) extinction cross section to the geometric cross section πa^2. Suppose that $Q_{\rm ext} \approx 1$. What would be the visual extinction A_V (in magnitudes!) between the Sun and the Galactic Center (assumed to be $8.5\,{\rm kpc}$ away)?

(d) Now assume that 30% of the gas and dust mass is in spherical molecular clouds of radius 15 pc and mean density $n({\rm H_2}) = 100\,{\rm cm^{-3}}$.
What would be the mass of one such cloud?
How many such molecular clouds would there be in the Galaxy?

(e) With 30% of the gas and dust mass in molecular clouds as in (d), what is the expectation value for the visual extinction A_V to the Galactic Center?

(f) With 30% of the material in molecular clouds as in (d), what is the expectation value for the number of molecular clouds that will be intersected by the line of sight to the Galactic center?
What is the probability that zero molecular clouds will be intersected? [Hint: the number of molecular clouds in the Galaxy is large, and they occupy a small fraction of the volume, so think of this as a "Poisson process", where the presence or absence of each molecular cloud on the line-of-sight is treated as an independent event (like the number of radioactive decays in a fixed time interval).]

(g) If the line of sight to the Galactic center happens not to intersect any molecular clouds, and if the atomic hydrogen and associated dust are distributed uniformly throughout the disk volume, what will be the visual extinction to the Galactic center?

Solution:

(a)

$$n_{\rm H} = \frac{4\times10^9\,M_\odot/1.4m_{\rm H}}{\pi R_{\rm disk}^2 H} = 0.82\,{\rm cm^{-3}} \quad .$$

(b)

$$\langle n_{\rm dust}\rangle = \frac{0.007\times 4\times10^9\,M_\odot}{(4\pi/3)a^3\rho\times\pi R_{\rm disk}^2 H} = 1.60\times10^{-12}\,{\rm cm^{-3}} \quad .$$

(c)

$$\begin{aligned}\tau &= \langle n_{\rm dust}\rangle Q_{\rm ext}\pi a^2 L \\ &= 1.60\times10^{-12}\,{\rm cm}^{-3}\times\pi(10^{-5}\,{\rm cm})^2\times 8.5\times 3.086\times10^{21}\,{\rm cm} \\ &= 13.2 \quad . \\ A_V &= \frac{2.5}{\ln 10}\tau = 1.086\tau = 14.3\ {\rm mag}\ .\end{aligned}$$

(d)

$$\begin{aligned}M_{\rm cloud} &= (4\pi/3)R_{\rm cloud}^3\times 1.4\times 2n({\rm H_2})m_{\rm H} = 9.8\times10^4\,M_\odot \quad . \\ N_{\rm cloud} &= 0.3M_{\rm ISM}/M_{\rm cloud} = 1.23\times10^4 \quad .\end{aligned}$$

(e) If clouds are randomly-distributed,d then expectation value is same as if gas and dust are uniformly-distributed, as in (c): $\langle A_V\rangle = 14.3$ mag.

(f) Let ν = number of clouds intersected. The expectation value for ν is

$$\begin{aligned}\langle\nu\rangle &= \mu_{\rm cl} = \left(\frac{N_{\rm cloud}}{\pi R_{\rm disk}^2 H}\right)\pi R_{\rm cloud}^2 L \\ &= N_{\rm cloud}\left(\frac{R_{\rm cloud}}{R_{\rm disk}}\right)^2\left(\frac{L}{H}\right) = 0.521 \quad .\end{aligned}$$

$${\rm Poisson:}\qquad P(\nu) = \frac{\mu_{\rm cl}^\nu\,{\rm e}^{-\mu_{\rm cl}}}{\nu!}\ \rightarrow\ P(\nu=0) = {\rm e}^{-\mu_{\rm cl}} = 0.594 \quad .$$

(g) If $\nu = 0$, then the dust in smoothly-distributed H I *will produce extinction equal to 0.7 of value found in (c): $A_V = 10.0$ mag.*

1.3 The "very local" interstellar medium has $n_{\rm H} \approx 0.22\,{\rm cm}^{-3}$ (Lallement et al. 2004: Astr. & Astrophys. 426, 875; Slavin & Frisch 2007: Sp. Sci. Revs. 130, 409). The Sun is moving at $v_W = 26 \pm 1\,{\rm km\,s}^{-1}$ relative to this local gas (Möbius et al. 2004: Astr. & Astrophys. 426, 897).

Suppose that this gas has He/H=0.1, and contains dust particles with total mass equal to 0.5% of the mass of the gas. Suppose these particles are of radius $a = 0.1\,\mu{\rm m}$ and density $\rho = 2\,{\rm g\,cm}^{-3}$, and we wish to design a spacecraft to collect them for study.

How large a collecting area A should this spacecraft have in order to have an expected collection rate of 1 interstellar grain per hour? Neglect the motion of the spacecraft relative to the Sun, and assume that the interstellar grains are unaffected by solar gravity, radiation pressure, and the solar wind (and interplanetary magnetic field).

Solution:

$$\begin{aligned}
n_{\rm dust} &= \frac{0.005 \times 1.4 n_{\rm H} m_{\rm H}}{[(4\pi/3)a^3\rho]} \\
&= 3.07\times10^{-13}\,{\rm cm}^{-3} \quad , \\
\text{Collision rate } \dot{N} &= n_{\rm dust} v_W A \\
A &= \frac{(1/3600\,{\rm s})}{(n_{\rm dust} v_W)} = 348\,{\rm cm}^2 \quad .
\end{aligned}$$

1.5 Suppose that large rocky objects from interstellar space pass within $1\,\mathrm{AU}$ of the Sun at a rate of 1 per year, with mean speed (at infinity) $v_{rock} = 20\,\mathrm{km\,s^{-1}}$. The objects are irregular, but suppose that they have solid volumes equal (on average) to spheres with radius $50\,\mathrm{m}$.

(a) If the rock itself has a mass density of $3\,\mathrm{g\,cm^{-3}}$, and 75% of the mass in the rock is contributed by the elements Mg, Si, and Fe, estimate $\langle\rho_{\rm rocks}^{\rm MgSiFe}\rangle$, the mean mass density in the ISM of Mg, Si, and Fe contained in such rocky objects. For simplicity, neglect effects of gravitational focusing by the Sun.

(b) If the mean density of H in the ISM is $\langle n_{\rm H}\rangle = 1\,\mathrm{cm^{-3}}$, and Mg, Si, Fe together contribute a mass equal to 0.4% of the H mass, estimate the fraction f of the interstellar Mg, Si, and Fe that is contained in these large rocky objects.

Solution:

(a) Neglecting gravitational focusing, the arrival rate $\dot{N}$ is related to the number density of rocks n_{rock} by

$$\begin{aligned}
\dot{N} &= n_{rock} v_{rock} \pi\,\mathrm{AU}^2 \\
n_{rock} &= \frac{\dot{N}}{\pi\,\mathrm{AU}^2 v_{rock}} \\
&= \frac{1}{3.16\times10^7\,\mathrm{s}} \times \frac{1}{\pi(1.5\times10^{13}\,\mathrm{cm})^2 \times 2\times10^6\,\mathrm{cm\,s^{-1}}} \\
&= 2.24\times10^{-41}\,\mathrm{cm^{-3}} \quad .
\end{aligned}$$

The mean density of Mg, Si, Fe present in such rocks, with volume-equivalent radius $r = 50\,\mathrm{m} = 5\times10^3\,\mathrm{cm}$, is

$$\begin{aligned}
\langle\rho_{\rm rocks}^{\rm MgSiFe}\rangle &= n_{rock} \times 0.75 \times 3\,\mathrm{g\,cm^{-3}} \times \frac{4\pi}{3} \times (5\times10^3\,\mathrm{cm})^3 \\
&= 2.64\times10^{-29}\,\mathrm{g\,cm^{-3}} \quad .
\end{aligned}$$

(b) The fraction f of the interstellar Mg, Si, and Fe locked up in such rocks is

$$\begin{aligned}
f &= \frac{\langle\rho_{\rm rocks}^{\rm MgSiFe}\rangle}{0.004\langle n_{\rm H}\rangle m_{\rm H}} \\
&= \frac{2.64\times10^{-29}\,\mathrm{g\,cm^{-3}}}{0.004 \times 1\,\mathrm{cm^{-3}} \times 1.67\times10^{-24}\,\mathrm{g}} \\
&= 3.95\times10^{-3} \quad .
\end{aligned}$$

1.7 The "very local" interstellar medium has $n_{\rm H} \approx 0.22\,{\rm cm}^{-3}$ (Lallement et al. 2004: Astr. & Astrophys. 426, 875; Slavin & Frisch 2007: Sp. Sci. Revs. 130, 409). The Sun is moving at $v_W \approx 26\,{\rm km\,s}^{-1}$ relative to this local gas (Möbius et al. 2004: Astr. & Astrophys. 426, 897).

(a) Suppose that this gas has He/H=0.1, and contains dust particles with total mass equal to 0.5% of the mass of the gas, in the form of dust grains of radius $a = 0.1\,\mu{\rm m}$ and density $\rho = 2\,{\rm g\,cm}^{-3}$.

(b) We wish to design a spacecraft to collect interstellar dust for study. The spacecraft will travel outwards from the Sun with at a speed sufficient to escape from the Solar system. Suppose that the heliocentric velocity of the spacecraft will be $v_{\rm spacecraft} = 30\,{\rm km\,s}^{-1}$ when it is far enough from the Sun that Solar gravity can be neglected. It will be traveling ***toward*** the incoming "interstellar wind".

How large a collecting area A is required to have an expected collection rate of 1 interstellar grain per 24 hours when it is far from the Sun?

Solution:
(a)

$$n_{\rm dust} = \frac{0.005 \times 1.4 n_{\rm H} m_{\rm H}}{[(4\pi/3)a^3\rho]} = 3.07\times10^{-13}\,{\rm cm}^{-3} \quad ,$$

(b)

$$\text{Collision rate } \dot{N} = n_{\rm dust} \times (v_W + v_{\rm spacecraft}) \times A$$

$$\begin{aligned}\text{Collecting area } A &= \frac{\dot{N}}{n_{\rm dust}(v_W + v_{\rm spacecraft})} \\ &= \frac{1}{24 \times 3600\,{\rm s} \times 3.07\times10^{-13}\,{\rm cm}^{-3} \times (26+30)\,{\rm km\,s}^{-1}} \\ &= 6.73\,{\rm cm}^2 \quad .\end{aligned}$$

Chapter 2. Collisional Processes

2.1 Consider an electron-proton plasma at temperature T. Let $t_s(e-e)$ be the time scale for 90 degree scattering of one electron with kinetic energy $\sim kT$ by encounters with other electrons.
The electron-proton mass ratio $m_p/m_e = 1836$. The following time scales t_x will differ from $t_s(e-e)$ by factors $(m_p/m_e)^\alpha$ and factors of order unity; ignore the latter, so that $t_x \approx (m_p/m_e)^\alpha \times t_s(e-e)$.

Identify the exponent α for each of the following processes; in each case, assume the process to be acting alone. It is not necessary to do any derivations – just give a one-sentence justification for each answer.

(a) 90 degree scattering of one electron by encounters with protons.

(b) 90 degree scattering of one proton by encounters with electrons.

(c) 90 degree scattering of one proton by encounters with other protons.

(d) exchange of energy from one electron to other electrons.

(e) exchange of energy from one electron to protons.

(f) exchange of energy from one proton to electrons.

(g) exchange of energy from one proton to other protons.

Solution:

(a) 90 degree scattering of one electron by encounters with protons:
$\alpha = 0$*; cross section similar, and electron-proton velocity difference is similar to electron-electron velocity difference.*

(b) 90 degree scattering of one proton by encounters with electrons:
$\alpha = 1$*; proton transfers only* $\sim 2m_e/m_p$ *of its momentum in a head-on scattering by an electron.*

(c) 90 degree scattering of one proton by encounters with other protons:
$\alpha = 1/2$*; proton speeds are smaller than electron speeds by a factor* $(m_e/m_p)^{-1/2}$.

(d) exchange of energy from one electron to other electrons:
$\alpha = 0$.

(e) exchange of energy from one electron to protons:
$\alpha = 1$*; scattering of electron by stationary proton will transfer only* $\sim(m_e/m_p)$ *of electron kinetic energy to proton.*

(f) exchange of energy from one proton to electrons:
$\alpha = 1$*; essentially same as* (e).

(g) exchange of energy from one proton to other protons:
$\alpha = 1/2$*; protons move more slowly than electrons.*

2.3 Consider a dust grain of radius a, and mass $M \gg m_{\rm H}$, where $m_{\rm H}$ is the mass of an H atom. Suppose that the grain is initially at rest in a gas of H atoms with number density $n_{\rm H}$ and temperature T. Assume the grain is large compared to the radius of an H atom. Suppose that the H atoms "stick" to the grain when they collide with it, so that all of their momentum is transferred to the grain, and that they subsequently "evaporate" from the grain with no change in the grain velocity during the evaporation.

(a) What is the mean *speed* $\langle v_{\rm H}\rangle$ of the H atoms (in terms of $m_{\rm H}$, T, and Boltzmann's constant $k_{\rm B}$)?

(b) Calculate the time τ_M for the grain to be hit by its own mass M in gas atoms. Express τ_M in terms of M, a, $n_{\rm H}$, and $\langle v_{\rm H}\rangle$.

(c) Evaluate $\langle v_{\rm H}\rangle$ and τ_M for a grain of radius $a = 10^{-5}\,{\rm cm}$ and density $\rho = 3\,{\rm g\,cm^{-3}}$, in a gas with $n_{\rm H} = 30\,{\rm cm^{-3}}$ and $T = 10^2\,{\rm K}$.

(d) If the collisions are random, the grain velocity will undergo a random walk. Estimate the *initial* rate of increase $(dE/dt)_0$ of the grain kinetic energy E due to these random collisions. Express $(dE/dt)_0$ in terms of $n_{\rm H}$, $m_{\rm H}$, $k_{\rm B}T$, a, and M. [Hint: think of the random walk that the grain momentum $\vec{p}$ undergoes, starting from the initial state $\vec{p} = 0$. What is the rate at which $\langle p^2\rangle$ increases?]

(e) Eventually the grain motion will be "thermalized", with time-averaged kinetic energy $\langle E\rangle = (3/2)k_{\rm B}T$. Calculate the timescale

$$\tau_E \equiv \frac{(3/2)k_{\rm B}T}{(dE/dt)_0}$$

for thermalization of the grain speed. Compare to τ_M calculated in (b).

Solution:

(a) The speed distribution function in a thermal gas is

$$f_v dv = \left(\frac{m_{\rm H}}{2\pi k_{\rm B}T}\right)^{3/2} e^{-m_{\rm H}v^2/2k_{\rm B}T} 4\pi v^2 dv \quad .$$

The mean speed of H atoms in a thermal gas is

$$\begin{aligned}
\langle v_{\rm H}\rangle &\equiv \int_0^\infty v f_v dv \\
&= \left(\frac{m_{\rm H}}{2\pi k_{\rm B}T}\right)^{3/2} \int_0^\infty v\, e^{-m_{\rm H}v^2/2k_{\rm B}T} 4\pi v^2 dv \\
&= \left(\frac{m_{\rm H}}{2\pi k_{\rm B}T}\right)^{3/2} \frac{4\pi}{2} \left(\frac{2k_{\rm B}T}{m_{\rm H}}\right)^2 \int_0^\infty e^{-x} x\, dx \\
&= \left(\frac{8k_{\rm B}T}{\pi m_{\rm H}}\right)^{1/2}
\end{aligned}$$

(b) The collision cross section is just the grain geometric cross section πa^2. The collision rate is just $\dot{N} = n_{\rm H}\langle v_{\rm H}\rangle \pi a^2$. Therefore

$$\tau_M \equiv \frac{M}{n_{\rm H}\langle v_{\rm H}\rangle m_{\rm H}\pi a^2}$$

(c)

$$\begin{aligned}
\langle v_{\rm H}\rangle &= \left(\frac{8k_{\rm B}T}{\pi m_{\rm H}}\right)^{1/2} = 1.45\times 10^5\,{\rm cm\,s^{-1}} = 1.45\,{\rm km\,s^{-1}}\\
\tau_M &= \frac{(4\pi/3)\rho a^3}{n_{\rm H}\langle v_{\rm H}\rangle m_{\rm H}\pi a^2}\\
&= \frac{4\rho a}{3n_{\rm H}m_{\rm H}\langle v_{\rm H}\rangle} = 5.50\times 10^{12}\,{\rm s} = 1.74\times 10^5\,{\rm yr}
\end{aligned}$$

(d) A collision with velocity $\vec{v}$ transfers a momentum $m_{\rm H}\vec{v}$ to the grain. The vector momentum $\vec{p}$ of the grain undergoes a random walk with a series of (vector) steps $m_{\rm H}\vec{v}_1, m_{\rm H}\vec{v}_2, m_{\rm H}\vec{v}_3, ...$ Because it is a random walk, the expectation value of the square of the momentum increases linearly with time:

$$\begin{aligned}
\left(\frac{d}{dt}\langle p^2\rangle\right)_0 &= \int_0^\infty (n_{\rm H}v)\pi a^2 (m_{\rm H}v)^2 f_v dv\\
&= n_{\rm H}\pi a^2 m_{\rm H}^2 \int_0^\infty v^3 f_v dv\\
&= n_{\rm H}\pi a^2 m_{\rm H}^2 4\pi \left(\frac{m_{\rm H}}{2\pi k_{\rm B}T}\right)^{3/2} \int_0^\infty e^{-m_{\rm H}v^2/2k_{\rm B}T} v^5 dv\\
&= n_{\rm H}\pi a^2 m_{\rm H}^2 4\pi \left(\frac{m_{\rm H}}{2\pi k_{\rm B}T}\right)^{3/2} \frac{1}{2}\left(\frac{2k_{\rm B}T}{m_{\rm H}}\right)^3 \int_0^\infty e^{-x}x^2 dx\\
&= n_{\rm H}\pi a^2 \left(\frac{8k_{\rm B}T}{\pi m_{\rm H}}\right)^{1/2} 4 m_{\rm H}k_{\rm B}T\\
\left(\frac{dE}{dt}\right)_0 &= \left(\frac{d}{dt}\frac{\langle p^2\rangle}{2M}\right)_0 = n_{\rm H}\pi a^2\langle v_{\rm H}\rangle 2k_{\rm B}T\frac{m_{\rm H}}{M}
\end{aligned}$$

(e)

$$\begin{aligned}
\tau_E &\equiv \frac{(3/2)k_{\rm B}T}{(dE/dt)_0} \\
&= \frac{(3/2)k_{\rm B}T}{n_{\rm H}\pi a^2 \langle v_{\rm H}\rangle 2k_{\rm B}T(m_{\rm H}/M)} \\
&= \frac{(3/2)(4\pi/3)\rho a^3}{n_{\rm H}\pi a^2 \langle v_{\rm H}\rangle 2m_{\rm H}} \\
&= \frac{\rho a}{n_{\rm H}m_{\rm H}\langle v_{\rm H}\rangle} \\
&= \frac{\rho a}{n_{\rm H}m_{\rm H}(8k_{\rm B}T/\pi m_{\rm H})^{1/2}} = 4.13\times 10^{12}\,{\rm s} = 1.31\times 10^{5}\,{\rm yr}
\end{aligned}$$

$$\frac{\tau_E}{\tau_M} = \tau_E \times \frac{1}{\tau_M} = \frac{\rho a}{n_{\rm H}m_{\rm H}\langle v_{\rm H}\rangle}\frac{3n_{\rm H}m_{\rm H}\langle v_{\rm H}\rangle}{4\rho a} = \frac{3}{4}$$

The thermalization time is approximately the same as the time to be hit by mass of gas equal to mass of grain.

2.5 Consider a cloud of partially-ionized hydrogen with $n(\mathrm{H}^0) = 20\,\mathrm{cm}^{-3}$, $n(\mathrm{H}^+) = n_e = 0.01\,\mathrm{cm}^{-3}$, and $T = 100\,\mathrm{K}$. Consider an electron injected into the gas with kinetic energy $E_0 = 1\,\mathrm{eV}$. We will refer to it as the "fast" electron.

(a) What is the speed v_0 of the fast electron?

(b) If the electron-neutral elastic scattering cross section is given by eq. (2.40):

$$\sigma_{\mathrm{mt}} = 7.3\times10^{-16}\left(\frac{E_0}{0.01\,\mathrm{eV}}\right)^{0.18}\ \mathrm{cm}^2$$

calculate t_{scat}, where t_{scat}^{-1} is the probability per unit time for elastic scattering of the fast electron by the neutral H atoms.

(c) A result from elementary mechanics:

> If a particle of mass m_1 and kinetic energy E_0 undergoes a head-on elastic collision with a particle of mass m_2 that was initially at rest, the kinetic energy E_2 of particle 2 after the collision is just $E_2 = 4m_1m_2E_0/(m_1+m_2)^2$.

Using this result, if the electron undergoes a head-on elastic scattering with a hydrogen atom that was initially at rest, what fraction $f_{\max}$ of the electron kinetic energy is transfered to the H atom?

(d) If elastic scattering off H atoms were the only process acting, and if the average scattering event transferred 50% as much energy as in head-on scattering, what would be the initial time scale $t_E = E_0/|dE/dt|_{E_0}$ for the electron to share its energy with the H atoms?

(e) Now consider elastic scattering of the fast electron by the other (thermal) free electrons in the gas. Equation (2.19) from the textbook gives the energy loss time for a fast particle of mass m_1, velocity v_1, charge Z_1e moving through a gas of particles of mass m_2 and charge Z_2e:

$$\begin{aligned} t_{\mathrm{loss}} &= \frac{m_1m_2v_1^3}{8\pi n_2 Z_1^2Z_2^2e^4\ln\Lambda} \\ \ln\Lambda &= 22.1 + \ln\left[\left(\frac{E_1}{kT}\right)\left(\frac{T}{10^4\,\mathrm{K}}\right)^{3/2}\left(\frac{\mathrm{cm}^{-3}}{n_e}\right)\right] \end{aligned}$$

If the only energy loss process was scattering of the fast electrons by the thermal electrons in the gas, evaluate the energy loss time t_{loss} for the fast electron.

Solution:

(a)

$$v_0 = \left(\frac{2E_0}{m_e}\right)^{1/2} = 5.93\times10^7\,\mathrm{cm\,s}^{-1}$$

(b)

$$\begin{aligned} t_{\mathrm{scat}}^{-1} &= n(\mathrm{H}^0)\sigma_{\mathrm{mt}}v_0 \\ &= 20\,\mathrm{cm}^{-3}\times7.3\times10^{-16}\times(100)^{0.18}\,\mathrm{cm}^2\times5.93\times10^7\,\mathrm{cm\,s}^{-1} \\ &= 1.98\times10^{-6}\,\mathrm{s}^{-1} \\ t_{\mathrm{scat}} &= 5.04\times10^5\,\mathrm{s} \end{aligned}$$

(c)

$$
\begin{aligned}
f_{\max} &= \frac{4m_e m_{\rm H}}{(m_e+m_{\rm H})^2} \\
&= \frac{(4m_e/m_{\rm H})}{(1+m_e/m_{\rm H})^2} = \frac{4/1836}{(1+1/1836)^2} = 2.18\times10^{-3}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\frac{dE}{dt} &= -\frac{0.5 f_{\max} E_0}{t_{\rm scat}} \\
t_E = \frac{E_0}{|dE/dt|_{E_0}} &= \frac{t_{\rm scat}}{0.5 f_{\max}} = \frac{t_{\rm scat}}{0.5\times2.18\times10^{-3}} \\
&= 4.63\times10^8\,{\rm s}
\end{aligned}
$$

(e) The particle has

$$
\begin{aligned}
\frac{E}{kT} &= \frac{1.602\times10^{-12}\,{\rm erg}}{1.38\times10^{-14}\,{\rm erg}} = 116 \\
\ln\Lambda &= 22.1 + \ln\left[116\times(0.01)^{3/2}\times\frac{1}{0.01}\right] = 22.1+\ln(11.6) = 24.6 \\
t_{\rm loss} &= \frac{m_e^2 v_1^3}{8\pi n_e e^4\ln\Lambda} = \frac{2^{3/2} m_e^{1/2} E_0^{3/2}}{8\pi n_e e^4 \ln\Lambda} = \frac{m_e^{1/2}E_0^{3/2}}{2^{3/2}\pi n_e e^4\ln\Lambda} \\
&= \frac{(9.11\times10^{-28}\,{\rm g})^{1/2}\times(1.60\times10^{-12}\,{\rm erg})^{3/2}}{2^{3/2}\times\pi\times(0.01\,{\rm cm}^{-3})\times(4.80\times10^{-10}\,{\rm esu})^4\times24.6} \\
&= 5.26\times10^5\,{\rm s}
\end{aligned}
$$

Comparing with the time $t_E = 4.63\times10^8\,{\rm s}$ found above, we see that Coulomb scattering off the thermal electrons is much more important (for energy loss) than elastic scattering off the H atoms.

Chapter 3. Statistical Mechanics and Thermodynamic Equilibrium

3.1 Consider a "$n\alpha$" radio recombination line for $n \gg 1$, and assume the emission to be optically-thin. Assume that the probability per unit time of emitting a photon is given by the Einstein $A-$coefficient

$$A_{n+1\to n} \approx \frac{6.130\times 10^9\,\mathrm{s}^{-1}}{(n+0.7)^5} \quad .$$

Relate the ratio b_{n+1}/b_n to the observed line intensity ratio $I_{n+1\to n}/I_{n\to n-1}$. Assume that $I_\mathrm{H}/n^2 k_\mathrm{B}T \ll 1$. Retain the term of leading order in the small parameter $1/n$.

Solution:

$$\begin{aligned}
\nu_{n+1\to n} &= I_\mathrm{H}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = 2I_\mathrm{H}\frac{(n+0.5)}{n^2(n+1)^2} \\
\frac{I_{n+1\to n}}{I_{n\to n-1}} &= \frac{N(n+1)}{N(n)} \times \frac{A_{n+1\to n}}{A_{n\to n-1}} \times \frac{h\nu_{n+1\to n}}{h\nu_{n\to n-1}} \\
&= \frac{(n+1)^2 b_{n+1} e^{I_\mathrm{H}/(n+1)^2 k_\mathrm{B}T}}{n^2\;\, b_n\;\, e^{I_\mathrm{H}/n^2 k_\mathrm{B}T}} \times \frac{(n+0.7)^{-5}}{(n-0.3)^{-5}} \times \frac{(n+0.5)n^{-2}(n+1)^{-2}}{(n-0.5)(n-1)^{-2}n^{-2}} \\
&= \frac{b_{n+1}}{b_n} \times \frac{(n+1)^2}{n^2}\frac{e^{I_\mathrm{H}/(n+1)^2 k_\mathrm{B}T}}{e^{I_\mathrm{H}/n^2 k_\mathrm{B}T}} \times \frac{(n+0.7)^{-5}}{(n-0.3)^{-5}} \times \frac{(n+0.5)n^{-2}(n+1)^{-2}}{(n-0.5)(n-1)^{-2}n^{-2}} \\
&\approx \frac{b_{n+1}}{b_n} \times \frac{1 + I_\mathrm{H}/(n+1)^2 k_\mathrm{B}T}{1 + I_\mathrm{H}/n^2 k_\mathrm{B}T} \times \frac{(n+0.7)^{-5}}{(n-0.3)^{-5}} \times \frac{(n+0.5)n^{-2}}{(n-0.5)(n-1)^{-2}} \\
\frac{b_{n+1}}{b_n} &= \frac{I_{n+1\to n}}{I_{n\to n-1}} \times \frac{(n-0.3)^{-5}(n-0.5)(n-1)^{-2}}{(n+0.7)^{-5}(n+0.5)\,n^{-2}} + \mathcal{O}\left(\frac{I_\mathrm{H}}{n^3 k_\mathrm{B}T}\right) \\
&\approx \frac{I_{n+1\to n}}{I_{n\to n-1}} \times \frac{(1-0.3/n)^{-5}(1-0.5/n)(1-1/n)^{-2}}{(1+0.7/n)^{-5}(1+0.5/n)} + \mathcal{O}\left(\frac{I_\mathrm{H}}{n^3 k_\mathrm{B}T}\right) \\
&\approx \frac{I_{n+1\to n}}{I_{n\to n-1}} \times \left[1 + \frac{6}{n} + \mathcal{O}\left(n^{-2}\right)\right] \quad .
\end{aligned}$$

3.3 Consider a path of length L with electron density n_e and gas kinetic temperature T. Let the population of the high-n levels of H be characterized by departure coefficients b_n.

If the medium is optically-thin, and the only radiative transitions are spontaneous decays, the integrated line intensity for an $n\alpha$ (i.e., $n+1 \rightarrow n$) transition is

$$I(n\alpha) = \frac{A(n\alpha)}{4\pi} h\nu_{n\alpha} \int_0^L ds\, n[\mathrm{H}(n+1)]$$

where $A(n\alpha)$ is the Einstein A-coefficient, and $n[\mathrm{H}(n+1)]$ is the volume density of H atoms in quantum state $n+1$. Assume that $A(n\alpha)$ is accurately approximated by

$$A(n\alpha) \approx \frac{A_0}{(n+0.7)^5}$$

where $A_0 \equiv 6.130 \times 10^9\,\mathrm{s}^{-1}$.

(a) Obtain an expression for $I(n\alpha)$ in terms of $T_4 \equiv T/10^4\,\mathrm{K}$, quantum number n, departure coefficient b_{n+1}, and the "emission measure"

$$EM \equiv \int_0^L ds\, n(\mathrm{H}^+) n_e \quad .$$

Solution: From eq. (3.45), we have

$$n[\mathrm{H}(n+1)] = b_{n+1} \frac{(n+1)^2 h^3}{(2\pi m_e k_\mathrm{B} T)^{3/2}} e^{I_\mathrm{H}/(n+1)^2 k_\mathrm{B} T} n(\mathrm{H}^+) n_e$$

and

$$h\nu_{n\alpha} = \frac{I_\mathrm{H}}{n^2} - \frac{I_\mathrm{H}}{(n+1)^2} = I_\mathrm{H} \frac{(2n+1)}{n^2(n+1)^2}$$

Thus,

$$\begin{aligned}
I(n\alpha) &= \frac{A(n\alpha)}{4\pi} h\nu_{n\alpha} \int_0^L ds\, n[\mathrm{H}(n+1)] \\
&= \frac{A(n\alpha)}{4\pi} h\nu_{n\alpha} \int_0^L ds\, b_{n+1} n(\mathrm{H}^+) n_e \frac{(n+1)^2 h^3}{(2\pi m_e k_\mathrm{B} T)^{3/2}} e^{I_\mathrm{H}/(n+1)^2 k_\mathrm{B} T} \\
&= \frac{A_0}{4\pi} \frac{(2n+1) I_\mathrm{H}}{n^2 (n+0.7)^5} e^{I_\mathrm{H}/(n+1)^2 k_\mathrm{B} T} \frac{h^3}{(2\pi m_e k_\mathrm{B} T)^{3/2}} b_{n+1} \int_0^L ds\, n(\mathrm{H}^+) n_e \\
&= 1.36 \times 10^{-5} T_4^{-3/2} \frac{(2n+1)}{n^2 (n+0.7)^5} e^{15.78/(n+1)^2 T_4} b_{n+1} \frac{EM}{\mathrm{cm}^{-6}\,\mathrm{pc}} \frac{\mathrm{erg}}{\mathrm{cm}^2\,\mathrm{s}\,\mathrm{sr}}
\end{aligned}$$

(b) Evaluate $I(166\alpha)/b_{167}$ for $EM = 10^6\,\mathrm{cm}^{-6}\,\mathrm{pc}$, and $T_4 = 1$.
Solution:

$$e^{I_{\mathrm{H}}/(n+1)^2 k_{\mathrm{B}} T} = e^{15.78/(n+1)^2 T_4} = e^{0.000566} = 1.000566$$

$$\frac{I(166\alpha)}{b_{167}} = 1.28 \times 10^{-12}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,sr^{-1}}$$

3.5 Suppose that Rydberg levels of hydrogen with quantum number $100 \le n \le n_{\rm max}$ are in LTE at $T = 5000\,{\rm K}$ with protons and electrons, with $n({\rm H}^+) = n_e = 1\,{\rm cm}^{-3}$. Calculate the ratio

$$\frac{1}{n({\rm H}^+)} \sum_{100}^{n_{\rm max}} n[{\rm H}(n)]$$

and evaluate it for $n_{\rm max} = 10^3$. Make approximations as appropriate.

Solution: From Eq. (3.34):

$$\begin{aligned}
n_{\rm LTE}[{\rm H}(n)] &= n^2 \frac{h^3}{(2\pi m_e k_{\rm B} T)^{3/2}} n_e n({\rm H}^+) e^{I_{\rm H}/k_{\rm B}T} \\
&\approx n^2 \frac{h^3}{(2\pi m_e k_{\rm B} T)^{3/2}} n_e n({\rm H}^+) \left[1 + \frac{I_{\rm H}}{n^2 k_{\rm B} T}\right] \\
\frac{1}{n({\rm H}^+)} \sum_{n=100}^{n_{\rm max}} n[{\rm H}(n)] &\approx \frac{h^3}{(2\pi m_e k_{\rm B} T)^{3/2}} n_e \sum_{n=100}^{n_{\rm max}} \left[n^2 + \frac{I_{\rm H}}{k_{\rm B} T}\right] \\
&\approx \frac{h^3}{(2\pi m_e k_{\rm B} T)^{3/2}} n_e \left[\frac{(n_{\rm max}^3 - 10^6)}{3} + (n_{\rm max} - 10^2) \frac{I_{\rm H}}{k_{\rm B} T}\right] \\
&= \frac{1.17 \times 10^{-21}}{(T/5000\,{\rm K})^{3/2}} \frac{n_e}{{\rm cm}^{-3}} \left[\frac{(10^9 - 10^6)}{3} + (10^3 - 10^2) 31.56 \left(\frac{5000\,{\rm K}}{T}\right)\right] \\
&\approx 3.90 \times 10^{-13} \left(\frac{5000\,{\rm K}}{T}\right)^{3/2} \left(\frac{n_e}{{\rm cm}^{-3}}\right)
\end{aligned}$$

Evidently the high-n Rydberg levels make a negligible contribution to the hydrogen budget provided $n_{\rm max} \lesssim 10^7$.

Chapter 4. Energy Levels of Atoms and Ions

4.1 Classify the following emission lines as either (i) *Permitted*, (ii) *Intercombination*, or (iii) *Forbidden*, and give your reason.

(a) **C III** : $1s^22s2p\ ^3\mathrm{P}_1^\mathrm{o} \rightarrow 1s^22s^2\ ^1\mathrm{S}_0$ 1908.7 Å

(b) **O III** : $1s^22s^22p^2\ ^1\mathrm{D}_2 \rightarrow 1s^22s^22p^2\ ^3\mathrm{P}_2$ 5008.2 Å

(c) **O III** : $1s^22s^22p^2\ ^1\mathrm{S}_0 \rightarrow 1s^22s^22p^2\ ^1\mathrm{D}_2$ 4364.4 Å

(d) **O III** : $1s^22s2p^3\ ^5\mathrm{S}_2^\mathrm{o} \rightarrow 1s^22s^22p^2\ ^3\mathrm{P}_1$ 1660.8 Å

(e) **O III** : $1s^22s^22p^2\ ^3\mathrm{P}_1 \rightarrow 1s^22s^22p^2\ ^3\mathrm{P}_0$ 88.36 μm

(f) **C IV** : $1s^22p\ ^2\mathrm{P}_{3/2}^\mathrm{o} \rightarrow 1s^22s\ ^2\mathrm{S}_{1/2}$ 1550.8 Å

(g) **Ne II** : $1s^22s^22p^5\ ^2\mathrm{P}_{1/2}^\mathrm{o} \rightarrow 1s^22s^22p^5\ ^2\mathrm{P}_{3/2}^\mathrm{o}$ 12.814 μm

(h) **O I** : $1s^22s^22p^33s\ ^3\mathrm{S}_1^\mathrm{o} \rightarrow 1s^22s^22p^4\ ^3\mathrm{P}_2$ 1302.2 Å

Solution:

(a) C III : $1s^22s2p\ ^3\mathrm{P}_1^\mathrm{o} \rightarrow 1s^22s^2\ ^1\mathrm{S}_0$ 1908.7 Å:
Intercombination: Satisfies electric dipole selection rules except for change in spin.

(b) O III : $1s^22s^22p^2\ ^1\mathrm{D}_2 \rightarrow 1s^22s^22p^2\ ^3\mathrm{P}_2$ 5008.2 Å:
Forbidden: 1. No change in parity (because orbitals are unchanged). 2. Change in spin.

(c) O III : $1s^22s^22p^2\ ^1\mathrm{S}_0 \rightarrow 1s^22s^22p^2\ ^1\mathrm{D}_2$ 4364.4 Å:
Forbidden: 1. No change in parity (because orbitals are unchanged). 2. $|\Delta L| = 2$.

(d) O III : $1s^22s2p^3\ ^5\mathrm{S}_2^\mathrm{o} \rightarrow 1s^22s^22p^2\ ^3\mathrm{P}_1$ 1660.8 Å:
Intercombination: Satisfied electric dipole selection rules except for change in spin.

(e) O III : $1s^22s^22p^2\ ^3\mathrm{P}_1 \rightarrow 1s^22s^22p^2\ ^3\mathrm{P}_0$ 88.36 μm:
Forbidden (fine structure transition): 1. No change in parity (because orbitals are unchanged).

(f) C IV : $1s^22p\ ^2\mathrm{P}_{3/2}^\mathrm{o} \rightarrow 1s^22s\ ^2\mathrm{S}_{1/2}$ 1550.8 Å:
Permitted: satisfies electric dipole selection rules.

(g) Ne II : $1s^22s^22p^5\ ^2\mathrm{P}_{1/2}^\mathrm{o} \rightarrow 1s^22s^22p^5\ ^2\mathrm{P}_{3/2}^\mathrm{o}$ 12.814 μm:
Forbidden (fine structure transition): 1. No change in parity (because orbitals are unchanged).

(h) O I : $1s^22s^22p^33s\ ^3\mathrm{S}_1^\mathrm{o} \rightarrow 1s^22s^22p^4\ ^3\mathrm{P}_2$ 1302.2 Å:
Permitted: satisfies electric dipole selection rules.

Chapter 5. Energy Levels of Molecules

5.1 Both H_2 and HD have similar internuclear separation $r_0 \approx 0.741$ Å. Assume that the molecules can be approximated as rigid rotors.

(a) Calculate $[E(v=0,J)-E(v=0,J=0)]/k_B$ for H_2 for $J=1$, $J=2$, and $J=3$.
(b) Calculate $[E(v=0,J)-E(v=0,J=0)]/k_B$ for HD for $J=1$, $J=2$, and $J=3$.
(c) Because H_2 has no electric dipole moment, $\Delta J = \pm 1$ transitions are forbidden, and instead the only radiative transitions are electric quadrupole transitions with $\Delta J = 0, \pm 2$. Calculate the wavelengths of the $J=2 \to 0$ and $J=3 \to 1$ transitions of H_2
(d) Because HD has a (small) electric dipole moment, it has (weak) electric dipole transitions. What is the longest-wavelength spontaneous decay for HD in the $v = 0$ vibrational level?

Solution:

(a) Assume a rigid rotor. Calculate $I = m_1 m_2 r_0^2/(m_1+m_2)$ *from the given value of* r_0 *and the known masses:*

$$\begin{aligned} E/k &= \frac{J(J+1)\hbar^2}{2Ik_B} \\ &\approx \frac{J(J+1)\hbar^2}{[2k_B m_1 m_2 r_0^2/(m_1+m_2)]} \\ &= \frac{J(J+1)\hbar^2}{[k_B m_H r_0^2]} \\ &= J(J+1)\times 87.7\,\mathrm{K} \\ &= 175\,\mathrm{K},\ 526\,\mathrm{K},\ 1052\,\mathrm{K}\ \ \text{for}\ \ J=1,\ 2,\ 3\ . \end{aligned}$$

Note: the more accurate value of $B_0/k = 85.37\,\mathrm{K}$ *from Table 5.1 gives* $E/k = 171\,\mathrm{K},\ 512\,\mathrm{K},\ 1024\,\mathrm{K}$.

(b)

$$\begin{aligned} E/k_B &\approx J(J+1)\hbar^2/[2k_B m_1 m_2 r_0^2/(m_1+m_2)] \\ &= J(J+1)\hbar^2/[4k_B m_H r_0^2/3] \\ &= J(J+1)\times 65.7\,\mathrm{K} \\ &= 131\,\mathrm{K},\ 394\,\mathrm{K},\ 789\,\mathrm{K}\ \ \text{for}\ \ J=1,\ 2,\ 3\ . \end{aligned}$$

Or: $B_0(\mathrm{HD})/k = (3/4)B_0(\mathrm{H_2})/k = 64.02\,\mathrm{K} \to E_J/k = 128\,\mathrm{K},\ 384\,\mathrm{K},\ 768\,\mathrm{K}$.

(c) Using $hc/k = 1.439\,\mathrm{K\,cm}$*,*

$$\begin{aligned} \lambda(2\to 0) &= \frac{hc}{\Delta E} = 1.439\,\mathrm{cm}/(526-0) = 27.4\,\mu\mathrm{m} \quad , \\ \lambda(3\to 1) &\approx 1.439\,\mathrm{cm}/(1052-175) = 16.4\,\mu\mathrm{m} \quad . \end{aligned}$$

(The actual wavelengths of these transitions obtained are $28.22\,\mu\mathrm{m}$ *and* $17.03\,\mu\mathrm{m}$*.)*

(d) For HD, the longest-wavelength electric dipole transition is at

$$\lambda(1\to 0) \approx 1.439\,\mathrm{cm}/(131-0) = 110\,\mu\mathrm{m} \quad .$$

(The actual wavelength of this transition is $112.1\,\mu\mathrm{m}$*.)*

5.3 Most interstellar CO is $^{12}C^{16}O$. The $J = 1 \rightarrow 0$ transition is at $\nu = 115.27\,\mathrm{GHz}$, or $\lambda = 0.261\,\mathrm{cm}$, and the $v = 1 \rightarrow 0$ transition is at $\lambda = 4.61\,\mu\mathrm{m}$ (ignoring rotational effects).

(a) Estimate the frequencies of the $J = 1 \rightarrow 0$ transitions in $^{13}C^{16}O$ and $^{12}C^{17}O$.

(b) Estimate the wavelengths of the $v = 1 \rightarrow 0$ transitions in $^{13}C^{16}O$ and $^{12}C^{17}O$. Ignore rotational effects.

(c) Suppose that the $^{13}C^{16}O$ $J{=}1-0$ line were mistaken for the $^{12}C^{16}O$ $J{=}1-0$ line. What would be the error in the inferred radial velocity of the emitting gas?

(d) What is $\Delta E/k_B$, where ΔE is the difference in "zero-point energy" between $^{12}C^{16}O$ and $^{13}C^{16}O$, and k_B is Boltzmann's constant?

Solution:

(a) The rotational energy is $E_J = J(J+1)\hbar^2/(2m_r r_0^2)$, where m_r is the reduced mass, and r_0 is the internuclear separation. We can suppose that r_0 is unaffected by the isotopic substitution. Thus, assuming r_0 to be the same for the different isotopologues of CO:

$$\frac{\nu(^{13}\mathrm{C}^{16}\mathrm{O}\;J{=}1-0\,)}{\nu(^{12}\mathrm{C}^{16}\mathrm{O}\;J{=}1-0\,)} = \frac{m_r(^{12}\mathrm{CO})}{m_r(^{13}\mathrm{CO})} = \frac{12\times 16/28}{13\times 16/29} = \frac{12\times 29}{13\times 28} = 0.956$$

$$\nu(^{13}\mathrm{C}^{16}\mathrm{O}\;J{=}1-0\,) = 0.956\times 115.3\,\mathrm{GHz} \;=\; 110.2\,\mathrm{GHz}$$

$$\frac{\nu(^{12}\mathrm{C}^{17}\mathrm{O}\;J{=}1-0\,)}{\nu(^{12}\mathrm{C}^{16}\mathrm{O}\;J{=}1-0\,)} = \frac{12\times 16/28}{12\times 17/29} = \frac{16\times 29}{17\times 28} = 0.975$$

$$\nu(^{12}\mathrm{C}^{17}\mathrm{O}\;J{=}1-0\,) = 0.975\times 115.3\,\mathrm{GHz} \;=\; 112.4\,\mathrm{GHz}$$

(b) The vibrational energy is $E_v = (v+\frac{1}{2})\hbar\omega_0$ where the vibrational frequency for a simple harmonic oscillator is $\omega_0 = \sqrt{k/m_r}$, where k is the "spring constant" characterizing the variation of electronic energy for variation of the internuclear separation around the equilibrium position, and m_r is the usual reduced mass. Thus, assuming k to be the same for the different isotopologues of CO:

$$\frac{\lambda(^{13}\mathrm{C}^{16}\mathrm{O}\;v{=}1-0)}{\lambda(^{12}\mathrm{C}^{16}\mathrm{O}\;v{=}1-0)} \;=\; \sqrt{\frac{m_r(^{13}\mathrm{C}^{16}\mathrm{O})}{m_r(^{12}\mathrm{C}^{16}\mathrm{O})}} \;=\; \sqrt{\frac{13\times 16/29}{12\times 16/28}} \;=\; 1.023$$

$$\lambda(^{13}\mathrm{C}^{16}\mathrm{O}\;v{=}1-0) \;=\; 1.023\times 4.61\,\mu\mathrm{m} = 4.72\,\mu\mathrm{m}$$

$$\lambda(^{12}\mathrm{C}^{17}\mathrm{O}\;v{=}1-0) \;=\; \sqrt{\frac{17/29}{16/28}}\times 4.61\,\mu\mathrm{m} = 4.67\,\mu\mathrm{m} \quad .$$

(c) We have seen in (b) that $\nu(^{13}\mathrm{C}^{16}\mathrm{O}\;J=1-0\,)/\nu(^{12}\mathrm{C}^{16}\mathrm{O}\;J=1-0\,) = 0.956$. Hence the inferred radial velocity would be $(1+v/c) = 1/0.956 = 1.046$, or $v = 0.046c =$ 13,800 $\mathrm{km\,s^{-1}}$.

(d) *The zero point energy is* $(1/2)\hbar\omega_0$. *Thus the difference in zero-point energy between* ${}^{12}\mathrm{C}^{16}\mathrm{O}$ *and* ${}^{13}\mathrm{C}^{16}\mathrm{O}$ *is*

$$\frac{\hbar\omega_0({}^{12}\mathrm{C}^{16}\mathrm{O}) - \hbar\omega_0({}^{13}\mathrm{C}^{16}\mathrm{O})}{2k_B} = \frac{hc}{2k_B\lambda_0}\left[1 - \sqrt{\frac{m_r({}^{12}\mathrm{C}^{16}\mathrm{O})}{m_r({}^{13}\mathrm{C}^{16}\mathrm{O})}}\right] = \frac{hc}{2k_B\lambda_0}\left[1 - \sqrt{\frac{12\times16/28}{13\times16/29}}\right]$$

$$= \frac{1.4388\,\mathrm{K\,cm}}{2\times4.61\,\mu\mathrm{m}}\,[1 - 0.9778] = 34.7\,\mathrm{K} \quad .$$

Chapter 6. Spontaneous Emission, Stimulated Emission, and Absorption

6.1 A hydrogen atom with principal quantum number n has energy $E_n = -I_{\rm H}/n^2$ where $I_{\rm H} = 13.602\,{\rm eV}$ is the ionization energy of hydrogen. A radiative transition from level $n+1 \to n$ is referred to as "$n\alpha$"; a radiative transition from level $n+2 \to n$ is referred to as "$n\beta$". E.g., the 1α transition is the same as Lyman alpha, and the 2α transition is the same as Balmer α (also known as Hα).

(a) Show that the frequency of the $n\alpha$ transition is given by

$$\nu_{n\to n+1} = \frac{C(n+0.5)}{\left[(n+0.5)^2 - 0.25\right]^2} \quad .$$

What is the value of C (in Hz)?

(b) For $n \gg 1$, it is reasonable to neglect the term 0.25 in the denominator, so from here on approximate

$$\nu_{n\to n+1} \approx C(n+0.5)^{-3} \quad .$$

Now suppose that we want to observe 21cm radiation from gas at redshift $z = 9$, redshifted to frequency $\nu = 142.04$ MHz. Our Galaxy will also be producing hydrogen recombination radiation. What are the frequencies and n values of the $n\alpha$ transition just above, and just below, 142.04 MHz?

(c) Suppose that the high-n levels of hydrogen are found in ionized gas with an electron temperature $T = 8000\,{\rm K}$, with the hydrogen having one-dimensional velocity dispersion $\sigma_v = 10\,{\rm km\,s^{-1}}$. What will be the FWHM linewidth (in MHz) of the $n\alpha$ transitions near 142 MHz? Compare this linewidth to the frequency difference $(\nu_{n+1\to n} - \nu_{n+2\to n+1})$ between adjacent $n\alpha$ lines near 142 MHz.

(d) Find the frequency and n value for the $n\beta$ transition just below 142 MHz, and just above 142 MHz.

Solution:

(a)

$$\begin{aligned}
\nu_{n\alpha} &= \frac{1}{h}\left[\frac{I_{\rm H}}{n^2} - \frac{I_{\rm H}}{(n+1)^2}\right] = \frac{I_{\rm H}}{h}\left[\frac{(n+1)^2 - n^2}{n^2(n+1)^2}\right] \\
&= \frac{I_{\rm H}}{h}\frac{(2n+1)}{[(n+\frac{1}{2}-\frac{1}{2})(n+\frac{1}{2}+\frac{1}{2})]^2} = \frac{2I_{\rm H}}{h}\frac{(n+\frac{1}{2})}{[(n+\frac{1}{2})^2-\frac{1}{4}]^2} \\
&= \frac{C(n+\frac{1}{2})}{[(n+\frac{1}{2})^2-\frac{1}{4}]^2} \qquad C \equiv \frac{2I_{\rm H}}{h} = 6.578\times10^{15}\,{\rm Hz} \quad .
\end{aligned}$$

(b)

$$\begin{aligned}
\nu_{n\alpha} &\approx \frac{C}{(n+\frac{1}{2})^3} \approx 142.04\,{\rm MHz} \\
n &\approx \left(\frac{C}{142.04\,{\rm MHz}}\right)^{1/3} - \frac{1}{2} \approx 358.6 \quad .
\end{aligned}$$

Therefore the two $n\alpha$ *transitions closest to* $142.04\,\mathrm{MHz}$ *are* $n = 358$ *and* 359*:*

$$\nu_{358\alpha} = 142.77\,\mathrm{MHz}$$
$$\nu_{359\alpha} = 141.58\,\mathrm{MHz} \quad .$$

(c)
$$(\mathrm{FWHM})_v = \sqrt{8\ln 2}\,\sigma_v = 23.5\,\mathrm{km\,s^{-1}}$$
$$(\mathrm{FWHM})_\nu = \left[\frac{(\mathrm{FWHM})_v}{c}\right]\nu_{n\alpha} = \frac{23.5}{3\times10^5}142\,\mathrm{MHz} = 0.0111\,\mathrm{MHz} = 11.1\,\mathrm{kHz}\,.$$

This is only ~1% of $\nu_{358\alpha} - \nu_{359\alpha} = 1.19\,\mathrm{MHz}\,.$

(d) $n\beta = (n+2 \to n)$ *transition:*

$$\nu_{n\beta} = \frac{I_\mathrm{H}}{h}\left[\frac{1}{n^2} - \frac{1}{(n+2)^2}\right] = \frac{I_\mathrm{H}}{h}\frac{4(n+1)}{n^2(n+2)^2} \approx \frac{4I_\mathrm{H}}{h}\frac{1}{(n+1)^3}$$
$$n \approx \left(\frac{4\times 3.289\times10^{15}}{1.42\times10^8}\right)^{1/3} - 1 \approx 451.5$$
$$\nu_{452\beta} = 141.52\,\mathrm{MHz} \quad , \quad \nu_{451\beta} = 142.47\,\mathrm{MHz} \quad .$$

Chapter 7. Radiative Transfer

7.1 A local H I cloud is interposed between us and the cosmic microwave background with temperature $T_{\rm CMB} = 2.7255\,{\rm K}$. Suppose that the H I in the cloud has a spin temperature $T_{\rm spin} = 50\,{\rm K}$, and that the optical depth at line-center (of the 21 cm line) is $\tau = 0.1$. The cloud is extended. We observe the cloud with a radio telescope with a beam that is small compared to the angular extent of the cloud.

(a) What will be the (absolute) brightness temperature T_B at line-center of the 21 cm line? Express your answer in deg K. You may assume that $h\nu \ll k_{\rm B}T_B$.

(b) What will be the (absolute) intensity at line-center of the 21 cm line? Express your answer in $\,{\rm Jy\,sr^{-1}}$.

Solution:

(a) In the limit $h\nu \ll k_{\rm B}T$, the solution to the equation of radiative transfer is just

$$
\begin{aligned}
I_\nu &= B_\nu(T_{\rm CMB})e^{-\tau} + B_\nu(T_{\rm spin})\left(1 - e^{-\tau}\right) \\
&\approx \frac{2k_{\rm B}}{\lambda^2}\left[T_{\rm CMB}e^{-\tau} + T_{\rm spin}(1 - e^{-\tau})\right] \\
T_B = \frac{\lambda^2}{2k_{\rm B}}I_\nu &= T_{\rm CMB}\,e^{-\tau} + T_{\rm spin}\left(1 - e^{-\tau}\right) \\
&= 2.73\,{\rm K}e^{-0.1} + 50\,{\rm K}\left(1 - e^{-0.1}\right) \\
&= 7.22\,{\rm K}
\end{aligned}
$$

(b) In the limit $h\nu/k_{\rm B}T \ll 1$, we have

$$
\begin{aligned}
I_\nu &= \frac{2k_{\rm B}T_B}{\lambda^2} \\
&= \frac{2k_{\rm B}}{(21.11\,{\rm cm})^2} \times 7.22\,{\rm K} = 4.48\times 10^{-18}\,{\rm erg\,cm^{-2}\,s^{-1}\,Hz^{-1}\,sr^{-1}} \\
&= 4.48\times 10^{5}\,{\rm Jy\,sr^{-1}} \quad .
\end{aligned}
$$

7.3 Suppose that we have a molecule with three energy levels – denoted 0, 1, 2 – ordered according to increasing energy, $E_0 < E_1 < E_2$. Let g_0, g_1, g_2 be the degeneracies of the levels. Supppose that there is radiation present with $h\nu = E_2 - E_0$, due to an external source plus emission in the $2 \to 0$ transition.

Let ζ_{02} be the absorption probability per unit time for a molecule in level 0, with a transition to level 2. Let A_{20}, A_{21}, and A_{10} be the Einstein A coefficients for decays 2→0, 2→1, and 1→0 by spontaneous emission of a photon. Ignore collisional processes.

(a) Ignoring possible absorption of photons in the $2 \to 1$ and $1 \to 0$ transitions, obtain an expression for the ratio n_1/n_0, where n_i is the number density of molecules in level i.

(b) How large must ζ_{02} be for this molecule to act as a maser in the 1→0 transition?

(c) Is it possible for this system to have maser emission in the 2→1 transition? If so, what conditions must be satisfied?

Solution:

(a) Radiation with $h\nu = E_2 - E_0$ will produce both absorption and stimulated emission. If ζ_{02} is the probability per time for a molecule in level 0 to absorb a photon, it is easy to show that the probability per time for a molecule in level 2 to undergo stimulated emission is $(g_0/g_2)\zeta_{02}$.

The steady-state level populations must, therefore, satisfy

$$\begin{aligned}
\text{level 2 in} = \text{out}: \quad n_0\zeta_{02} &= n_2\left[A_{21} + A_{20} + (g_0/g_2)\zeta_{02}\right] \\
\frac{n_2}{n_0} &= \frac{\zeta_{02}}{A_{21} + A_{20} + (g_0/g_2)\zeta_{02}} \\
\text{level 1 in} = \text{out}: \quad n_2 A_{21} &= n_1 A_{10} \\
\frac{n_2}{n_1} &= \frac{A_{10}}{A_{21}} \\
\frac{n_2}{n_0} \div \frac{n_2}{n_1}: \quad \frac{n_1}{n_0} &= \frac{\zeta_{02}}{A_{21} + A_{20} + (g_0/g_2)\zeta_{02}} \times \frac{A_{21}}{A_{10}} \quad .
\end{aligned}$$

(b) Must have

$$\begin{aligned}
\frac{n_1}{n_0} &> \frac{g_1}{g_0} \\
\frac{\zeta_{02}}{A_{21} + A_{20} + (g_0/g_2)\zeta_{02}} \times \frac{A_{21}}{A_{10}} &> \frac{g_1}{g_0} \\
\zeta_{02} &> \frac{g_1}{g_0}\frac{A_{10}(A_{21} + A_{20})}{\left[A_{21} - (g_1/g_2)A_{10}\right]} \quad .
\end{aligned}$$

(c) To mase in the $2 \to 1$ *transition,*

$$\text{must have} \quad \frac{n_2}{n_1} > \frac{g_2}{g_1}$$

$$\text{which will occur provided} \quad \frac{A_{10}}{A_{21}} > \frac{g_2}{g_1} \quad .$$

7.5 A supernova remnant (SNR) is emitting a sychrotron continuum with a brightness temperature $T_{B,\rm SNR} = 700\,\rm K$ near 21 cm. An extended H I cloud is interposed between us and the SNR. Suppose that the H I in the cloud has a spin temperature $T_{\rm spin} = 100\,\rm K$, and that the optical depth of the H I $\lambda = 21.11\,\rm cm$ line at line-center is $\tau = 0.2$.
We observe the SNR through the cloud. The radio telescope has a beam that is small compared to the angular extent of the SNR and the cloud.

(a) What will be the (absolute) brightness temperature T_B at line-center of the 21 cm line? Express your answer in deg K. You may assume that $h\nu \ll k_{\rm B}T_{\rm spin} \ll k_{\rm B}T_{B,\rm SNR}$. The cosmic background radiation can be neglected.

(b) What will be the (absolute) intensity at line-center of the 21 cm line? Express your answer in $\rm Jy\,sr^{-1}$.

Solution:

(a) The solution to the equation of radiative transfer is

$$I_\nu = B_\nu(T_{B,\rm SNR})\,e^{-\tau} + B_\nu(T_{\rm spin})\left(1-e^{-\tau}\right)$$

$$\approx \frac{2k_{\rm B}}{\lambda^2}\left[T_{B,\rm SNR},e^{-\tau} + T_{\rm spin}(1-e^{-\tau})\right]$$

$$T_B = \frac{\lambda^2}{2k_{\rm B}} I_\nu = T_{B,\rm SNR}\,e^{-\tau} + T_{\rm spin}\left(1-e^{-\tau}\right)$$

$$= 700\,{\rm K}\,e^{-0.2} + 100\,{\rm K}\left(1-e^{-0.2}\right)$$

$$= 573.1\,{\rm K} + 18.1\,{\rm K} = 591\,{\rm K}$$

(b) In the limit $h\nu/k_{\rm B}T_B \ll 1$, we have

$$I_\nu = \frac{2k_{\rm B}T_B}{\lambda^2}$$

$$= \frac{2k_{\rm B}}{(21.11\,{\rm cm})^2} \times 591\,{\rm K} = 3.66\times10^{-16}\,{\rm erg\,cm^{-2}\,s^{-1}\,Hz^{-1}\,sr^{-1}}$$

$$= 3.66\times10^{7}\,{\rm Jy\,sr^{-1}} \quad .$$

Chapter 8. H I 21-cm Emission and Absorption

8.1 H I 21 cm emission observations (if optically-thin) measure the amount n_u of H I in the hyperfine excited state. In Eq. (8.3) it was assumed that exactly 75% of the HI is in the excited state, so that $n(\mathrm{H\,I}) = (4/3) \times n_u$.

(a) What is the fractional error in the assumption that $n(\mathrm{H\,I}) = (4/3)\times n_u$ if $T_{\rm spin} = 100\,\mathrm{K}$?

(b) What if $T_{\rm spin} = 20\,\mathrm{K}$?

Solution: Let $x \equiv h\nu/k_{\rm B}T_{\rm spin} = 0.06816\,\mathrm{K}/T_{\rm spin} \ll 1$. *Then*

$$\begin{aligned}
\frac{n_\ell + n_u}{n_u} &= \frac{g_\ell + g_u \mathrm{e}^{-x}}{g_u \mathrm{e}^{-x}} = \frac{g_\ell \mathrm{e}^{x} + g_u}{g_u} \\
&\approx \frac{g_\ell (1+x) + g_u}{g_u} \\
&\approx \frac{g_\ell + g_u}{g_u}\left[1 + \frac{g_\ell x}{g_\ell + g_u}\right] = \frac{4}{3}\left[1 + \frac{x}{4}\right] \quad .
\end{aligned}$$

Thus the actual amount of hydrogen is larger than $(4/3)n_u$ *by a factor*

$$\begin{aligned}
\left[1 + \frac{x}{4}\right] &= \left[1 + \frac{6.8\times 10^{-4}}{4}\right] = \left[1 + 1.7 \times 10^{-4}\right] \quad \text{for } T_{\rm spin} = 100\,\mathrm{K}, \\
&\ \left[1 + \frac{3.4\times 10^{-3}}{4}\right] = \left[1 + 8.5 \times 10^{-4}\right] \quad \text{for } T_{\rm spin} = 20\,\mathrm{K}\ .
\end{aligned}$$

8.2 Calculate the oscillator strength $f_{\ell u}$ for the H I 21 cm transition.

Solution:

$$f_{\ell u} = \frac{g_u}{g_\ell} A_{ul} \frac{m_e c \lambda_{u\ell}^2}{8\pi^2 e^2} = 5.779 \times 10^{-12} \quad .$$

where we have set $g_u/g_\ell = 3$, $\lambda = 21.106\,\mathrm{cm}$, $A_{u\ell} = 2.8843 \times 10^{-15}\,\mathrm{s}^{-1}$.

8.3 An extragalactic radio "point source" (unresolved by the beam of the radio telescope) is observed to have an emission feature. The observed flux density is approximately constant at $F_\nu = 0.01\,\mathrm{Jy}$ from $1299.9\,\mathrm{MHz}$ to $1300.1\,\mathrm{MHz}$, with a negligible continuum below $1299.9\,\mathrm{MHz}$ and above $1300.1\,\mathrm{MHz}$.

The emission feature is interpreted as the 21 cm line of H I.

(a) What is the redshift of the galaxy?

(b) For a Hubble constant of $H = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, estimate the "luminosity distance" D_L to the galaxy. (Assume a simple, uniform "Hubble flow" in Euclidean space – don't worry about relativistic corrections.)

(c) If self-absorption can be ignored, what is the mass of H I in the galaxy?

(d) If the galaxy is a disk of radius $R = 20\,\mathrm{kpc}$, what is the average H I column density $N(\mathrm{H\,I})$ in $\mathrm{cm^{-2}}$?

(e) What can be said about the velocity distribution of the H I in the galaxy's rest frame?

Solution:

(a) $z = \lambda_{\mathrm{obs}}/\lambda_{\mathrm{rest}} - 1 = \nu_{\mathrm{rest}}/\nu_{\mathrm{obs}} - 1 = (1420\,\mathrm{MHz}/1300\,\mathrm{MHz}) - 1 = 0.0923.$

(b) $v = zc = 27.7\times10^3\,\mathrm{km\,s^{-1}} \to D = v/H_0 = 396\,\mathrm{Mpc}.$ *We take this to be the "luminosity distance" D_L,*

(c) The integrated line flux is $F_{\mathrm{obs}} = \int F_\nu d\nu = 0.01\,\mathrm{Jy}\times0.2\,\mathrm{MHz} = 0.002\,\mathrm{Jy\,MHz}.$ *From Eq. (8.19) we have*

(d)

$$M_{\mathrm{H\,I}} = 4.95\times10^7\,M_\odot\times(396)^2\times0.002 = 1.55\times10^{10}\,M_\odot$$

$$N(\mathrm{H\,I}) = \frac{1.55\times10^{10}\,M_\odot/m_{\mathrm{H}}}{\pi(20\,\mathrm{kpc})^2} = 1.54\times10^{21}\,\mathrm{cm^{-2}}$$

(e) The line is broadened by $\Delta\nu/\nu = 0.2\,\mathrm{MHz}/1300\,\mathrm{MHz}$, *corresponding to a velocity range in the galaxy's rest frame*

$$\Delta v = (\Delta\nu/\nu)\times c = (0.2/1300)\times3\times10^5\,\mathrm{km\,s^{-1}} = 46.2\,\mathrm{km\,s^{-1}} \quad ,$$

so that the H I extends $\pm23\,\mathrm{km\,s^{-1}}$ *around the velocity centroid. This of course only constrains the component of the velocity along the line-of-sight. Because it is a small number even compared to* $\sqrt{GM_{\mathrm{H\,I}}/R} = 58\,\mathrm{km\,s^{-1}}$ *(and the galaxy presumably has additional mass in stars and dark matter) the small observed range in radial velocities of the H I implies that this must be a disk galaxy observed nearly face-on.*

Chapter 9. Absorption Lines: The Curve of Growth

9.1 Suppose that we observe a radio-bright QSO and detect absorption lines from Milky Way gas in its spectra. The 21 cm line is seen in optically-thin absorption with a profile with $\mathrm{FWHM(H\,I)} = 10\,\mathrm{km\,s^{-1}}$. We also have high-resolution observations of the Na I doublet lines referred to as "D_1" (5898 Å) and "D_2" (5892 Å) [see Table 9.3] in absorption. The Na I $\mathrm{D_2}$ $\lambda 5892$ Å line width is $\mathrm{FWHM(Na\,I\,D_2)} = 5\,\mathrm{km\,s^{-1}}$. The line profiles are the result of a combination of thermal broadening plus turbulence with a Gaussian velocity distribution with one-dimensional velocity dispersion $\sigma_{v,\mathrm{turb}}$.

You will want to employ the following theorem: If the turbulence has a Gaussian velocity distribution, the overall velocity distribution function of atoms of mass M will be Gaussian, with one-dimensional velocity dispersion

$$\sigma_v^2 = \sigma_{v,\mathrm{turb}}^2 + \frac{k_\mathrm{B}T}{M} \quad .$$

(a) If the Na I $\mathrm{D_2}$ line is optically thin, estimate the kinetic temperature T and $\sigma_{v,\mathrm{turb}}$.

(b) Now suppose that the observed Na I D doublet ratio $W_2/W_1 < 2$. What can be said about T and σ_v^{turb}?

Solution:

(a) Let FWHM_j *be the FWHM of line* j*, where* $j = 1$ *for* $\mathrm{H\,I\,21cm}$ *and* $j = 2$ *for* $\mathrm{Na\,I\,D_2}$*. If both lines are optically thin, then*

$$\begin{aligned}
\sigma_{v,j}^2 &= \frac{(\mathrm{FWHM}_j)^2}{8\ln 2} = \sigma_{v,\mathrm{turb}}^2 + \frac{k_\mathrm{B}T}{m_j} \\
\sigma_{v,1}^2 - \sigma_{v,2}^2 &= \frac{k_\mathrm{B}T}{m_1} - \frac{k_\mathrm{B}T}{m_2} = \frac{(m_2 - m_1)}{m_1 m_2} k_\mathrm{B}T \\
T &= \frac{1}{k_\mathrm{B}} \frac{m_1 m_2}{(m_2 - m_1)} (\sigma_{v,1}^2 - \sigma_{v,2}^2) = \frac{1}{k_\mathrm{B}} \frac{m_1 m_2}{(m_2 - m_1)} \frac{(\mathrm{FWHM}_1^2 - \mathrm{FWHM}_2^2)}{8\ln 2} \\
&= \frac{23 m_p}{22 k_\mathrm{B}} \times \frac{(100 - 25)(\,\mathrm{km\,s^{-1}})^2}{8\ln 2} \\
&= 1710\,\mathrm{K} \\
\sigma_{v,\mathrm{turb}}^2 &= \frac{m_2 \sigma_{v,2}^2 - m_1 \sigma_{v,1}^2}{(m_2 - m_1)} = \frac{m_2 \mathrm{FWHM}_2^2 - m_1 \mathrm{FWHM}_1^2}{(m_2 - m_1)(8\ln 2)} \\
\sigma_{v,\mathrm{turb}} &= \left(\frac{23 \times (5)^2 - 1 \times (10)^2}{22 \times 8\ln 2} \right)^{1/2} \mathrm{km\,s^{-1}} = 1.97\,\mathrm{km\,s^{-1}}
\end{aligned}$$

(b) If $W_2/W_1 < 2$*, then at least the stronger* $\mathrm{Na\,I\,D_2}$ *line is not optically-thin, and will have*

a profile with $\mathrm{FWHM}_2 > \sqrt{8\ln 2}\,\sigma_v$*:*

$$\begin{aligned}
\sigma_{v,2} &< \frac{\mathrm{FWHM}_2}{\sqrt{8\ln 2}} \\
T &= \frac{1}{k_\mathrm{B}}\frac{m_1 m_2}{(m_2 - m_1)}(\sigma_{v,1}^2 - \sigma_{v,2}^2) \\
&> \frac{1}{k_\mathrm{B}}\frac{m_1 m_2}{(m_2 - m_1)}\frac{(\mathrm{FWHM}_1^2 - \mathrm{FWHM}_2^2)}{8\ln 2} \\
T &> 1710\,\mathrm{K} \\
\textit{and} \qquad \sigma_{v,\mathrm{turb}} &= \left(\frac{m_2\sigma_{v,2}^2 - m_1\sigma_{v,1}^2}{(m_2 - m_1)}\right)^{1/2} \\
&< \left(\frac{m_2\mathrm{FWHM}_2^2 - m_1\mathrm{FWHM}_1^2}{(m_2 - m_1)(8\ln 2)}\right)^{1/2} = \left(\frac{23\times(5)^2 - 1\times(10)^2}{22\times 8\ln 2}\right)^{1/2}\ \mathrm{km\,s^{-1}} \\
\sigma_{v,\mathrm{turb}} &< 1.97\,\mathrm{km\,s^{-1}}.
\end{aligned}$$

9.3 An absorption line, assumed to be H I Lyman α, is measured to have a dimensionless equivalent width $W = (2.00 \pm 0.10) \times 10^{-4}$. Suppose that the velocity profile is a Gaussian with $b \approx 5\,\mathrm{km\,s^{-1}}$. If b is known exactly, estimate the uncertainty in $N_\ell f_{\ell u} \lambda_{\ell u}$ arising from the $\pm 5\%$ uncertainty in W.

Solution: *If the line-center optical depth $\tau_0 \ll 1$, we would have $W = \sqrt{\pi}(b/c)\tau_0 = 2.95 \times 10^{-5} \tau_0$. But $W = 2 \times 10^{-4}$, and therefore $\tau_0 > 1$. So we are not on the linear (optically-thin) part of the curve-of-growth.*

Suppose that we are on the "flat" portion of the curve-of-growth. Then, from Eq. (9.19):

$$N_\ell f_{\ell u} \lambda_{\ell u} \approx 46.29\, b\, \exp[(cW/2b)^2]\,\mathrm{cm^{-2}\,s} \quad .$$

But if

$$\begin{array}{lrcl} & 1.9 \times 10^{-4} < & W & < 2.1 \times 10^{-4} \qquad and\ \ b = 5\,\mathrm{km\,s^{-1}} \\ then & 1.29 \times 10^{14} < & \exp[(cW/2b)^2] & < 1.73 \times 10^{17} \\ and & 2.99 \times 10^{21}\,\mathrm{cm^{-1}} < & N_\ell f_{\ell u} \lambda_{\ell u} & < 4.00 \times 10^{24}\,\mathrm{cm^{-1}} \quad , \end{array}$$

or $N_\ell f_{\ell u} \lambda_{\ell u} > 2.99 \times 10^{21}\,\mathrm{cm^{-1}}$. We now check to see whether we are actually on the flat portion of the curve of growth. From Eq. (9.9):

$$\begin{aligned} \tau_0 &= 1.497 \times 10^{-2}\,\mathrm{cm^2\,s^{-1}} N_\ell f_{\ell u} \lambda_{\ell u} / b \\ &> 1.497 \times 10^{-2} \times 2.99 \times 10^{21} / 5 \times 10^5 = 8.95 \times 10^{13} \quad . \end{aligned}$$

The transition from the flat regime to the square-root regime occurs near line-center optical depth $\tau_{\rm damp}$ given by Eq. (9.25). If the line in question is Lyman α with $\gamma_{u\ell} \lambda_{\ell u} = 7616\,\mathrm{cm\,s^{-1}}$, then Eq. (9.25) gives

$$\tau_{\rm damp} \approx 93 \times 5 \times \ln(134 \times 5) \approx 3030 \quad .$$

But we find $\tau_0 \gg \tau_{\rm damp}$; therefore we are <u>not</u> *on the flat portion of the curve of growth, and instead must be on the square-root portion of the c.o.g.*

If we now use the asymptotic result Eq. (9.24) with $\gamma_{\ell u} \lambda_{\ell u}$ for H Lyman α, we obtain

$$N_\ell = 2.759 \times 10^{24} W^2\,\mathrm{cm^{-2}} = (1.10 \pm 0.11) \times 10^{17}\,\mathrm{cm^{-2}} \quad .$$

From (9.10), the line-center optical depth $\tau_0 = (1.67 \pm 0.17) \times 10^4$. Above we estimated $\tau_{\rm damp} \approx 3030$; since $\tau_0 > \tau_{\rm damp}$, we confirm that we are in the "damping" portion of the curve-of-growth, where $N_\ell \propto W^2$.
Therefore, a $\pm 5\%$ uncertainty in W results in a $\pm 10\%$ uncertainty in N_ℓ.

9.5 A quasar (PKS0237-23) at a redshift $z_Q = 2.22$ is observed to have an absorption feature in its spectrum produced by Si II ions at a redshift $z_G = 1.36$ The absorption line is due to the allowed transition Si II ${}^2\mathrm{P}^{\mathrm{o}}_{1/2} \rightarrow {}^2\mathrm{S}_{1/2}$ (see the energy level diagram on p. 493) at a rest wavelength $\lambda = 1527\,\text{Å}$ (at an observed wavelength $\lambda_{\rm obs} = 3604\,\text{Å}$).

The ${}^2\mathrm{P}^{\mathrm{o}}_{1/2} \rightarrow {}^2\mathrm{S}_{1/2}$ feature has an observed equivalent width $W_{\lambda,\rm obs} = 2\,\text{Å}$. The conventional interpretation is that this absorption feature is produced in an intervening galaxy.

(a) What is the column density $N(\mathrm{Si\,II}\,{}^2\mathrm{P}^{\mathrm{o}}_{1/2})$ of Si II in the ground state? Assume the line to be optically thin (what condition does this impose on the velocity dispersion of the SiII in the intervening galaxy?). Required atomic data can be found in the text (Table 9.5).

(b) The quasar spectrum shows no trace of absorption in the ${}^2\mathrm{P}^{\mathrm{o}}_{3/2} \rightarrow {}^2\mathrm{S}_{1/2}$ transition of Si II at $\lambda = 1533\,\text{Å}$. If the upper limit on the observed equivalent width is $(W_\lambda)_{\rm obs} < 1\,\text{Å}$, what is the corresponding upper limit on the column density $N(\mathrm{Si\,II}\,{}^2\mathrm{P}^{\mathrm{o}}_{3/2})$ in the intervening galaxy?

(c) Given your result from (b) on the upper bound for $N(\mathrm{Si\,II}\,{}^2\mathrm{P}^{\mathrm{o}}_{3/2})$, what limit can be placed on the electron density n_e in the intervening galaxy if the kinetic temperature is assumed to be $10^4\,\mathrm{K}$? The Einstein A coefficient is $A({}^2\mathrm{P}^{\mathrm{o}}_{3/2} \rightarrow {}^2\mathrm{P}^{\mathrm{o}}_{1/2}) = 2.13 \times 10^{-4}\,\mathrm{s}^{-1}$, and the electron collision strength is $\Omega({}^2\mathrm{P}^{\mathrm{o}}_{3/2}, {}^2\mathrm{P}^{\mathrm{o}}_{1/2}) = 4.45$ (see Table F1 on p. 496). (Ignore the existence of the ${}^2\mathrm{S}_{1/2}$ state in this and (d) below; i.e., treat the two fine-structure states as a two-level system. Assume the interstellar radiation field in the intervening galaxy to be not too wildly dissimilar to that in our Galaxy.)

(d) Can any useful limit be placed on n_e if the kinetic temperature is assumed to be $10^2\,\mathrm{K}$ rather than $10^4\,\mathrm{K}$?

Solution:

(a) $\ell = {}^2\mathrm{P}^{\mathrm{o}}_{1/2}$, $u = {}^2\mathrm{S}_{1/2}$, $f_{\ell u} = 0.133$, $\lambda_{\ell u} = 1.5267 \times 10^{-5}\,\mathrm{cm}$.

$W = (W_\lambda)_{\rm obs}/\lambda_{\rm obs} = 2\,\text{Å}/3604\,\text{Å} = 5.55 \times 10^{-4}$.

If optically-thin, $N_\ell = 1.130 \times 10^{12} W/(f_{\ell u}\lambda_{\ell u}) = 3.09 \times 10^{14}\,\mathrm{cm}^{-2}$.

If optically-thin, then $W \approx \sqrt{\pi}(b/c)\tau_0$ *(Eq. 9.11). Therefore, to be optically-thin, must have*

$$\frac{b}{c} \gtrsim \frac{W}{\sqrt{\pi}} \quad : \quad b \gtrsim 94\,\mathrm{km\,s}^{-1} \quad .$$

(b) $\lambda_{\ell u} = 1.5335 \times 10^{-5}\,\mathrm{cm}$, $f_{\ell u} = 0.133$,

$W < [1\,\text{Å}/(1 + z_G)]/1533.5\,\text{Å} = 2.76 \times 10^{-4}$.

$N_\ell = [1.13 \times 10^{12}\,\mathrm{cm}^{-1}/(f_{\ell u}\lambda_{\ell u})]W < 1.53 \times 10^{14}\,\mathrm{cm}^{-2}$.

(c) $N_1/N_0 < 1.53 \times 10^{14}\,\mathrm{cm}^{-2}/3.09 \times 10^{14}\,\mathrm{cm}^{-2} = 0.495$,

From Table 9.5, $E_1/hc = 287.2\,\mathrm{cm}^{-1}$, $\lambda_{10} = 34.82\,\mu\mathrm{m}$. *If the radiation field is similar to that in our galaxy, the photon occupation number* $n_\gamma(\lambda = 34.82\,\mu\mathrm{m}) \approx 10^{-8} \ll 1$ *(see Fig. 12.1), and we can neglect both absorption and stimulated emission.*

Consider collisional excitation of the fine-structure excited state, with $E_{10}/k_{\rm B} = 413\,{\rm K}$, $g_0 = 2$, $g_1 = 4$. *In steady-state:*

$$\begin{aligned}
N_0 n_e k_{01} &= N_1(n_e k_{10} + A_{10}) \\
n_e &= \frac{(A_{10}/k_{10})}{[(N_0/N_1)/(g_0/g_1)]\,{\rm e}^{-E_{10}/k_{\rm B}T} - 1} \quad . \\
where\;\; k_{10} &= \frac{8.629\times 10^{-8}}{\sqrt{T_4}}\frac{\Omega_{10}}{g_1}\,{\rm cm}^3\,{\rm s}^{-1} = 9.60\times 10^{-8} T_4^{-1/2}\,{\rm cm}^3\,{\rm s}^{-1} \\
n_e &= \frac{2.13\times 10^{-4}/9.60\times 10^{-8} T_4^{-1/2}}{2(N_0/N_1)\,{\rm e}^{-413/T} - 1}\;\;{\rm cm}^{-3} \\
&< \frac{2.22\times 10^3\,{\rm cm}^{-3}}{4.04\,{\rm e}^{-.0413} - 1} \quad \text{for}\;\; T_4 = 1 \\
n_e &< \frac{2.22\times 10^3\,{\rm cm}^{-3}}{3.877 - 1} \\
n_e &< 770\,{\rm cm}^{-3} \quad .
\end{aligned}$$

(d) *From the analysis in (c), we have*

$$n_e = \frac{2.22\times 10^3 T_4^{1/2}\,{\rm cm}^{-3}}{[4.04{\rm e}^{-413/T} - 1]} \quad .$$

But if $T = 10^2\,{\rm K}$, *then the denominator*

$$[4.04\,{\rm e}^{-413/T} - 1] = [0.0650 - 1] < 0 \;\; .$$

The bound on n_e *becomes unphysical, and therefore there is* **no useful limit**. *The situation here is that if* $T = 10^2\,{\rm K}$, *even at high densities the ratio* N_1/N_0 *is consistent with the upper limit* $N_1/N_0 < 0.495$.

9.7 An absorption line is observed in the spectrum of a quasar at an observed wavelength $\lambda = 5000.\,\text{Å}$. The absorption is produced by an intergalactic cloud of gas somewhere between us and the quasar. The observer measures an equivalent width $W_\lambda = 1.0 \times 10^{-2}\,\text{Å}$. The absorption line is resolved, with an observed $\mathrm{FWHM}_\lambda = 0.50\,\text{Å}$.

The line is assumed to be H I Lyman α, with rest wavelength $\lambda_0 = 1215.7\,\text{Å}$ and oscillator strength $f_{\ell u} = 0.4164$.

(a) What is the redshift z of the absorber?

(b) What is the column density of H I in the absorbing cloud?

(c) In the rest frame of the cloud, the H I has a one-dimensional velocity distribution $\propto e^{-(\Delta v/b)^2}$. What is the value of b for this cloud?

Solution:

(a)

$$\begin{aligned}\lambda_{\rm obs} &= (1+z) \times \lambda_0 \\ z &= \frac{\lambda_{\rm obs}}{\lambda_0} - 1 = \frac{5000.}{1215.7} - 1 = 3.11\end{aligned}$$

(b) The line has $W_\lambda \ll \mathrm{FWHM}_\lambda$, so it is optically-thin. The dimensionless equivalent width

$$W = \frac{W_\lambda}{\lambda} = \frac{10^{-2}\,\text{Å}}{5000\,\text{Å}} = 2.00 \times 10^{-6}$$

In the optically-thin regime, we use eq. (9.15) from the textbook:

$$\begin{aligned}N_\ell &= 1.130 \times 10^{12}\,\mathrm{cm^{-1}} \frac{W}{f_{\ell u}\lambda_{\ell u}} \\ &= 1.130 \times 10^{12}\,\mathrm{cm^{-1}} \frac{2.00 \times 10^{-6}}{0.4164 \times 1.216 \times 10^{-5}\,\mathrm{cm}} \\ &= 4.46 \times 10^{11}\,\mathrm{cm^{-2}} \quad .\end{aligned}$$

(c) In general,

$$\frac{\mathrm{FWHM}_\lambda}{\lambda} = \frac{\mathrm{FWHM}_v}{c} \quad .$$

Hence

$$\begin{aligned}\mathrm{FWHM}_v &= c \times (\mathrm{FWHM}_\lambda/\lambda) \\ &= 3 \times 10^5\,\mathrm{km\,s^{-1}} \times (0.5\,\text{Å}/5000\,\text{Å}) = 30\,\mathrm{km\,s^{-1}} \quad .\end{aligned}$$

For an optically-thin line with a Gaussian profile,

$$
\begin{aligned}
0.5 &= \exp(-[(\mathrm{FWHM}_v/2)/b]^2) \\
\ln 2 &= [(\mathrm{FWHM}_v/2)/b]^2 \\
b &= \frac{\mathrm{FWHM}}{2\sqrt{\ln 2}} \\
&= \frac{30\,\mathrm{km\,s^{-1}}}{2\sqrt{\ln 2}} = 18.0\,\mathrm{km\,s^{-1}} \quad .
\end{aligned}
$$

9.9 The CH^+ molecule has an absorption line at $\lambda = 4233\,\text{Å}$ with an oscillator strength $f_{\ell u} = 0.0060$ out of the ground state ℓ. An absorption line is observed at this wavelength with an equivalent width $W_\lambda = 0.010\,\text{Å}$, and a FWHM of $10\,\mathrm{km\,s^{-1}}$. What is the column density of ground-state CH^+ on this line-of-sight? Single-digit accuracy is sufficient.

Solution:

$$\lambda = 4233\,\text{Å} \quad , \quad f_{\ell u} = 0.0060$$

$$W = \frac{W_\lambda}{\lambda} = \frac{0.010}{4233} = 2.4 \times 10^{-6}$$

$$\frac{(\mathrm{FWHM})_v}{c} = \frac{10\,\mathrm{km\,s^{-1}}}{c} = 3.33\times 10^{-5}$$

Because $W \ll (\mathrm{FWHM})_v/c$, *the line is optically-thin, and we can use the optically-thin result*

$$N_\ell = \frac{W}{8.85\times 10^{-13}\,\mathrm{cm}\, f_{\ell u}\lambda_{\ell u}} = \frac{2.4\times 10^{-6}}{8.85\times 10^{-13}\,\mathrm{cm} \times 0.0060 \times 4.233\times 10^{-5}\,\mathrm{cm}}$$
$$= 1.05 \times 10^{13}\,\mathrm{cm}^{-2}$$

9.11 Suppose that an H atom in the $3p$ level is at rest in an H I cloud of density $n(\mathrm{H}) = 20\,\mathrm{cm}^{-3}$ and kinetic temperature $T = 100\,\mathrm{K}$. Assume that the motions of the other H atoms in the cloud are purely thermal. Assume the cloud to be infinite in extent, and pure H (no dust, etc.).

If the H($3p$) emits a Lyman β photon, what is the mean free path of this photon before it is absorbed by another H atom? The wavelength of Lyman β is 1025.7Å. The oscillator strength for the Lyman β transition is $f_{1s,3p} = 0.0791$.

Solution: The emitting atom is at rest, so the photon will be emitted at "line-center", $\nu = \nu_0$ (ignoring the very small intrinsic linewidth). The H photoabsorption cross section at line-center for H atoms with a 1-dimensional velocity dispersion $\sigma_v = \sqrt{k_{\mathrm{B}}T/m_{\mathrm{H}}}$ is:

$$\sigma_{\ell u}(\nu_0) = \frac{\pi e^2}{m_e c} f_{\ell u} \phi_\nu(\nu_0)$$

$$\phi_\nu(\nu) = \frac{1}{\sqrt{2\pi}} \frac{\lambda_0}{\sigma_v} e^{-v^2/(2\sigma_v^2)} \qquad \text{where} \qquad v = \frac{(\nu - \nu_0)}{\nu_0} c$$

$$\phi_\nu(\nu_0) = \frac{1}{\sqrt{2\pi}} \frac{\lambda_0}{\sqrt{k_{\mathrm{B}}T/m_{\mathrm{H}}}}$$

$$\begin{aligned}
\sigma_{\ell u}(\nu_0) &= \frac{\pi e^2}{m_e c} \frac{\lambda_0 f_{\ell u}}{\sqrt{2\pi k_{\mathrm{B}}T/m_{\mathrm{H}}}} \\
&= \frac{\pi (4.80\times10^{-10})^2\,\mathrm{erg\,cm}}{9.11\times10^{-28}\,\mathrm{g} \times 3.00\times10^{10}\,\mathrm{cm\,s^{-1}}} \frac{1.026\times10^{-5}\,\mathrm{cm} \times 0.0791}{\sqrt{2\pi \times 1.38\times10^{-16}\,\mathrm{erg\,K^{-1}} \times 10^2\,\mathrm{K}/(1.67\times10^{-24}\,\mathrm{g})}} \\
&= 9.43 \times 10^{-14}\,\mathrm{cm}^2 \quad .
\end{aligned}$$

The mean free path is

$$\begin{aligned}
\mathrm{mfp} &= \frac{1}{n(\mathrm{H}^0)\sigma_{\ell u}(\nu_0)} \\
&= \frac{1}{20\,\mathrm{cm}^{-3} \times 9.43 \times 10^{-14}\,\mathrm{cm}^2} \\
&= 5.30 \times 10^{11}\,\mathrm{cm} \quad .
\end{aligned}$$

Chapter 10. Emission and Absorption by a Thermal Plasma

10.1 The brightest part of the Orion H II region has an emission measure $EM \approx 5\times10^6\,\mathrm{cm}^{-6}\,\mathrm{pc}$. Assume an electron temperature $T_e = 10^4\,\mathrm{K}$.

(a) What is the optical depth τ due to free-free absorption at $\lambda = 1\,\mathrm{cm}$ ($\nu = 30\,\mathrm{GHz}$)?

(b) What is the optical depth τ due to free-free absorption at $\lambda = 21.1\,\mathrm{cm}$ ($\nu = 1420\,\mathrm{MHz}$)?

(c) Suppose that there is atomic hydrogen on the far side of the H II region with a column density $N(\mathrm{H\,I}) = 10^{21}\,\mathrm{cm}^{-2}$ and a spin temperature $T_{\rm spin} = 1000\,\mathrm{K}$. Calculate the observed strength of the 21 cm line (where "line" is the excess above the continuum), integrated over the line profile, and expressed in the usual "antenna temperature-velocity" units of $\mathrm{K\,km\,s^{-1}}$. Assume the line to be broad enough to be optically thin.

(d) A radio telescope observes the brightest part of the H II region. Calculate the dimensionless "equivalent width" W of the 21 cm line, and also calculate the "velocity" equivalent width $W_V \equiv c \times W$ and the "frequency" equivalent width $W_\nu \equiv \nu \times W$.
Note: the dimensionless equivalent width of an emission line is defined to be

$$W \equiv \int \frac{[I_\nu - I_\nu^{(c)}]}{I_\nu^{(c)}}\frac{d\nu}{\nu}$$

where $I_\nu^{(c)}$ is the "continuum" level of the free-free emission on either side of the 21 cm line.

Solution:

(a) $$\tau_{\rm ff} = \int \kappa_{\rm ff} ds$$

$$= \frac{\kappa_{\rm ff}}{n_i n_e}\int n_i n_e ds = 1.09\times10^{-25} T_4^{-1.323}\nu_9^{-2.118}\,\mathrm{cm}^5 \times \mathrm{EM} \qquad [\kappa_{\rm ff} \text{ from eq. (10.16)}]$$

$$= 1.09\times10^{-25} \times 30^{-2.118} \times 5\times10^6 \times 3.086\times10^{18}$$

$$= 1.25\times10^{-3} \quad .$$

(b) $\tau = 1.09\times10^{-25} \times 1.42^{-2.118} \times 5\times10^6 \times 3.086\times10^{18} = 0.800$.

(c) We have $h\nu \ll k_{\rm B}T_e$ and $h\nu \ll k_{\rm B}T_{\rm spin}$. The equation of radiative transfer can then be written

$$\frac{dT_A}{d\tau} = -T_A + T_{\rm exc} \quad ,$$

where $T_{\rm exc}$ is the excitation temperature characterizing the emitting and absorbing medium. We have two isothermal reqions: the H I *region ($T_{\rm spin} = 10^3\,\mathrm{K}$) and the* H II *region ($T_e = 10^4\,\mathrm{K}$). The* H I *region has a line-center optical depth (from Eq. 8.11)*

$$\tau_1 = 2.190\frac{N(\mathrm{H\,I})}{10^{21}\,\mathrm{cm}^{-2}}\frac{100\,\mathrm{K}}{T_{\rm spin}}\frac{\mathrm{km\,s}^{-1}}{\sigma_V} \quad .$$

The $\mathrm{H\,I}$ *has* $\sigma_V > (k_\mathrm{B} \times 10^3\,\mathrm{K}/m_\mathrm{H})^{1/2} = 2.87\,\mathrm{km\,s^{-1}}$. *Therefore* $\tau_1 < 0.0762$*, and we may assume the* $\mathrm{H\,I}$ *emission itself is optically-thin, with the power/area in the 21 cm line leaving the* $\mathrm{H\,I}$ *and entering the* $\mathrm{H\,II}$ *given by (see Eq. 8.16)*

$$\int [T_A - T_A(0)]\,du = 54.9\,\mathrm{K\,km\,s^{-1}} \frac{N(\mathrm{H\,I})}{10^{20}\,\mathrm{cm^{-2}}} = 549\,\mathrm{K\,km\,s^{-1}} \quad .$$

This radiation will be attenuated as it passes through the $\mathrm{H\,II}$ *gas, with an optical depth* $\tau = 0.800$*. Thus, after passing through the* $\mathrm{H\,II}$ *gas, the 21 cm line flux, after subtracting the continuum, is*

$$\begin{aligned} \int [T_A - T_A(\mathrm{continuum})]\,du &= 549\,\mathrm{e}^{-0.80}\,\mathrm{K\,km\,s^{-1}} \\ &= 247\,\mathrm{K\,km\,s^{-1}} \quad . \end{aligned}$$

(d) The free-free intensity off-line has $T_A = 10^4\,\mathrm{K}(1 - \mathrm{e}^{-0.80}) = 5510\,\mathrm{K}$*. Therefore the equivalent width of the 21-cm line is*

$$\begin{aligned} W_V &= \frac{247\,\mathrm{K\,km\,s^{-1}}}{5510\,\mathrm{K}} = 0.0448\,\mathrm{km\,s^{-1}} \quad , \\ W &\equiv \frac{1}{c} W_V = 1.49 \times 10^{-7} \quad , \\ W_\nu &\equiv \nu W = 202\,\mathrm{Hz} \quad . \end{aligned}$$

10.3 We are hoping to observe 21 cm emission from redshift $z \approx 9$ and need to model the "Galactic foreground" produced by a slab of partially-ionized hydrogen (at redshift 0) at temperature T. Consider a H $n\alpha$ line originating in this slab.

(a) For what n will the H $n\alpha$ line be near $142\,\mathrm{MHz}$?

(b) Suppose that $\beta_{n\alpha}$ [defined by Eq. (10.30)] is negative, and suppose that the optical depth $\tau_{n\alpha}$ is a small negative number. Suppose that just beyond the slab of partially-ionized hydrogen, there is a region producing synchrotron emission with antenna temperature $T_{A,0}$ at 142 MHz.
If the hydrogen in the slab is "isothermal" (or perhaps we should say "iso-excited"), then the *exact* solution to the equation of radiative transfer is simply

$$I_\nu = I_{\nu,0}\mathrm{e}^{-\tau_\nu} + B_\nu(T_{\mathrm{exc}})\left(1 - \mathrm{e}^{-\tau_\nu}\right) \quad ,$$

where recall that the *definition* of T_{exc} is such that

$$\frac{n_u}{n_\ell} \equiv \frac{g_u}{g_\ell}\mathrm{e}^{-E_{u\ell}/k_{\mathrm{B}}T_{\mathrm{exc}}} \quad .$$

Assuming that $|h\nu/k_{\mathrm{B}}T_{\mathrm{exc}}| \ll 1$ and $|\tau_\nu| \ll 1$, show that

$$T_A(\nu) \approx T_{A,0}\mathrm{e}^{-\tau_\nu} + T_{\mathrm{exc}}\tau_\nu \quad .$$

(c) Recall the definition of $\beta_{n\alpha}$:

$$\beta_{n\alpha} \equiv \frac{1 - (n_u g_\ell)/(n_\ell g_u)}{1 - \exp(-h\nu/k_{\mathrm{B}}T)} \quad ,$$

where $u = n+1$ and $\ell = n$. If $|h\nu/k_{\mathrm{B}}T| \ll 1$, $|h\nu/k_{\mathrm{B}}T_{\mathrm{exc}}| \ll 1$, and $|\tau_\nu| \ll 1$, show that

$$T_A(\nu) = T_{A,0}\mathrm{e}^{-\tau_\nu} + \beta_{n\alpha}^{-1}T\tau_\nu \quad .$$

(d) The sky-averaged synchrotron background is approximately given by Eq. (12.3). What is the sky-averaged antenna temperature T_{A0} of the synchrotron background at 142 MHz?

(e) Suppose that the difference in antenna temperature "on-line" vs. "off-line" is

$$\Delta T_A = [T_{A,0}\mathrm{e}^{-\tau_\nu} + \beta_{n\alpha}^{-1}T\tau_\nu] - T_{A,0} = T_{A,0}(\mathrm{e}^{-\tau_\nu} - 1) + \beta_{n\alpha}^{-1}T\tau_\nu \quad .$$

If $\tau_\nu = -10^{-7}$, $\beta_{n\alpha} = -100$, $T = 10^4\,\mathrm{K}$, and $T_{A,0} = 500\,\mathrm{K}$, calculate the "antenna temperature" ΔT_A of the line. Which is larger – the amplification of the synchrotron emission, or the contribution $\beta_{n\alpha}^{-1}T\tau_\nu$ that is independent of the synchrotron emission? [For comparison, the redshifted 21cm line is expected to have $\Delta T_A \approx +15$ mK if the universe was reionized by radiation from massive "Pop III" stars, and the fluctuations in antenna temperature due to "minihalos" at $z \approx 9$ are expected to be of order ~ 1 mK (Furlanetto et al. 2006: Physics Reports 433, 181, Fig. 12)].

Solution:

(a) From Eq. (10.26), $n = 358$ *and* 359 *have* $\nu_{n\alpha} = 142.7$ *and* $141.5\,\mathrm{MHz}$

(b) From the definition of T_A (Eq. 7.6),

$$T_A(\nu) = T_{A,0}(\nu)\mathrm{e}^{-\tau_\nu} + \frac{h\nu}{k_\mathrm{B}}\frac{1}{\mathrm{e}^{h\nu/k_\mathrm{B}T_\mathrm{exc}} - 1}\left(1 - \mathrm{e}^{-\tau_\nu}\right)$$
$$\approx T_{A,0}\mathrm{e}^{-\tau_\nu} + T_\mathrm{exc}\tau_\nu \quad .$$

(c)

$$\beta_{n\alpha} \equiv \frac{1 - \mathrm{e}^{-h\nu/k_\mathrm{B}T_\mathrm{exc}}}{1 - \mathrm{e}^{-h\nu/k_\mathrm{B}T}} \approx \frac{h\nu/k_\mathrm{B}T_\mathrm{exc}}{h\nu/k_\mathrm{B}T} = \frac{T}{T_\mathrm{exc}} \quad \to \quad T_\mathrm{exc} = \frac{T}{\beta_{n\alpha}}$$

$$T_A(\nu) \approx T_{A0}(\nu)\mathrm{e}^{-\tau_\nu} + \frac{T\tau_\nu}{\beta_{n\alpha}} \quad .$$

(d) $T_{A0} \approx 677\,\mathrm{K}$ *at* $\nu = 142\,\mathrm{MHz}$.

(e)

$$\begin{aligned}\Delta T_A &= T_{A0}\left(\mathrm{e}^{-\tau_\nu} - 1\right) + \beta_{n\alpha}^{-1}T\tau_\nu \\ &= 500 \times 10^{-7} \quad + \quad (-10^{-2}) \times 10^4 \times (-10^{-7}) \\ &= \quad 5\times10^{-5} \quad + \quad 1\times10^{-5} \\ &= \quad 0.06\,\mathrm{mK} \quad .\end{aligned}$$

The amplification term $T_{A0}\,(\mathrm{e}^{-\tau_\nu} - 1) = 0.05\,\mathrm{mK}$*, while the spontaneous emission term* $\beta_{n\alpha}^{-1}T\tau_\nu = 0.01\,\mathrm{mK}$*. Therefore, amplification of the sychrotron emission is dominant.*

10.5 Consider an ionized wind flowing outward from a point source. Suppose that the temperature $T = 10^4 T_4\,\mathrm{K}$ and the electron density n_e varies as

$$n(r) = n_0 \left(\frac{R_0}{r}\right)^2 \quad .$$

R_0 is simply some reference radius, with n_0 the electron density at that radius. Assume the attenuation coefficient κ_ν to have the simple power-law dependence on frequency $\nu = \nu_9\,\mathrm{GHz}$ and temperature given by Eq. (10.8):

$$\kappa_\nu = \frac{A}{R_0}\left(\frac{n}{n_0}\right)^2 \nu_9^{-2.12}$$
$$A \equiv 1.09 \times 10^{-25} n_0^2 R_0 T_4^{-1.32}\,\mathrm{cm}^5 \quad .$$

(a) Let $\tau(R)$ be the attenuation optical depth along a radial path from $r = R$ to $r = \infty$. Define $R_p(\nu)$ to be the radius where $\tau = 2/3$. Obtain an expression for R_p/R_0 in terms of A and the frequency $\nu_9 \equiv \nu/\,\mathrm{GHz}$.

(b) When viewed at frequency ν by a distant observer, the wind will have a "photosphere" at radius R_p. Suppose that the emission from this photosphere at frequency ν can be approximated as a blackbody. Assume we are in the Rayleigh-Jeans limit ($h\nu \ll k_{\mathrm{B}}T$). The observer is at distance D.
Obtain an expression for the flux density $F_\nu^{(\mathrm{photo})}$ of the "photospheric emission".
The "spectral index" β is defined by $F_\nu^{(\mathrm{photo})} \propto \nu^\beta$. Obtain β.

(c) In addition to the "photospheric" emission, there will be additional emission from the optically-thin wind outside the photosphere. Assume the emissivity to have the simple power-law dependence given by Eq. (10.8):

$$4\pi j_\nu = B\left(\frac{n}{n_0}\right)^2 \nu_9^{-0.12}$$
$$B = 4\pi \times 3.35 \times 10^{-40} T_4^{-0.32} n_0^2\,\mathrm{erg\,cm^3\,s^{-1}\,Hz^{-1}} \quad .$$

If the wind extends to infinity, and absorption can be entirely ignored (ignore the fact that the "photosphere" blocks radiation from some of the material on the far side), calculate the flux density $F_\nu^{(\mathrm{outer})}$ from this extended emission in terms of B, R_0, and D. What is the spectral index of $F_\nu^{(\mathrm{outer})}$?

(d) According to this approximate treatment, what is the ratio $F_\nu^{(\mathrm{outer})}/F_\nu^{(\mathrm{photo})}$?

(e) By neglecting absorption, the above treatment has overestimated $F_\nu^{(\mathrm{outer})}$, so we now neglect $F_\nu^{(\mathrm{photo})}$ and take $F_\nu \approx F_\nu^{(\mathrm{outer})}$. If the wind has a mass-loss rate $\dot{M}_{\mathrm{w}}$ and velocity v_{w}, show that

$$F_\nu^{(\mathrm{outer})} \approx 0.013\,\mathrm{Jy}\left(\frac{\mathrm{kpc}}{D}\right)^2 T_4^{0.12}\left(\frac{\dot{M}}{10^{-6}\,M_\odot\,\mathrm{yr}^{-1}}\right)^{4/3}\left(\frac{20\,\mathrm{km\,s^{-1}}}{v_{\mathrm{w}}}\right)^{4/3} \nu_9^{0.59} \quad .$$

Assume H to be fully ionized but He to be neutral, so that $n = \rho/1.4m_{\mathrm{H}}$.

Solution:

(a)

$$\frac{2}{3} = \int_{R_p}^{\infty} dr\ \kappa_\nu(r) = \int_{R_p}^{\infty} dr \frac{A}{R_0}\left(\frac{R_0}{r}\right)^4 \nu_9^{-2.12}$$

$$\frac{2}{3} = A\nu_9^{-2.12}\int_{R_p/R_0}^{\infty} dx\ \ x^{-4} = \frac{A\nu_9^{-2.12}}{3}\left(\frac{R_0}{R_p}\right)^3$$

$$\frac{R_p}{R_0} = \left(\frac{A}{2}\right)^{\frac{1}{3}} \nu_9^{\frac{-2.12}{3}}$$

(b) The photosphere subtends a solid angle $\Omega_p = \pi(R_p/D)^2$. The flux density is

$$F_\nu^{(\mathrm{photo})} = B_\nu(T)\times\Omega_p = \frac{2k_{\mathrm{B}}T\nu^2}{c^2}\pi\left(\frac{R_p}{D}\right)^2$$

$$= \frac{2k_{\mathrm{B}}T\nu^2}{c^2}\pi\left(\frac{R_0}{D}\right)^2\left(\frac{R_p}{R_0}\right)^2 = \frac{2\pi k_{\mathrm{B}}T}{c^2}10^{18}\,\mathrm{s}^{-2}\left(\frac{R_0}{D}\right)^2\left(\frac{R_p}{R_0}\right)^{2-\gamma}\nu_9^2$$

$$F_\nu^{(\mathrm{photo})} = \frac{2\pi k_{\mathrm{B}}T}{c^2}10^{18}\,\mathrm{s}^{-2}\left(\frac{R_0}{D}\right)^2\left(\frac{A}{2}\right)^{\frac{2}{3}}\nu_9^{\frac{1.76}{3}} \quad .$$

The spectral index $\beta = 1.76/3 = 0.59$.

(c)

$$F_\nu^{(\mathrm{outer})} \approx \frac{1}{4\pi D^2}\int_{R_p}^{\infty} 4\pi r^2 dr\ 4\pi j_\nu = \frac{1}{4\pi D^2}\int_{R_p}^{\infty} 4\pi r^2 dr B\left(\frac{n}{n_0}\right)^2 \nu_9^{-0.12}$$

$$= \frac{B}{D^2}\nu_9^{-0.12}\int_{R_p}^{\infty} r^2 dr\left(\frac{n}{n_0}\right)^2 = \frac{BR_0^3}{D^2}\nu_9^{-0.12}\int_{R_p/R_0}^{\infty} x^2\ dx\ x^{-4} = \frac{BR_0^3}{D^2}\nu_9^{-0.12}\left(\frac{R_p}{R_0}\right)^{-1}$$

$$= \frac{BR_0^3}{D^2}\nu_9^{-0.12}\left(\frac{A}{2}\right)^{-\frac{1}{3}}\nu_9^{\frac{2.12}{3}}$$

$$F_\nu^{(\mathrm{outer})} = \frac{BR_0^3}{D^2}\left(\frac{A}{2}\right)^{-\frac{1}{3}}\nu_9^{\frac{1.76}{3}}$$

The spectral index of $F_\nu^{(\mathrm{outer})}$ is identical to the spectral index of $F_\nu^{(\mathrm{photo})}$ [N.B. this is not a coincidence.]

(d) The ratio

$$\frac{F_\nu^{(\mathrm{outer})}}{F_\nu^{(\mathrm{photo})}} = \frac{BR_0^3D^{-2}(A/2)^{-1/3}\nu_9^{\beta}}{2\pi k_{\mathrm{B}}T\ c^{-2}\ 10^{18}\,\mathrm{s}^{-2}\ R_0^2D^{-2}(A/2)^{2/3}\nu_9^{\beta}} = \frac{BR_0}{2\pi k_{\mathrm{B}}T}\frac{c^2}{10^{18}\,\mathrm{s}^{-2}}\times\frac{2}{A}$$

$$= \frac{4\pi\times 3.35\times10^{-40}T_4^{-0.32}n_0^2\,\mathrm{erg\,cm^3\,s^{-1}\,Hz^{-1}}R_0\times 9\times10^{20}\,\mathrm{cm^2\,s^{-2}}\times 2}{2\pi\times1.38\times10^{-12}T_4\,\mathrm{erg}\times10^{18}\,\mathrm{s}^{-2}\times1.09\times10^{-25}n_0^2R_0T_4^{-1.32}\,\mathrm{cm}^5} \quad = \quad 8 \quad .$$

(e) From (c) we have

$$
\begin{aligned}
F_\nu^{(\rm outer)} &= \frac{BR_0^3}{D^2} 2^{1/3} A^{-1/3} \nu_9^{0.59} \\
&= 4\pi\times 3.35\times 10^{-40} \frac{\rm erg\,cm^3}{\rm s\,Hz} T_4^{-0.32} n_0^2 \frac{R_0^3}{D^2} 2^{1/3} \left(1.09\times 10^{-25} n_0^2 T_4^{-1.32} R_0\,{\rm cm}^5\right)^{-1/3} \nu_9^{0.59} \\
&= 4\pi\times 3.35\times 10^{-40}\,{\rm erg\,cm^3\,s^{-1}\,Hz^{-1}} \frac{2^{1/3}}{D^2} \left(1.09\times 10^{-25}\,{\rm cm}^5\right)^{-1/3} \left(n_0 R_0^2\right)^{4/3} T_4^{0.12} \nu_9^{0.59} \\
n_0 &= \left(\frac{\dot{M}_{\rm w}}{4\pi R_0^2\, 1.4 m_{\rm H} v_{\rm w}}\right) \\
F_\nu^{(\rm outer)} &= 4\pi\times 3.35\times 10^{-40} \frac{\rm erg\,cm^3}{\rm s\,Hz} \frac{2^{1/3}}{D^2} \left(1.09\times 10^{-25}\,{\rm cm}^5\right)^{-1/3} \left(\frac{\dot{M}_{\rm w}}{4\pi 1.4 m_{\rm H} v_{\rm w}}\right)^{4/3} T_4^{0.12} \nu_9^{0.59} \\
&= 1.3\times 10^{-25}\,{\rm erg\,cm^{-2}\,s^{-1}\,Hz^{-1}} \left(\frac{\rm kpc}{D}\right)^2 \left(\frac{\dot{M}_{\rm w}}{10^{-6}\,M_\odot\,{\rm yr}^{-1}}\right)^{4/3} \left(\frac{v_{\rm w}}{20\,{\rm km\,s^{-1}}}\right)^{-4/3} T_4^{0.12} \nu_9^{0.59} \\
&= 0.013\,{\rm Jy} \left(\frac{\rm kpc}{D}\right)^2 \left(\frac{\dot{M}_{\rm w}}{10^{-6}\,M_\odot\,{\rm yr}^{-1}}\right)^{4/3} \left(\frac{v_{\rm w}}{20\,{\rm km\,s^{-1}}}\right)^{-4/3} T_4^{0.12} \nu_9^{0.59} \quad .
\end{aligned}
$$

10.7 Consider an H I cloud of column density $N_{\rm H} = 10^{21}\,{\rm cm}^{-2}$, H nucleon density $n_{\rm H} = 30\,{\rm cm}^{-3}$, temperature $T = 100\,{\rm K}$, and fractional ionization $n_e/n_{\rm H} = 10^{-3}$.

(a) What is the emission measure $EM = \int n_e n_i ds$?

(b) What is the surface brightness (i.e., specific intensity) of free-free emission from the cloud at $\nu = 5\,{\rm GHz}$? Assume the cloud to be optically-thin at $5\,{\rm GHz}$, and take $g_{\rm ff} \approx 1.83$. Express your answer in ${\rm Jy\,sr}^{-1}$.

Solution:

(a) For free-free emission, it doesn't matter whether the ions are protons or $\rm C^+$. *The emission measure is*

$$\begin{aligned} EM &= \int n_i n_e ds = (n_e/n_{\rm H})^2 \int {n_{\rm H}}^2 ds \\ &= (n_e/n_{\rm H})^2 n_{\rm H} N_{\rm H} \qquad\qquad \text{(if } n_{\rm H} \text{ is uniform)} \\ &= 10^{-6} \times 30\,{\rm cm}^{-3} \times 10^{21}\,{\rm cm}^{-2} = 3\times 10^{16}\,{\rm cm}^{-5} \quad . \end{aligned}$$

(b) The emission intensity is
(see Eq. 10.2 in the book; note that $h\nu \ll k_{\rm B}T$ *so that* $e^{-h\nu/k_{\rm B}T} \approx 1$*):*

$$\begin{aligned} I_\nu &= \int j_\nu ds \\ &= 5.44\times 10^{-41} g_{\rm ff} T_4^{-1/2} e^{-h\nu/k_{\rm B}T}\,{\rm erg\,cm^3\,s^{-1}\,sr^{-1}\,Hz^{-1}} \int n_i n_e ds \\ &= 5.44\times 10^{-41} \times 1.83 \times 10 \times 1 \times 3\times 10^{16}\,{\rm erg\,cm^{-2}\,s^{-1}\,sr^{-1}\,Hz^{-1}} \\ &= 2.99\times 10^{-23}\,{\rm erg\,cm^{-2}\,s^{-1}\,sr^{-1}\,Hz^{-1}} \\ &= 2.99\,{\rm Jy\,sr^{-1}} \quad . \end{aligned}$$

Chapter 11. Propagation of Radio Waves through the ISM

11.1 The pulses from a pulsar arrive later at low frequencies than at high frequencies. Suppose that the arrival times at $1420\,\mathrm{MHz}$ and $1610\,\mathrm{MHz}$ differ by $\Delta t(1420\,\mathrm{MHz}, 1610\,\mathrm{MHz}) = 0.0913\,\mathrm{s}$.

(a) What is the "dispersion measure" for this pulsar?

(b) If the pulsar is assumed to be at a distance $D = 6\,\mathrm{kpc}$, what is the mean electron density $\langle n_e \rangle$ along the path to the pulsar?

Solution:
(a) From Eq. (11.11):

$$\frac{dt_{\mathrm{arrival}}}{d(\nu_9^{-2})} = 4.146\times10^{-3}\frac{DM}{\mathrm{cm^{-3}\,pc}}$$

$$DM = \frac{\Delta t}{[4.146\times10^{-3}(1/1.42^2 - 1/1.61^2)\,\mathrm{s}]}$$

$$= 200\,\mathrm{cm^{-3}\,pc} \quad .$$

(b) $\langle n_e \rangle = DM/D = 200\,\mathrm{cm^{-3}\,pc}/6000\,\mathrm{pc} = 0.0333\,\mathrm{cm^{-3}}$.

11.3 A fast radio burst (FRB) occurs in a galaxy at redshift $z_{\rm FRB}$. The pulse arrival is delayed at low frequencies because of dispersion contributed by electrons along the path [including electrons in the Milky Way, the intergalactic medium (IGM), and the host galaxy of the FRB]. The observed DM is

$$DM_{\rm obs} = -\frac{\pi m_e c}{e^2}\nu_{\rm obs}^3 \frac{dt_{\rm arrival}}{d\nu_{\rm obs}} \quad .$$

For $z \lesssim 7$ (i.e., after reionization), assume the electron density in the IGM to be

$$\begin{aligned} n_e &= n_0(1+z)^3 \\ n_0 &\approx 1.1\times 10^{-7}\,{\rm cm}^{-3} \quad , \end{aligned}$$

(corresponding to an IGM containing $\sim$50% of the baryons in the Universe). Assume a Hubble constant $H_0 = 70\,{\rm km\,s^{-1}\,Mpc^{-1}}$.

If the redshift is not too large, we can assume a simple Hubble flow, with redshift proportional to distance, $cdz = H_0 dr$.

(a) At low redshift $z \ll 1$, show that the contribution of the IGM to the observed dispersion measure is

$$DM_{\rm IGM} = 471\left(\frac{n_0}{1.1\times 10^{-7}\,{\rm cm}^{-3}}\right) z_{\rm FRB}\,{\rm cm}^{-3}\,{\rm pc}$$

(b) At larger redshifts, one needs to take into account both the change in density of the universe and redshifting of the radiation in the pulse as it travels from the FRB to us. Show that the contribution of the IGM to DM is

$$DM_{\rm IGM} = 471\left(\frac{n_0}{1.1\times 10^{-7}\,{\rm cm}^{-3}}\right) \frac{[(1+z_{\rm FRB})^2 - 1]}{2}\,{\rm cm}^{-3}\,{\rm pc}$$

[To keep things simple, continue to assume a simple Hubble flow, $cdz = H_0 dr$.]

Solution:

(a) If we ignore both redshifting along the path, and the change of density along the path, this is easy:

$$\begin{aligned} DM_{\rm IGM} &= \int_0^{cz_{\rm FRB}/H_0} n_e dr \\ &\approx n_0 \frac{cz_{\rm FRB}}{H_0} \\ &= 471\left(\frac{n_0}{1.1\times 10^{-7}\,{\rm cm}^{-3}}\right)\, z_{\rm FRB}\,{\rm cm}^{-3}\,{\rm pc} \quad . \end{aligned}$$

(b) *At larger redshift, we need to calculate the time delay. At redshift z, the local frequency of the radiation is $\nu = \nu_{\rm obs}(1+z)$, and the electron density is $n_0(1+z)^3$.*

$$
\begin{aligned}
[\Delta t_{\rm arrival}]_{\rm IGM} &= \int dr \, \frac{d\Delta t}{dr} \\
&= \int \frac{cdz}{H_0} \, \frac{n_e e^2}{2\pi m_e c \nu^2} = \int_0^{z_{\rm FRB}} \frac{cdz}{H_0} \frac{n_0(1+z)^3 e^2}{2\pi m_e c \, \nu_{\rm obs}^2 (1+z)^2} \\
&= \frac{c}{H_0} \frac{n_0 e^2}{2\pi m_e c} \frac{1}{\nu_{\rm obs}^2} \int_0^{z_{\rm FRB}} dz(1+z) \\
&= \frac{c}{H_0} \frac{n_0 e^2}{2\pi m_e c} \frac{1}{\nu_{\rm obs}^2} \frac{[(1+z_{\rm FRB})^2 - 1]}{2} \\
DM_{\rm IGM} &= -\frac{\pi m_e c}{e^2} \nu_{\rm obs}^3 \left(\frac{d\Delta t_{\rm arrival}}{d\nu_{\rm obs}} \right)_{\rm IGM} = n_0 \frac{c}{H_0} \frac{[(1+z_{\rm FRB})^2 - 1]}{2} \\
&= 471 \left(\frac{n_0}{1.1 \times 10^{-7}\,{\rm cm}^{-3}} \right) \frac{[(1+z_{\rm FRB})^2 - 1]}{2} \,{\rm cm}^{-3}\,{\rm pc} \quad .
\end{aligned}
$$

Chapter 12. Interstellar Radiation Fields

12.1 After the Sun, Sirius (α Canis Majoris) is the brightest star in our sky. It is actually a binary; Sirius A and Sirius B. Sirius A is spectral type A1V, with mass $2.1\,M_\odot$; Sirius B is a (much fainter) white dwarf, with mass $0.98\,M_\odot$.

The Sirius system has luminosity $L = 25\,L_\odot$, and is at a distance $D = 2.6\,\mathrm{pc}$. What is the energy density u due to radiation from Sirius alone at the location of the Sun? What fraction of the local starlight background energy density is contributed by Sirius alone?

Solution:

$$u = \frac{L}{4\pi D^2 c} = 4.0\times10^{-15}\,\mathrm{erg\,cm^{-3}} \quad .$$

From Table 12.1, the total starlight background is $1.05\times10^{-12}\,\mathrm{erg\,cm^{-3}}$. *Thus Sirius contributes only 0.004 = 0.4% of the local starlight background.*

Chapter 13. Ionization Processes

13.1 Define E_x to be the energy at which the photoelectric cross section (13.4) for a hydrogenic ion is equal to the Compton scattering cross section, $\sigma_T = (8\pi/3)(e^2/m_e c^2)^2$.

(a) Express E_x/I_H in terms of Z and the fine structure constant $\alpha \equiv e^2/\hbar c = 1/137.04$

(b) For hydrogen, calculate E_x in eV.

Solution:

(a) Using $m_e c^2 = 2I_\mathrm{H}/\alpha^2$ *and* $e^2 = 2I_\mathrm{H} a_0$

$$\frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = \frac{2^8}{3Z^2}\alpha\pi a_0^2 \left(\frac{Z^2 I_\mathrm{H}}{E_x}\right)^{3.5}$$

$$\frac{8\pi}{3}\frac{4I_\mathrm{H}^2 a_0^2}{4I_\mathrm{H}^2/\alpha^4} = \frac{2^8}{3Z^2}\alpha\pi a_0^2 Z^7 \left(\frac{I_\mathrm{H}}{E_x}\right)^{3.5}$$

$$\frac{E_x}{I_\mathrm{H}} = \frac{(2Z)^{10/7}}{\alpha^{6/7}} = 183 Z^{10/7} \quad .$$

(b) $E_x = 183 Z^{10/7} I_\mathrm{H} = 183 I_\mathrm{H} = 2480\,\mathrm{eV}$.

13.3 From Figure 13.2, one sees that the photoionization cross section for neutral Si can be approximated by

$$\sigma(h\nu) \approx 7 \times 10^{-17} \left(\frac{h\nu}{8.15\,\mathrm{eV}}\right)^{-3.5} \mathrm{cm}^2$$

for $8.15\,\mathrm{eV} < h\nu < 13.6\,\mathrm{eV}$. Suppose that the energy density of starlight in an H I cloud (see Fig. 12.2) can be approximated by

$$\nu u_\nu \approx 9 \times 10^{-14} \left(\frac{h\nu}{8.15\,\mathrm{eV}}\right)^{-1} \mathrm{erg\,cm}^{-3}$$

for $8.15\,\mathrm{eV} < h\nu < 13.6\,\mathrm{eV}$, and $u_\nu \approx 0$ for $h\nu > 13.6\,\mathrm{eV}$.

Calculate the photoionization rate ζ for an Si atom.

Solution: The photon density is the energy density divided by the energy per photon. The photons move at the speed of light c. *Hence, noting that* $\nu u_\nu = E(du/dE)$*:*

$$\begin{aligned}
\zeta &\approx \int_{8.15\,\mathrm{eV}}^{13.6\,\mathrm{eV}} dE\, \frac{du/dE}{E}\, c\,\sigma(E) = \int_{8.15\,\mathrm{eV}}^{13.6\,\mathrm{eV}} dE\, \frac{1}{E^2} \left(E\frac{du}{dE}\right) c\,\sigma(E) \\
&= \int_{8.15\,\mathrm{eV}}^{13.6\,\mathrm{eV}} dE \frac{1}{E^2} 9\times10^{-14}\,\mathrm{erg\,cm}^{-3} \left(\frac{E}{8.15\,\mathrm{eV}}\right)^{-1} 3\times10^{10}\,\mathrm{cm\,s}^{-1}\times7\times10^{-17}\,\mathrm{cm}^2 \left(\frac{E}{8.15\,\mathrm{eV}}\right)^{-3.5} \\
&= \int_1^{13.6/8.15} \frac{1}{8.15\,\mathrm{eV}} \frac{dx}{x^2}\, 9\times10^{-14}\,\mathrm{erg\,cm}^{-3}\, x^{-1}\, 3\times10^{10}\,\mathrm{cm\,s}^{-1}\times7\times10^{-17}\,\mathrm{cm}^2\, x^{-3.5} \\
&= \frac{9\times3\times7\times10^{-21}\,\mathrm{erg\,s}^{-1}}{8.15\,\mathrm{eV}} \int_1^{13.6/8.15} dx\, x^{-6.5} \\
&= 1.45 \times 10^{-8}\,\mathrm{s}^{-1} \frac{1}{5.5} \left[1 - \left(\frac{8.15}{13.6}\right)^{5.5}\right] \\
&= 2.47 \times 10^{-9}\,\mathrm{s}^{-1}
\end{aligned}$$

which is in reasonable agreement with the result ($2.77\times10^{-9}\,\mathrm{s}^{-1}$) *of a more careful calculation in Table 13.1.*

Chapter 14. Recombination of Ions with Electrons

14.1 Suppose that an electron recombines into the $n = 5$, $\ell = 4$ (also known as $5g$) level of hydrogen. What is the probability that an Hα photon will be emitted during the radiative cascade starting from $(n, \ell) = (5, 4)$?

Solution: The electric dipole selection rule limits transitions to $\Delta\ell = 0, \pm 1$. Therefore a level $(n, \ell) = (N, N-1)$, has only one allowed radiative decay path: a $(N-1)\alpha$ transition to $(n, \ell) = (N-1, N-2)$. Thus the decay chain will be

$$5g \to 4f \to 3d \to 2p \to 1s \ .$$

Therefore 100% of the radiative cascades from the $5g$ level will include emission of an Hα photon in the $3d \to 2p$ transition.

14.3 For case B recombination at $T = 10^4\,\mathrm{K}$, estimate $j(\mathrm{Ly}\,\alpha)/j(\mathrm{H}\alpha)$ for $n_e = 10^2\,\mathrm{cm}^{-3}$, $10^3\,\mathrm{cm}^{-3}$, and $10^4\,\mathrm{cm}^{-3}$. Here j is the power radiated per unit volume, where "radiated" is interpreted as creation of "new" photons (i.e., scattering is not included). Note that j is a *local* property – it does not take into account whether or not the photons will "escape" the emitting region.

Solution: Recombinations that populate 2p will produce Lyα. Recombinations that populate 2s will either decay by 2 photon emission or will be collisionally converted to 2p, followed by emission of Lyα. We can write

$$\frac{j(\mathrm{Ly}\,\alpha)}{j(\mathrm{H}\alpha)} = \frac{27}{5} \times \frac{\alpha_B}{\alpha_{\mathrm{eff},H\alpha}} \times \left[f(2p) + [1 - f(2p)]\left(\frac{n_e}{n_e + n_{e,\mathrm{crit}}}\right)\right]$$

$$\approx 11.95\left[0.675 + 0.325\left(\frac{n_e}{n_e + 1.55\times10^4\,\mathrm{cm}^{-3}}\right)\right] \qquad \text{at } T = 10^4\,\mathrm{K}$$

where

- $\alpha_B \approx 2.59\times10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ *(Table 14.2);*
- $\alpha_{\mathrm{eff,H}\alpha} \approx 1.17\times10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ *[Eq. (14.8)];*
- $f(2p) \approx 0.675$ *is the fraction of case B recombinations that populate* $2p$ *(see Table 14.3);*
- $n_{e,\mathrm{crit}} = A(2s \to 1s)/(q_{p,2s\to2p} + q_{e,2s\to2p}) \approx 1.55\times10^4\,\mathrm{cm}^{-3}$ *[see book errata] is the critical density for* $2s \to 2p$ *collisional transitions.*

Thus,

$$\begin{aligned} j(\mathrm{Ly}\,\alpha)/j(\mathrm{H}\alpha) &= 8.09 && \text{at } n_e = 10^2\,\mathrm{cm}^{-3} \\ &= 8.30 && \text{at } n_e = 10^3\,\mathrm{cm}^{-3} \\ &= 9.59 && \text{at } n_e = 10^4\,\mathrm{cm}^{-3} \quad . \end{aligned}$$

14.5 In the standard Big Bang model, H and He were nearly fully ionized at redshifts $z \gtrsim 2000$. As the expanding Universe cooled, the rates for photoionization and collisional ionization dropped and the gas began to recombine. According to current estimates of the baryon density, the hydrogen fractional ionization was $x = 0.5$ at a redshift and temperature

$$z_{0.5} \approx 1250 \quad , \quad T_{0.5} \approx 3410\,\mathrm{K} \quad .$$

At this temperature, collisional ionization and photoionization are still important, and the Saha equation provides a good approximation to the ionization fraction. As the temperature and density continue to drop, the Universe is expanding too rapidly to maintain thermodynamic equilibrium, and a kinetic calculation is necessary. According to detailed calculations (Grin & Hirata 2010; Phys. Rev. D 81, 083005), the fractional ionization has dropped to $x = 0.01$ at redshift

$$z_{0.01} = 880 \quad , \quad T_{0.01} = 2400\,\mathrm{K} \quad .$$

From this point on, let us assume that photoionization and collisional ionization can be neglected, and consider only radiative recombination.

According to current estimates of cosmological parameters
($H_0 = 70.2\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $\Omega_{\mathrm{baryon}} = 0.0458$, $\Omega_{\mathrm{dark\ matter}} = 0.229$, $\Omega_\Lambda = 0.725$: Komatsu et al. 2010; arXiv1001.4538), the age of the Universe at redshift $z \gtrsim 10$ is

$$t(z) \approx \frac{17\,\mathrm{Gyr}}{(1+z)^{3/2}} \quad ,$$

and the age of the Universe when $x = 0.01$ was

$$t_{0.01} \approx \frac{17\,\mathrm{Gyr}}{(1+z_{0.01})^{3/2}} = 6.5\times 10^5\,\mathrm{yr} \quad .$$

According to current estimates of the baryon density based on nucleosynthetic constraints, the hydrogen density $n_{\mathrm{H}} = n(\mathrm{H}^0) + n(\mathrm{H}^+)$ in the expanding Universe is

$$n_{\mathrm{H}}(t) = n_{\mathrm{H},0.01}\left(\frac{1+z}{1+z_{0.1}}\right)^3 = n_{\mathrm{H},0.01}\left(\frac{t}{t_{0.01}}\right)^{-2} \quad ,$$

where the H density at $z_{0.01} = 880$ was

$$n_{\mathrm{H},0.01} = 130\ \mathrm{cm^{-3}} \quad .$$

Because of Compton scattering, the temperature of the free electrons remains coupled to the radiation field until quite late times. If we assume that this coupling persists throughout the main phase of recombination, then the electron temperature evolves as

$$T_e(z) = T_{0.01}\left(\frac{1+z}{1+z_{0.01}}\right) = T_{0.01}\left(\frac{t}{t_{0.01}}\right)^{-2/3} \quad .$$

Suppose that for $t > t_{0.01}$ (i.e., $z < z_{0.01}$), no further ionization takes place, and the ionized fraction x continues to drop due to radiative recombination. Suppose that the rate coefficient for radiative recombination for $T < 2400\,\mathrm{K}$ can be written

$$\alpha_B = 7.8\times 10^{-13}(T_e/2400\,\mathrm{K})^{-0.75}\,\mathrm{cm^3\,s^{-1}} \quad .$$

Note: Ignore helium in this problem.

(a) Obtain an equation for $dx/d\tau$, where $x \equiv n(\mathrm{H}^+)/n_{\mathrm{H}}$ is the hydrogen fractional ionization, and $\tau \equiv t/t_{0.01}$ is time in units of $t_{0.01}$. (Hint: do not let the expansion of the Universe confuse you. Remember that if there were no recombination, the fractional ionization would remain constant even as the Universe expands.)

(b) Solve the differential equation from part (a) to find the solution $x(t)$ for $t > t_{0.01}$.

(c) Assuming that photoionization and collisional ionization remain negligible, evaluate the fractional ionization x at redshift $z = 50$, $z = 100$ and $z = 15$. The nonzero ionization remaining at $z \lesssim 100$ is sometimes referred to as "ionization freezout".

(d) *WMAP* observations of polarization in the CMB appear to require partial reionization of the Universe at $z \approx 12$. Suppose that a large region is reionized by photoionization at $z = 12$. The photoionized gas will initially be at $T \approx 2\times 10^4\,\mathrm{K}$, with case B recombination coefficient $\alpha_B \approx 1.7{\times}10^{-13}\,\mathrm{cm^3\,s^{-1}}$. Compare the timescale for recombination at $z = 12$ to $t(z = 12)$, the age of the Universe at $z = 12$.

Solution:

(a) Let $x \equiv n(\mathrm{H}^+)/n_{\mathrm{H}}$ be the fractional ionization. Consider a comoving volume V. The rate of change of the number of free electrons is

$$\frac{d}{dt}(n_{\mathrm{H}} x V) = -\alpha {n_{\mathrm{H}}}^2 x^2 V$$

but $n_{\mathrm{H}} V = constant$, thus, noting that $n_{\mathrm{H}} \propto t^{-2}$ and $\alpha_B \propto T_e^{-3/4} \propto (t^{-2/3})^{-3/4} \propto t^{1/2}$:

$$\begin{aligned}
\frac{dx}{dt} &= -\alpha_B\, n_{\mathrm{H}}\, x^2 \\
&= -\alpha_{0.01}\left(\frac{t}{t_{0.01}}\right)^{1/2} n_{\mathrm{H},0.01}\left(\frac{t}{t_{0.01}}\right)^{-2} x^2 \\
\frac{dx}{d\tau} &= -\alpha_{0.01} n_{\mathrm{H},0.01} t_{0.01}\, x^2\, \tau^{-3/2}
\end{aligned}$$

where $\alpha_{0.01} \equiv 7.8\times 10^{-13}\,\mathrm{cm^3\,s^{-1}}$, and

$$\tau \equiv \frac{t}{t_{0.01}} = \left(\frac{1+z_{0.01}}{1+z}\right)^{3/2} \quad .$$

(b) This is readily integrated:

$$\begin{aligned}
x^{-2} dx &= -\alpha_{0.01} n_{\mathrm{H},0.01} t_{0.01}\; \tau^{-3/2}\, d\tau \\
\int_{0.01}^{x} \frac{du}{u^2} &= -\alpha_{0.01} n_{\mathrm{H},0.01} t_{0.01} \int_1^{\tau} u^{-3/2} du \\
x &= \frac{1}{10^2 + 2(\alpha_{0.01} n_{\mathrm{H},0.01} t_{0.01})(1-\tau^{-1/2})} \quad .
\end{aligned}$$

(c) *The dimensionless number*

$$\alpha_{0.01} n_{\mathrm{H},0.01} t_{0.01} = 7.8\times10^{-13}\,\mathrm{cm^3\,s^{-1}} \times 130\,\mathrm{cm^{-3}} \times 6.5\times10^5\,\mathrm{yr} = 2080$$

Thus,

$$x = \frac{1}{10^2 + 4160(1-\tau^{-1/2})} \qquad \text{for } \tau > 1$$

$$\tau = \left(\frac{881}{1+z}\right)^{3/2} \quad .$$

Thus:

$$z = 500: \quad \tau = (881/501)^{3/2} = 2.33\ , \quad x = 6.5\times10^{-3}$$
$$z = 100: \quad \tau = (881/101)^{3/2} = 25.8\ , \quad x = 2.9\times10^{-4}$$
$$z = 15: \quad \tau = (881/16)^{3/2} = 409.\ , \quad x = 2.5\times10^{-4} \quad .$$

Therefore, unless dark matter annihilation or decay produces ionization, the intergalactic medium has a residual ionization $x \approx 2.4\times10^{-4}$ *at late times, until new sources of ionization (stars, QSOs) appear.*

(d) *At* $z = 12$*, the age of the Universe is* $t \approx 17\,\mathrm{Gyr}/13^{1.5} \approx 3.6\times10^8\,\mathrm{yr}$*, and the density is* $n_{\mathrm{H}} = 4.2\times10^{-4}\,\mathrm{cm^{-3}}$.

In fully-ionized gas, the timescale for radiative recombination is $(n_{\mathrm{H}}\alpha_B)^{-1} = (4.2\times10^{-4} \times 1.7\times10^{-13}\,\mathrm{s^{-1}})^{-1} = 4.4\times10^8\,\mathrm{yr}$*, longer than the age of the Universe at* $z = 12$.

14.7 From observation of the K I absorption line at $7667\,\text{Å}$, an H I cloud is determined to have a column density $N(\text{K I}) = 1.0\times10^{13}\,\text{cm}^{-2}$. Starlight ionizes the K I at a rate (from Table 13.1) $\zeta(\text{K} + h\nu \to \text{K}^+ + e^-) = 6.85\times10^{-12}\,\text{s}^{-1}$. Assume that the electron density in the cloud is $n_e = 0.03\,\text{cm}^{-3}$, the gas temperature is $T = 100\,\text{K}$, and the radiative recombination rate coefficient $\alpha_{\text{rr}} \equiv \alpha(\text{K}^+ + e^- \to \text{K} + h\nu) = 1.11 \times 10^{-11}\,\text{cm}^3\,\text{s}^{-1}$. Assume that radiative recombination is the only process removing K II and producing K I (i.e., neglect grain-assisted recombination). Assume that higher ion stages (K III, K IV, ...) can be neglected.

(a) Estimate the total column density of gas-phase K (both K I and K II) on this sightline.

(b) Given that grain-assisted recombination has been neglected, is the above estimate for the total column density of gas-phase K a lower bound or an upper bound?

Solution:

(a) We assume ionizations $\text{K}^0 \to \text{K}^+$ = *recombinations* $\text{K}^+ \to \text{K}^0$*:*

$$\begin{aligned} n(\text{K}^0)\zeta &= n(\text{K}^+)\alpha_{\text{rr}}n_e \\ \frac{n(\text{K}^+)}{n(\text{K}^0)} &= \frac{\zeta}{\alpha_{\text{rr}}n_e} = \frac{6.85\times10^{-12}\,\text{s}^{-1}}{1.11\times10^{-11}\,\text{cm}^3\,\text{s}^{-1}\times0.03\,\text{cm}^{-3}} = 20.6 \quad . \end{aligned}$$

Thus

$$\begin{aligned} N(\text{K}^0) + N(\text{K}^+) &= (1 + 20.6)N(\text{K}^0) \\ &= 21.6 \times 1.0\times10^{13}\,\text{cm}^{-2} = 2.16\times10^{14}\,\text{cm}^{-2} \quad . \end{aligned}$$

(b) If there are other channels for neutralization of the K^+*, the column density* $N(\text{K}^+)$ *will have been* **overestimated***, and hence the total gas-phase column density of* K *will have been overestimated: the gas-phase column density estimate for the total column density of gas-phase K is an* **upper bound**.

14.9 Observation of $\text{He I}\,10833\,\text{Å}$ absorption during transit of some exoplanets has been interpreted as absorption by metastable $\text{He I}\,2^3S_1$ produced by recombination of He^+ (see textbook Figure 14.3). A simple toy model that attempts to explain this postulates that the exoplanet has a an extended atmosphere with $n_{\rm H} = 10^8\,\text{cm}^{-3}$, He/H=0.1, $n(\text{H}^+)/n_{\rm H} = 0.1$, $n(\text{He}^+)/n_{\rm He} = 0.1$, and $T = 2000\,\text{K}$, and thickness $L \approx 5\times10^9\,\text{cm}$.

The He ionization in this extended atmosphere is assumed to be maintained by $h\nu > 24.6\,\text{eV}$ radiation from the star. Suppose that the $h\nu > 24.6\,\text{eV}$ stellar photons have a typical energy $35\,\text{eV}$, and suppose that the He photoionization cross section is $\sigma = 4\times10^{-18}\,\text{cm}^2$ (see Figure 13.1a).

If the exoplanet has orbital radius $R = 0.05\,\text{AU}$, what must be the stellar luminosity in $h\nu > 24.6\,\text{eV}$ photons? Neglect attenuation of the ionizing radiation in the extended atmosphere.

[For comparison, the average spectrum of the Sun has a luminosity $\sim 10^{-6}L_\odot$ in $h\nu > 24.6\,\text{eV}$ photons.]

Solution: By assumption, the electron density is $n_e = 1.1\times10^7\,\text{cm}^{-3}$ *and the neutral He density is* $n(\text{He}^0) = 9\times10^6\,\text{cm}^{-3}$.

A reasonable approximation would be to simply assume the effective rate coefficient for He recombinaton is just the "case B" recombination rate $\alpha_B(\text{He})$ *of all recombinations except the ground state:*

$$\alpha_{\rm eff} \approx \alpha_B(\text{He}) = 2.72\times10^{-13}\,\text{cm}^3\,\text{s}^{-1}T_4^{-0.789} = 9.7\times10^{-13}\,\text{cm}^3\,\text{s}^{-1}$$

for $\text{He}^+ + e^- \rightarrow \text{He}^0$

A more sophisticated treatment allows for the fact that some fraction of the recombinations directly the the ground state of He^0 *will create photons that will be absorbed by H instead of He, so that the effective recombination rate coefficient for He is given by Eq. (14.17):*

$$\alpha_{\rm eff}(\text{He}) = \alpha_B(\text{He}) + y\alpha_{1s^2}(\text{He}) \ ,$$

where

$$y \approx \frac{1}{[1 + 6n(\text{He}^0)/n(\text{H}^0)]}$$

is the fraction of $h\nu \approx 25\,\text{eV}$ *He recombination photons (emitted in direct recombinaton to the* $1s^2$ *ground state) that are absorbed by H. For* $n(\text{He}^0) = 0.1n(\text{H}^0)$ *we have*

$$y \approx \frac{1}{1+0.6} \approx 0.6 \ .$$

Taking $\alpha_{1s^2}(\text{He})$ *and* $\alpha_B(\text{He})$ *from Eq. (14.14,14.15), Eq. (14.17) gives*

$$\begin{aligned}\alpha_{\rm eff}(\text{He}) &\approx \left[9.68\times10^{-13} + 0.6\times3.37\times10^{-13}\right]\,\text{cm}^3\,\text{s}^{-1} \\ &\approx 1.17\times10^{-12}\,\text{cm}^3\,\text{s}^{-1} \ .\end{aligned}$$

This is only ∼20% larger than $\alpha_B(\text{He})$*, so it is OK to ignore this correction given the crude level at which we are trying to "model" the excitation in this exoplanet atmosphere.*

The photon flux F required to maintain this ionization is determined by

$$\begin{aligned} F\,\sigma\,n(\mathrm{He}^0) &= \alpha_{\mathrm{eff}}(\mathrm{He})\,n_e\,n(\mathrm{He}^+) \\ F &= \frac{\alpha_{\mathrm{eff}}(\mathrm{He})n_e}{\sigma} \times \frac{n(\mathrm{He}^+)}{n(\mathrm{He}^0)} \\ F &= \frac{1.17\times10^{-12}\times1.1\times10^{7}}{4\times10^{-18}} \times \frac{0.1}{0.9}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1} \\ &= 3.58\times10^{11}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\ . \end{aligned}$$

If the typical photon energy is $35\,\mathrm{eV}$ *and the distance from the star is* $R = 0.05\,\mathrm{AU}$*, then the stellar luminosity in He-ionizing photons is*

$$\begin{aligned} L(h\nu > 24.6\,\mathrm{eV}) &= F\times35\,\mathrm{eV}\times4\pi(0.05\,\mathrm{AU})^2 \\ &= 1.42\times10^{26}\,\mathrm{erg\,s}^{-1} = 3.7\times10^{-8}\,L_\odot \end{aligned}$$

Note that the optical depth through the entire slab for $35\,\mathrm{eV}$ *photons will be modest:*

$$\begin{aligned} \tau &= n(\mathrm{He}^0)\sigma L \approx 9\times10^{6}\,\mathrm{cm}^{-3}\times4\times10^{-18}\,\mathrm{cm}^2\times5\times10^{9}\,\mathrm{cm} \\ &= 0.18\ , \end{aligned}$$

hence the He-ionizing flux will be only moderately attenuated as it passes through the gas – the flux incident on the star-facing side will be able to maintain similar levels of ionization as it propagates through the extended atmosphere.

Chapter 15. Photoionized Gas

15.1 A O9V star has luminosity $L = 10^{4.77}\,L_\odot$, emits $h\nu > 13.6\,\mathrm{eV}$ photons at a rate $Q_0 = 10^{48.06}\,\mathrm{s}^{-1}$, and emits $h\nu > 24.6\,\mathrm{eV}$ photons at a rate $Q_1 = 0.0145 Q_0$ (see Table 15.1). The star is surrounded by a steady-state H II region.

(a) If the ionized region has a uniform density $n_\mathrm{H} = 10^2\,\mathrm{cm}^{-3}$ and temperature $T = 10^4\,\mathrm{K}$, estimate the neutral fraction $n(\mathrm{H}^0)/n_\mathrm{H}$ at a distance $r = 0.9 R_{\mathrm{H\,II}}$ from the star, where $R_{\mathrm{H\,II}}$ is the radius of the zone where H is ionized. Assume that the gas is pure hydrogen, and that dust is negligible.

(b) Now assume that the gas has He/H≈0.1 by number. What will be the ratio $R_{\mathrm{He\,II}}/R_{\mathrm{H\,II}}$, where $R_{\mathrm{He\,II}}$ is the radius of the zone where helium is ionized? An answer accurate to 10% is OK – don't worry over details. State your assumptions.

Solution:

(a) At $T = 10^4\,\mathrm{K}$, $\alpha_B = 2.59\times10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$. *The dustless Strömgren radius*

$$R_{\mathrm{H\,II}} = \left(\frac{3Q_0}{4\pi {n_\mathrm{H}}^2 \alpha_B}\right)^{1/3} = 4.73\times10^{18}\,\mathrm{cm} \quad .$$

The flux of H*-ionizing photons at radius* $R = 0.9 R_{\mathrm{H\,II}}$ *is*

$$F = \frac{(1-0.9^3)Q_0}{4\pi(0.9R_{\mathrm{H\,II}})^2} = 1.37\times10^9\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1} \quad .$$

Take $\sigma \approx 3\times10^{-18}\,\mathrm{cm}^2$ *as an estimate of the* H *photoionization cross section averaged over the spectrum of ionizing photons at* $R = 0.9R_{\mathrm{H\,II}}$.
The photoionization rate $\zeta = F\sigma \approx 4.1\times10^{-9}\,\mathrm{s}^{-1}$.
In steady-state, the neutral fraction $x_0 \equiv n(\mathrm{H}^0)/n_\mathrm{H}$ *must satisfy*

$$\frac{x_0}{(1-x_0)^2} = \frac{n_\mathrm{H}\alpha_B}{\zeta} \approx 6.3\times10^{-3}$$

$$x_0 \approx 6.2\times10^{-3} \quad .$$

Alternatively, using eq. (15.3):

$$\begin{aligned}
\frac{x_0}{(1-x_0)^2} &= \frac{3y^2}{1-y^3}\frac{1}{\tau_{S0}} \\
\tau_{S0} &= 2880(Q_{0,49})^{1/3} n_2^{1/3} = 2880\times(10^{48.06-49})^{1/3} = 1400 \\
\frac{x_0}{(1-x_0)^2} &= \frac{3\times0.81}{1-0.729}\times\frac{1}{1400} = 6.4\times10^{-3}
\end{aligned}$$

(b) Recombination rates for H *and* He *are found in Table 14.7:* $\alpha_B = 2.59\times10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ *for* H, $\alpha = 2.72\times10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ *for* He. *If* $n_\mathrm{H} = 10^2\,\mathrm{cm}^{-3}$ *is constant, then* n_e *is larger*

by 10% in the He II *zone. The radii of the* He II *and* He I *zones are given by the ionization balance conditions*

$$Q_0 = n_{\rm H} \times n_{\rm H}\alpha_B({\rm H})\frac{4\pi}{3}R^3_{\rm H\,II} \quad ,$$

$$Q_1 = 0.1 n_{\rm H} \times 1.1 n_{\rm H}\alpha({\rm He})\frac{4\pi}{3}R^3_{\rm He\,II} \quad .$$

so that

$$\begin{aligned}
\frac{R^3_{\rm He\,II}}{R^3_{\rm H\,II}} &\approx \frac{Q_1}{Q_0}\frac{1\times 1}{0.1\times 1.1}\frac{\alpha_B({\rm H\,II})}{\alpha({\rm He\,II})} \\
&= 0.0145 \times \frac{1}{0.11}\frac{2.59\times 10^{-13}}{2.72\times 10^{-13}} = 0.101 \quad , \\
\frac{R_{\rm He\,II}}{R_{\rm H\,II}} &\approx (0.101)^{1/3} = 0.466 \quad \to \quad R_{\rm He\,II} \approx 2.20\times 10^{18}\,{\rm cm} \quad .
\end{aligned}$$

15.3 Consider a spherically-symmetric stellar wind with mass-loss rate $\dot{M}_w = 10^{-4} M_\odot\,\mathrm{yr}^{-1}$. and wind speed $v_w = 20\,\mathrm{km\,s}^{-1}$. Suppose the mass-loss continues steadily for $t_w = 10^3\,\mathrm{yr}$ and then stops, with the wind continuing to "coast" outwards. Suppose that after a time t, the central star suddenly becomes an ionizing source emitting hydrogen-ionizing photons at a rate Q_0, creating a "protoplanetary nebula".

(a) After time t, the outflowing wind has a spherical outer surface and a spherical inner "hole". What is the density just inside the outer surface?

(b) What is the density just outside the inner hole?

(c) Ignoring expansion of the nebula during the ionization process, what is the minimum value of Q_0 required to ionize the H throughout the nebula?

(d) What is the recombination time just inside the outer surface? Compare this to the $10^3\,\mathrm{yr}$ dynamical age.

Solution:

(a) The inner surface is at $r_1 = v_w t$, and the outer surface is at $r_2 = v_w(t_w + t)$. The density is

$$\rho(r_2) = \frac{\dot{M}}{4\pi r_2^2 v_w} = 6.30\times10^{-20}\left(\frac{10^3\,\mathrm{yr}}{t_w + t}\right)^2\ \mathrm{g\,cm}^{-3} \quad ,$$

$$n_{\mathrm{H}}(r_2) = \frac{\dot{M}_w}{4\pi r_2^2 v_w \mu} = 2.69\times10^{4}\left(\frac{10^3\,\mathrm{yr}}{t_w + t}\right)^2\ \mathrm{cm}^{-3} \quad ,$$

where $\mu = 1.4 m_{\mathrm{H}}$ is the mass per H *nucleon.*

(b)

$$\rho(r_1) = 6.30\times10^{-20}\left(\frac{10^3\,\mathrm{yr}}{t}\right)^2\ \mathrm{g\,cm}^{-3} \quad ,$$

$$n_{\mathrm{H}}(r_1) = n_{\mathrm{inner}} = \frac{\dot{M}_w}{4\pi r_1^2 v_w \mu} = 2.69\times10^{4}\left(\frac{10^3\,\mathrm{yr}}{t}\right)^2\ \mathrm{cm}^{-3} \quad .$$

(c) If the nebula were fully ionized, and we consider only H, *the recombination rate would*

be

$$
\begin{aligned}
\dot{N}_{\rm rec} &= \int_{r_1}^{r_2} \alpha_B {n_{\rm H}}^2 4\pi r^2 dr \\
&= \left(\frac{\dot{M}_w}{v_w\mu}\right)^2 \frac{\alpha_B}{4\pi} \int_{r_1}^{r_2} \frac{1}{r^4} r^2 dr = \left(\frac{\dot{M}_w}{v_w\mu}\right)^2 \frac{\alpha_B}{4\pi} \left[\frac{1}{r_1} - \frac{1}{r_2}\right] \\
&= \left(\frac{\dot{M}_w}{v_w\mu}\right)^2 \frac{\alpha_B}{4\pi} \left[\frac{1}{v_w t} - \frac{1}{v_w(t_w+t)}\right] \\
&= \left(\frac{\dot{M}_w}{\mu}\right)^2 \frac{\alpha_B}{4\pi v_w^3} \frac{1}{t(1+t/t_w)} \\
&= 6.0\times 10^{50} \left(\frac{\rm yr}{t}\right) \frac{1}{1+t/t_w}\,{\rm s}^{-1} \quad ,
\end{aligned}
$$

where we have taken $\alpha_B = 2.59\times 10^{-13}\,{\rm cm}^3\,{\rm s}^{-1}$ *from Table 14.7.*
To ionize the nebula, require $Q_0 > 6.0\times 10^{50}({\rm yr}/t)\frac{1}{1+t/t_w}\,{\rm s}^{-1}$*. For* $t \lesssim 10^2\,{\rm yr}$*, this is a very large* Q_0*. The protoplanetary nebula nucleus (a white dwarf) will have* $Q_0 \lesssim 10^{48}\,{\rm s}^{-1}$*, and therefore at early times will only be able to ionize an inner zone near the star.*

(d)

$$
\begin{aligned}
t_{\rm rec} &= \frac{1}{\alpha_B n_{\rm H}(r_2)} \\
&= \frac{4\pi r_2^2 v_w \mu}{\alpha_B \dot{M}_w} \\
&= \frac{4\pi v_w^3 \mu}{\alpha_B \dot{M}_w} t_w^2 \left(1+\frac{t}{t_w}\right)^2 \\
&= 1.43\times 10^8 \left(1+\frac{t}{t_w}\right)^2 {\rm s} = 4.5\,{\rm yr}\,(1+t/t_w)^2 \quad .
\end{aligned}
$$

For the values $t \lesssim t_w$ *of interest here, the recombination time is short compared to the dynamical age* $\sim t_w$*.*

15.5 Consider an H II region powered by a star emitting ionizing photons at a rate Q_0. Assume a pure hydrogen nebula (no He, no dust), and approximate the H II region by a Strömgren sphere with uniform density $n_{\rm H}$. Let α_B be the case B recombination rate coefficient, T be the gas temperature, and assume the ionizing stellar photons to be monoenergetic with energy $h\nu$. Assume that the hydrogen is nearly fully ionized. ($n_e \approx n_{\rm H}$).

(a) The star exerts radiation pressure on the gas – each ionizing photon, when absorbed, deposits a momentum $h\nu/c$. Assume that the only interaction of the photons with the gas is through photoionization (i.e., electron scattering and free-free absorption are neglected). Define a function $p_{\rm rad}$ by the differential equation $-dp_{\rm rad}/dr$ = radial force/volume exerted by the absorbed stellar radiation. If the density in the H II region is uniform, calculate $\Delta p_{\rm rad} \equiv p_{\rm rad}(0) - p_{\rm rad}(R_S)$, where R_S is the Strömgren radius. Write your result in terms of α_B, $n_{\rm H}$, Q_0, and $h\nu$.

(b) Obtain the ratio $\Delta p_{\rm rad}/2n_{\rm H}k_{\rm B}T$ in terms of Q_0, $n_{\rm H}$, α_B, $h\nu$, c, and $k_{\rm B}T$.

(c) Evaluate the ratio $\Delta p_{\rm rad}/2n_{\rm H}k_{\rm B}T$ for Orion Nebula-like conditions: $T = 10^4\,{\rm K}$, $n_{\rm H} = 3200\,{\rm cm}^{-3}$, $Q_0 \approx 6.5\times10^{48}\,{\rm s}^{-1}$, and $h\nu \approx 18\,{\rm eV}$.

(d) When this radiation pressure is taken into consideration, it is clear that a uniform density isothermal nebula would not be in dynamical equilibrium. If the nebula needs to be in dynamical equilibrium, will the gas pressure at the edge adjust to be larger or smaller than the gas pressure at the center?

Solution:

(a)

$$\frac{dp_{\rm rad}}{dr} = -\alpha_B n_e^2 \approx -\alpha_B n_{\rm H}^2 \frac{h\nu}{c}$$

$$\Delta p_{\rm rad} \equiv -\int_0^{R_S} \frac{dp_{\rm rad}}{dr} dr = \alpha_B n_{\rm H}^2 \frac{h\nu}{c} R_S = \alpha_B n_{\rm H}^2 \frac{h\nu}{c} \left(\frac{3Q_0}{4\pi\alpha_B n_{\rm H}^2}\right)^{1/3}$$

$$= \left(\frac{3}{4\pi}\right)^{1/3} \alpha_B^{2/3} n_{\rm H}^{4/3} Q_0^{1/3} \frac{h\nu}{c} \quad .$$

(b)

$$\frac{\Delta p_{\rm rad}}{2n_{\rm H}k_{\rm B}T} = \frac{1}{2}\left(\frac{3}{4\pi}\right)^{1/3} \alpha_B^{2/3} n_{\rm H}^{1/3} Q_0^{1/3} \frac{h\nu}{ck_{\rm B}T} \quad .$$

(c) Let $T_4 \equiv T/10^4\,{\rm K}$, $n_3 \equiv n_{\rm H}/10^3\,{\rm cm}^{-3}$, $Q_{0,48} \equiv Q_0/10^{48}\,{\rm s}^{-1}$. *We take* $\alpha_B = 2.54\times$

$10^{-13}T_4^{-0.816}\,\mathrm{cm^3\,s^{-1}}$. *Then*

$$
\begin{aligned}
\frac{\Delta p_{\rm rad}}{2n_{\rm H}k_{\rm B}T} &= \frac{1}{2}\left(\frac{3}{4\pi}\right)^{1/3}(2.54\times10^{-13}T_4^{-0.816})^{2/3}(10^3 n_3)^{1/3}\times \\
&\qquad \left(10^{48}Q_{0,48}\right)^{1/3}\times\frac{18\times1.602\times10^{-12}}{3\times10^{10}\times1.38\times10^{-12}}\times\frac{h\nu/18\,\mathrm{eV}}{T_4} \\
&= 0.0866\,Q_{0,48}^{1/3}\,n_3^{1/3}\,T_4^{-1.544}\left(\frac{h\nu}{18\,\mathrm{eV}}\right) \\
&= 0.0866\times(6.5)^{1/3}\times(3.2)^{1/3} \\
&= 0.238 \quad .
\end{aligned}
$$

(d) Radiation is trying to push the gas away from the star. Therefore there needs to be an additional pressure on the outside pushing inward: the gas pressure at the edge will be **greater** *than the gas pressure at the center.*

For Orion-like conditions, the gas pressure at the edge will be approximately twice the gas pressure at the center.

Note that we have neglected the pressure produced by resonantly-trapped Lyman alpha radiation, which will **add** *to* $\Delta p_{\rm rad}$.

15.7 Consider an H II region with uniform electron density n_e, powered by a star emitting ionizing photons at a rate Q_0. Neglect helium and neglect dust. Let α_B be the case B recombination rate coefficient, and let f_{2s} be the fraction of case B recombinations that populate the $2s$ level.

Suppose that the only processes depopulating the $n = 2$ levels are spontaneous decays and collisions with electrons. Let A_{2s} be the rate for spontaneous decay of the $2s$ level, and let $n_e q_{2s\to 2p}$ be the rate for $2s \to 2p$ collisional transitions.

(a) Obtain an expression for $N(\mathrm{H}\,2s)$, the column density from center to edge of H in the $2s$ level, as a function of Q_0, n_e, α_B, f_{2s}, A_{2s}, and $q_{2s\to 2p}$.

(b) Evaluate $N(\mathrm{H}\,2s)$ for $f_{2s} = 0.325$, $\alpha_B = 2.59\times 10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$, $A_{2s} = 8.21\,\mathrm{s}^{-1}$, $q_{2s\to 2p} = 5.31\times 10^{-4}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$, $Q_0 = 10^{48}\,\mathrm{s}^{-1}$, and $n_e = 10^4\,\mathrm{cm}^{-3}$.

Solution:

(a) *formation of H 2s = removal of H 2s*

$$\alpha_B n_e^2 f_{2s} = n(\mathrm{H}\,2s)\,(A_{2s} + n_e q_{2s\to 2p})$$

$$n(\mathrm{H}\,2s) = \frac{\alpha_B n_e^2 f_{2s}}{A_{2s} + n_e q_{2s\to 2p}}$$

$$N(\mathrm{H}\,2s) \equiv \int_0^R n(\mathrm{H}\,2s)dr = \frac{f_{2s}\alpha_B n_e^2 R}{A_{2s} + n_e q_{2s\to 2p}} \quad , \quad R = \left(\frac{3Q_0}{4\pi\alpha_B n_e^2}\right)^{1/3}$$

$$N(\mathrm{H}\,2s) = f_{2s}\left(\frac{3Q_0}{4\pi}\right)^{1/3}\frac{(\alpha_B n_e^2)^{2/3}}{A_{2s} + n_e q_{2s\to 2p}}$$

(b)

$$\begin{aligned} N(\mathrm{H}\,2s) &= f_{2s}\left(\frac{3Q_0}{4\pi}\right)^{1/3}\frac{(\alpha_B n_e^2)^{2/3}}{A_{1s} + n_e q_{2s\to 2p}} \\ &= 0.325\left(\frac{3\times 10^{48}\,\mathrm{s}^{-1}}{4\pi}\right)^{1/3}\frac{(2.59\times 10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}\times 10^8\,\mathrm{cm}^{-6})^{2/3}}{(8.21\,\mathrm{s}^{-1} + 10^4\,\mathrm{cm}^{-3}\times 5.31\times 10^{-4}\,\mathrm{cm}^3\,\mathrm{s}^{-1})} \\ &= 1.29\times 10^{11}\,\mathrm{cm}^{-2} \end{aligned}$$

15.9 An O7V star radiates $h\nu > 13.6\,\mathrm{eV}$ photons at a rate $Q_0 = 10^{48.75}\,\mathrm{s}^{-1}$. The star is surrounded by pure hydrogen gas (no He, no dust) with uniform H nucleon density $n_{\mathrm{H}} = 10^2\,\mathrm{cm}^{-3}$. Assume that the star has been shining long enough to achieve "steady-state" conditions. Assume that the ionized gas has temperature $T \approx 10^4\,\mathrm{K}$, and the case B recombination coefficient $\alpha_B = 2.54\times10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$.

(a) Estimate the radius R of the volume around the star where the ionized fraction will be close to 1.

(b) 45% of case B recombinations generate an Hα photon. What will be the Hα luminosity of the ionized gas?

(c) Estimate $n(\mathrm{H}^0)/n_{\mathrm{H}}$ at a distance $r = 0.8R$ from the star. Assume the ionizing radiation at this location to have a typical photon energy $h\nu \approx 15\,\mathrm{eV}$ and a typical photoabsorption cross section $\sigma_{\mathrm{pe}} \approx 5\times10^{-18}\,\mathrm{cm}^2$.

Solution:

(a) The ionized zone will have a radius R obtained by balancing photoionizations and recombinations:

$$
\begin{aligned}
Q_0 &= \frac{4\pi}{3}R^3 {n_{\mathrm{H}}}^2\alpha_B \\
R &= \left(\frac{3Q_0}{4\pi {n_{\mathrm{H}}}^2\alpha_B}\right)^{1/3} \\
&= \left(\frac{3\times10^{48.75}\,\mathrm{s}^{-1}}{4\pi\times10^4\,\mathrm{cm}^{-6}\times2.54\times10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}}\right)^{1/3} \\
&= 8.09\times10^{18}\,\mathrm{cm} \;=\; 2.62\,\mathrm{pc} \quad .
\end{aligned}
$$

(b) The Hα production rate will be $0.45Q_0$.

$$
\begin{aligned}
(h\nu)_{\mathrm{H}\alpha} &= I_{\mathrm{H}}\times\left(\frac{1}{2^2}-\frac{1}{3^2}\right) = \frac{5}{36}I_{\mathrm{H}} \\
L(\mathrm{H}\alpha) &= 0.45\times Q_0\times(h\nu)_{\mathrm{H}\alpha} \\
&= 0.45\times10^{48.75}\,\mathrm{s}^{-1}\times\frac{5}{36}\times13.6\times1.60\times10^{-12}\,\mathrm{erg} \\
&= 7.65\times10^{36}\,\mathrm{erg\,s}^{-1} = 2000\,L_\odot \quad .
\end{aligned}
$$

(c) A fraction $(r/R)^3$ of the ionizing photons radiated by the star are absorbed interior to radius r. Therefore, the rate of ionizing photons crossing the surface at $r = 0.8R = 6.47\times10^{18}\,\mathrm{cm}$ will be

$$
Q(r) = Q_0\times\left[1-(r/R)^3\right] = Q_0\times\left[1-0.8^3\right] = Q_0\times(1-0.51) = 0.49Q_0 \quad .
$$

Ionization-recombination balance requires

$$
\begin{aligned}
\frac{Q(r)}{4\pi r^2}\sigma_{\rm pe}(1-x)n_{\rm H} &= \alpha_{\rm B}(n_{\rm H}x)^2 \\
\frac{1-x}{x^2} &= \frac{4\pi r^2 \alpha_{\rm B} n_{\rm H}}{Q(r)\sigma_{\rm pe}} \\
&= \frac{4\pi \times (6.47\times 10^{18}\,{\rm cm})^2 \times 2.54\times 10^{-13}\,{\rm cm}^3\,{\rm s}^{-1} \times 10^2\,{\rm cm}^{-3}}{0.49\times 10^{48.75}\,{\rm s}^{-1} \times 5\times 10^{-18}\,{\rm cm}^2} \\
&= 9.69\times 10^{-4} \\
\frac{n({\rm H}^0)}{n_{\rm H}} = (1-x) &\approx 9.69\times 10^{-4} \quad .
\end{aligned}
$$

Chapter 16. Ionization in Predominantly Neutral Regions

16.1 The diffuse molecular cloud toward the bright star ζ Persei has $N(\mathrm{H}_3^+) = 8\times10^{13}\,\mathrm{cm}^{-2}$ and $N(\mathrm{H}_2) = 5\times10^{20}\,\mathrm{cm}^{-2}$. Estimate the abundance $n(\mathrm{OH}^+)/n_\mathrm{H}$ in the molecular region of this cloud if the gas-phase abundance $n(\mathrm{O})/n_\mathrm{H} \approx 4\times10^{-4}$.

Assume $n_\mathrm{H} \approx 10^2\,\mathrm{cm}^{-3}$ and $T \approx 60\,\mathrm{K}$. Assume that most of the hydrogen is molecular, so that $n(\mathrm{H}_2) \approx n_\mathrm{H}/2$. The rate coefficient for $\mathrm{OH}^+ + e^-$ is given in Table 14.8. Assume that the free electrons come primarily from photoionization of "metals" with $n(M+)/n_\mathrm{H} = x_M \approx 1.07\times10^{-4}$ (as per Eq. 16.3).

Solution: Because this is a diffuse cloud, we will assume that C, S, *etc. are photoionized by starlight, so that* $n_e \approx x_M n_\mathrm{H}$, *with* $x_M \approx 1.07\times10^{-4}$ *(see Eq. 16.3). Then, assuming the only important destruction channel for* OH^+ *is* $\mathrm{OH}^+ + e^- \rightarrow \mathrm{O} + \mathrm{H}$, *we balance formation and destruction of* OH^+*:*

$$
\begin{aligned}
k_a n(\mathrm{H}_3^+) n(\mathrm{O}) &= k_b n(\mathrm{OH}^+) n_e \\
k_a &= 8.4\times10^{-10}\,\mathrm{cm}^3\,\mathrm{s}^{-1} \quad \text{(from Eq. 16.11)} \\
k_b &= 6.50\times10^{-8} T_2^{-0.50}\,\mathrm{cm}^3\,\mathrm{s}^{-1} \quad \text{(from Table 14.8)} \\
&= 8.39\times10^{-8}\,\mathrm{cm}^3\,\mathrm{s}^{-1} \\
\frac{n(\mathrm{OH}^+)}{n_\mathrm{H}} &= \frac{k_a n(\mathrm{H}_3^+) n(\mathrm{O})}{k_b n_e n_\mathrm{H}} \\
&= \frac{n(\mathrm{H}_3^+)}{n(\mathrm{H}_2)} \frac{n(\mathrm{H}_2)}{n_\mathrm{H}} \frac{k_a n(\mathrm{O})/n_\mathrm{H}}{k_b x_M} \\
&= \frac{8\times10^{13}}{5\times10^{20}} \frac{n(\mathrm{H}_2)}{n_\mathrm{H}} \frac{8.4\times10^{-10} \times 4\times10^{-4}}{8.39\times10^{-8} \times 1.07\times10^{-4}} \\
&= 2.99\times10^{-9} \times \left(\frac{2n(\mathrm{H}_2)}{n_\mathrm{H}}\right) \quad .
\end{aligned}
$$

We don't actually know the H_2 *fraction in the region where the* H_3^+ *is present, but it is probably of order unity. Hence,* $n(\mathrm{OH}^+)/n_\mathrm{H} \approx 3\times10^{-9}$.

16.3 The ion H_3^+ can react with electrons ($\mathrm{H}_3^+ + e^- \rightarrow \mathrm{H}_2 + \mathrm{H}$ and $\mathrm{H}_3^+ + e^- \rightarrow 3\mathrm{H}$) or neutral atoms or molecules M ($\mathrm{H}_3^+ + M \rightarrow M\mathrm{H}^+ + \mathrm{H}_2$). If the eligible species M (e.g., O, C, S) have abundance $n(M)/n_{\mathrm{H}} = 3\times10^{-4}$, what is the fractional ionization x_e below which the destruction of H_3^+ is dominated by

$$\mathrm{H}_3^+ + M \rightarrow M\mathrm{H}^+ + \mathrm{H}_2 \ \ ?$$

Assume that this reaction proceeds with a typical ion-neutral rate coefficient $k \approx 2\times10^{-9}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$, and that the gas temperature $T = 30\,\mathrm{K}$. The rate coefficient for $\mathrm{H}_3^+ + e^- \rightarrow \mathrm{H}_2 + \mathrm{H}$ is $5.0 \times 10^{-8} T_2^{-0.48}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ and the rate coefficient for $\mathrm{H}_3^+ + e^- \rightarrow 3\mathrm{H}$ is $8.9 \times 10^{-8} T_2^{-0.48}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$.

Solution: The condition for destruction by electrons and destruction by neutral atoms or molecules being equally important is

$$\begin{aligned}
k_a n_e n(\mathrm{H}_3^+) &= k_b n(M) n(\mathrm{H}_3^+) \quad , \\
k_a &= 1.39\times10^{-7} T_2^{-0.48}\,\mathrm{cm}^3\,\mathrm{s}^{-1} = 2.48\times10^{-7}\,\mathrm{cm}^3\,\mathrm{s}^{-1} \quad , \\
k_b &= 2\times10^{-9}\,\mathrm{cm}^3\,\mathrm{s}^{-1} \quad , \\
x_e = \frac{n_e}{n_{\mathrm{H}}} &= \frac{k_b}{k_a} \times \frac{n(M)}{n_{\mathrm{H}}} \\
&= \frac{2\times10^{-9}}{2.48\times10^{-7}} \times 3\times10^{-4} \\
&= 2.42\times10^{-6} \quad .
\end{aligned}$$

16.5 Consider a region containing only partially-ionized hydrogen. Let ζ be the ionization rate per H atom, and let α be the recombination coefficient.

(a) Determine the steady-state ionization fraction x_{ss} in terms of $n_{\rm H} \equiv n({\rm H}^0) + n({\rm H}^+)$, ζ, and α. Express your answer in terms of the dimensionless parameter $\beta \equiv \zeta/(\alpha n_{\rm H})$

(b) What is the asymptotic behavior of x_{ss} for $\beta \ll 1$? Show the leading dependence on β.

(c) What is the asymptotic behavior of x_{ss} for $\beta \gg 1$? Show the leading dependence on β.

(d) Suppose that the fractional ionization at time $t = 0$ is given by $x(0) = x_{ss} + \delta$. If $|\delta| \ll x_{ss}$, determine the solution $x(t > 0)$, assuming $n_{\rm H}$, ζ, and α to be constant. (Hint: linearize around the steady state.)

Solution:

(a)

$$\begin{aligned}
\frac{d}{dt}(n_{\rm H}x) &= \zeta n_{\rm H}(1-x) - \alpha {n_{\rm H}}^2 x^2 \\
0 &= \zeta n_{\rm H}(1-x_{ss}) - \alpha {n_{\rm H}}^2 x_{ss}^2 \\
0 &= x_{ss}^2 + \frac{\zeta}{\alpha n_{\rm H}} x_{ss} - \frac{\zeta}{\alpha n_{\rm H}} \\
0 &= x_{ss}^2 + \beta x_{ss} - \beta \qquad\qquad \beta \equiv \frac{\zeta}{\alpha n_{\rm H}} \\
x_{ss} &= \frac{-\beta + (\beta^2 + 4\beta)^{1/2}}{2} = \frac{-\zeta + (\zeta^2 + 4 n_{\rm H}\alpha\zeta)^{1/2}}{2\alpha n_{\rm H}}
\end{aligned}$$

(b)

$$\begin{aligned}
x_{ss} &= \frac{-\beta + (\beta^2 + 4\beta)^{1/2}}{2} \\
&\to \beta^{1/2} \quad \text{for } \beta \ll 1
\end{aligned}$$

(c)

$$\begin{aligned}
x_{ss} &= \frac{-\beta + (\beta^2 + 4\beta)^{1/2}}{2} \\
&= \frac{-\beta + \beta\left(1 + \frac{4}{\beta}\right)^{1/2}}{2} \\
&\approx \frac{-\beta + \beta\left(1 + \frac{1/2}{1!}\left(\frac{4}{\beta}\right) + \frac{(1/2)(-1/2)}{2!}\left(\frac{4}{\beta}\right)^2 + \ldots\right)}{2} \\
&= 1 - \frac{1}{\beta} \quad \text{for } \beta \gg 1 \quad .
\end{aligned}$$

(d) Define $\delta(t) \equiv x(t) - x_{ss}$

$$\begin{aligned}
n_{\mathrm{H}}\dot{x} &= \zeta n_{\mathrm{H}} - \zeta n_{\mathrm{H}} x - \alpha {n_{\mathrm{H}}}^2 x^2 \\
n_{\mathrm{H}}\dot{\delta} &= \zeta n_{\mathrm{H}} - \zeta n_{\mathrm{H}}(x_{ss}+\delta) - \alpha {n_{\mathrm{H}}}^2 (x_{ss}^2 + 2x_{ss}\delta + \delta^2) \\
n_{\mathrm{H}}\dot{\delta} &= -\zeta n_{\mathrm{H}}\delta - 2\alpha {n_{\mathrm{H}}}^2 x_{ss}\delta + O(\delta^2) \\
\dot{\delta} &\approx -(\zeta + 2\alpha n_{\mathrm{H}} x_{ss})\delta \ = \ -\delta/\tau \qquad \text{where} \\
\tau &\equiv \frac{1}{\zeta + 2\alpha n_{\mathrm{H}} x_{ss}} = \frac{1}{\alpha n_{\mathrm{H}}(\beta + 2x_{ss})} = \frac{1}{(\zeta^2 + 4\alpha n_{\mathrm{H}}\zeta)^{1/2}} = \frac{1}{\zeta\sqrt{1+4/\beta}}
\end{aligned}$$

For $|\delta(t\!=\!0)| \ll 1$*, the general solution to this differential equation is*

$$\begin{aligned}
\delta(t) &\approx \delta(t\!=\!0)\, \mathrm{e}^{-t/\tau} \\
x(t) &\approx x_{ss} + [x(t\!=\!0) - x_{ss}]\, \mathrm{e}^{-t/\tau} \quad .
\end{aligned}$$

Chapter 17. Collisional Excitation

17.1 Consider the H I spin temperature in a region where the brightness temperature of the background radiation field at $\lambda \approx 21\,\mathrm{cm}$ is $T_{\rm rad}$, and the gas temperature is $T_{\rm gas}$. Suppose that $h\nu/kT_{\rm rad} \ll 1$, and $T_{\rm gas} > T_{\rm rad}$.

Consider only the two hyperfine levels of H I (i.e., ignore the effects of Lyman alpha excitation of the $2p$ levels).

Let the H nucleon density of the gas be $n_{\rm H}$, and define the "critical density" according to Eq. (17.7),

$$n_{\rm crit} \equiv (1+n_\gamma)\frac{A_{10}}{k_{10}} \quad .$$

(a) Using the approximation $h\nu/kT_{\rm rad} \ll 1$, obtain an expression for the spin temperature $T_{\rm spin}$ in terms of $T_{\rm rad}$, $T_{\rm gas}$, and the ratio $R \equiv n_{\rm H}/n_{\rm crit}$.

(b) Find the minimum value of R such that $T_{\rm spin} > (T_{\rm rad}+T_{\rm gas})/2$.
You may assume that $T_{\rm gas} \gg h\nu/k,\ \ T_{\rm rad} \gg h\nu/k,$ and $|T_{\rm gas}-T_{\rm rad}| \gg h\nu/k.$

Solution:

(a)

$$n_0\left(C_{01}+n_\gamma\frac{g_1}{g_0}A_{10}\right) = n_1\left(C_{10}+(1+n_\gamma)A_{10}\right)$$

$$R \equiv \frac{C_{10}}{(1+n_\gamma)A_{10}} = \frac{n_{\rm H}}{n_{\rm crit}} \qquad n_{\rm crit} \equiv \frac{(1+n_\gamma)A_{10}}{k_{10}}$$

$$C_{01} = \frac{g_1}{g_0}C_{10}\,\mathrm{e}^{-\theta/T_{\rm gas}} \qquad \theta \equiv h\nu/k$$

$$\begin{aligned}
\mathrm{e}^{-\theta/T_{\rm spin}} = \frac{g_0}{g_1}\frac{n_1}{n_0} &= \frac{C_{10}\,\mathrm{e}^{-\theta/T_{\rm gas}}+n_\gamma A_{10}}{C_{10}+(1+n_\gamma)A_{10}} \\
&= \frac{R(1+n_\gamma)\,\mathrm{e}^{-\theta/T_{\rm gas}}+n_\gamma}{(1+R)(1+n_\gamma)} \\
1-\frac{\theta}{T_{\rm spin}} &\approx \frac{R(1+n_\gamma)-(\theta/T_{\rm gas})R(1+n_\gamma)+n_\gamma}{(1+R)(1+n_\gamma)} \\
\frac{\theta}{T_{\rm spin}} &= \frac{1+(\theta/T_{\rm gas})R(1+n_\gamma)}{(1+R)(1+n_\gamma)} \\
T_{\rm spin} &\approx \frac{(1+R)(\theta+T_{\rm rad})T_{\rm gas}}{T_{\rm gas}+R(\theta+T_{\rm rad})} \qquad \text{using } n_\gamma = \frac{1}{e^{\theta/T_{\rm rad}}-1} \approx \frac{T_{\rm rad}}{\theta}-\frac{1}{2} \\
&= \frac{(1+R)T_{\rm gas}(T_{\rm rad}+\frac{1}{2}\theta)}{T_{\rm gas}+R(T_{\rm rad}+\frac{1}{2}\theta)} \approx \frac{(1+R)T_{\rm rad}T_{\rm gas}}{T_{\rm gas}+RT_{\rm rad}} \quad .
\end{aligned}$$

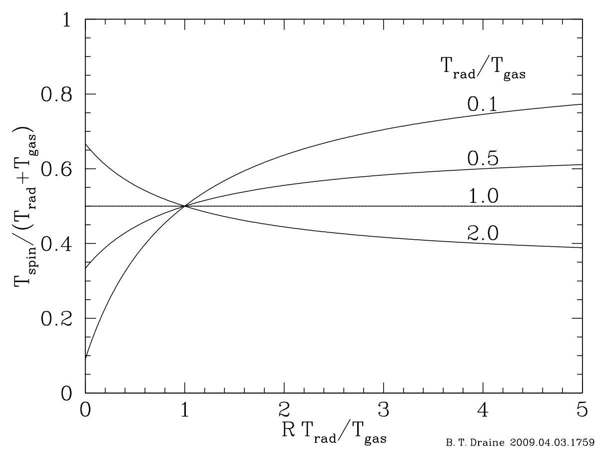
(b) Let $R_{\rm crit}$ be the value of R such that $T_{\rm spin} = (T_{\rm gas} + T_{\rm rad})/2$. Using the result from (a):

$$T_{\rm spin} = \frac{(1+R_{\rm crit})T_{\rm rad}T_{\rm gas}}{T_{\rm gas}+R_{\rm crit}T_{\rm rad}} = \frac{(T_{\rm rad}+T_{\rm gas})}{2}$$

$$R_{\rm crit} = \frac{T_{\rm gas}}{T_{\rm rad}} \quad .$$

If $T_{\rm gas}/T_{\rm rad} < 1$, then adding collisions <u>decreases</u> $T_{\rm spin}$, and we need $R < R_{\rm crit}$ to have $T_{\rm spin} > (T_{\rm gas} + T_{\rm rad})/2$.

If $T_{\rm gas}/T_{\rm rad} > 1$, then adding collisions <u>increases</u> $T_{\rm spin}$, and we need $R > R_{\rm crit}$ to have $T_{\rm spin} > (T_{\rm gas} + T_{\rm rad})/2$.



The figure shows $T_{\rm spin}/(T_{\rm rad} + T_{\rm gas})$ as a function of $RT_{\rm rad}/T_{\rm gas}$, where $R \equiv n_{\rm H}/n_{\rm crit}$.

17.3 When the proton spins in H_2 are **anti**parallel, we have "para"-H_2, which can have rotational angular momentum $J = 0, 2, 4, ...$. When the proton spins are parallel, we have "ortho"-H_2, for which only odd values of J are possible. Radiative transitions between ortho-H_2 and para-H_2 are strongly forbidden; para→ortho or ortho→para transitions occur only because of collisions.

Because the nuclear spins are only weakly-coupled to collision partners such as H atoms, the rate coefficients for ortho→para or para→ortho conversion are small.

Let $H_2(v, J)$ denote H_2 with vibrational and rotational quantum numbers (v, J). The rate coefficient for

$$\mathrm{H}_2(0,1) + \mathrm{H}_2(0,0) \rightarrow \mathrm{H}_2(0,0) + \mathrm{H}_2(0,0)$$

is estimated to be only $k_{10} = 1.56 \times 10^{-28}\,\mathrm{cm^3\,s^{-1}}$ (Huestis 2008: Plan. Sp. Sci. 56, 1733). The energy difference between $\mathrm{H}_2(0,1)$ and $\mathrm{H}_2(0,0)$ is $\Delta E/k = 170.5\,\mathrm{K}$.

(a) Use detailed balance to obtain the rate coefficient k_{01} for

$$\mathrm{H}_2(0,0) + \mathrm{H}_2(0,0) \rightarrow \mathrm{H}_2(0,1) + \mathrm{H}_2(0,0) \quad .$$

(b) In a molecular cloud with $n(\mathrm{H}_2) = 100\,\mathrm{cm^{-3}}$ and $T = 50\,\mathrm{K}$, what is the steady-state ratio of $n(J=1)/n(J=0)$ if only collisions with H_2 are acting?

(c) If the ortho-para ratio at $t = 0$ differs from the LTE value, small deviations from LTE abundances will decay exponentially on a time scale τ. Evaluate τ for $n(\mathrm{H}_2) = 10^6\,\mathrm{cm^{-3}}$ and $T = 50\,\mathrm{K}$, assuming that the only processes causing ortho-para conversion are

$$\mathrm{H}_2(0,0) + \mathrm{H}_2(0,0) \rightarrow \mathrm{H}_2(0,1) + \mathrm{H}_2(0,0)$$
$$\mathrm{H}_2(0,1) + \mathrm{H}_2(0,0) \rightarrow \mathrm{H}_2(0,0) + \mathrm{H}_2(0,0) \quad .$$

Solution:

(a) The level degeneracies are $g(J) = (2J+1)g_n(J)$, where $g_n(J) = 1$ for even J, and $g_n = 3$ for odd J. Thus $g(J=1) = 3 \times 3 = 9$, and

$$k_{01} = 9 \times 1.56 \times 10^{-28}\,e^{-170.5\,\mathrm{K}/T}\,\mathrm{cm^3\,s^{-1}} = 1.40 \times 10^{-27}\,e^{-170.5\,\mathrm{K}/T}\,\mathrm{cm^3\,s^{-1}} \quad .$$

(b) The steady-state solution will be LTE:

$$n(J=1)/n(J=0) = 9\,e^{-170.5/T} = 0.297 \quad .$$

(c) Let A and B denote the $J=0$ and $J=1$ states, and consider only the processes

$$\begin{array}{lcll} A + A & \rightarrow & A + B & \text{with rate coeff. } k_{01} \\ A + B & \rightarrow & A + A & \text{with rate coeff. } k_{10} \quad . \end{array}$$

Thus
$$\frac{dn_A}{dt} = -k_{01}n_A^2 + k_{10}n_A n_B \quad ,$$
$$\frac{dn_B}{dt} = k_{01}n_A^2 - k_{10}n_A n_B \quad .$$

Let $n \equiv n_A + n_B$. *It is easy to show that the steady-state solution is*

$$n_A^{(ss)} = \frac{k_{01}}{k_{01}+k_{10}} n \qquad , \qquad n_B^{(ss)} = \frac{k_{10}}{k_{01}+k_{10}} n \quad .$$

Define a dimensionless deviation $\delta \equiv (n_B - n_B^{(ss)})/n$. *Then* $n_B = n_B^{(ss)} + n\delta$, $n_A = n_A^{(ss)} - n\delta$. *Substitution into the equation for* dn_B/dt *yields*

$$\frac{d\delta}{dt} = -nk_{10}\delta + n(k_{01}+k_{10})\delta^2 \quad .$$

Small deviations $|\delta| \ll 1$ *decay exponentially, on a timescale*

$$\tau = (nk_{10})^{-1} = (10^6\,\mathrm{cm}^{-3} \times 1.56 \times 10^{-28}\,\mathrm{cm}^3\,\mathrm{s}^{-1})^{-1} = 6.41 \times 10^{21}\,\mathrm{s} = 2.0 \times 10^{14}\,\mathrm{yr} \quad .$$

This timescale is far too long to be of interest in the ISM! $\mathrm{H_2} - \mathrm{H_2}$ *collisions have a negligible effect on the ortho/para ratio in cold interstellar clouds. The ortho/para ratio is controlled by processes other than* $\mathrm{H_2} - \mathrm{H_2}$ *collisions.*

Chapter 18. Nebular Diagnostics

18.1 Derive the value of the constant C in the equation

$$\frac{n(\mathrm{O\,III})}{n(\mathrm{H}^+)} = C \times \frac{I([\mathrm{O\,III}]5008)}{I(\mathrm{H}\beta)} T_4^{-0.494-0.089\ln T_4}\, \mathrm{e}^{2.917/T_4} \quad .$$

in the low density limit. For what densities is your result valid?

Solution: Number the levels starting with 0 for the ground state (see Figure 18.1). For O III *we have*

$$\begin{aligned}
\Omega_{03} &= 0.243\, T_4^{0.120+0.031\ln T_4} && \text{from Table F.2, p. 497} \\
A_{31} &= 6.21 \times 10^{-3}\,\mathrm{s}^{-1} && \text{from https://physics.nist.gov/PhysRefData/ASD} \\
A_{32} &= 1.81 \times 10^{-2}\,\mathrm{s}^{-1} && \text{from https://physics.nist.gov/PhysRefData/ASD}
\end{aligned}$$

From Table 18.1, the critical densities for populating the excited fine structure levels of O III *are* $n_{\mathrm{crit}}(\mathrm{O\,III}\ {}^3\mathrm{P}_1) \approx 1700\,\mathrm{cm}^{-3}$ *and* $n_{\mathrm{crit}}(\mathrm{O\,III}\ {}^3\mathrm{P}_2) \approx 3800\,\mathrm{cm}^{-3}$. *The critical density for the* ${}^1\mathrm{D}_2$ *state is much higher,*
$n_{\mathrm{crit}}(\mathrm{O\,III}\ {}^1\mathrm{D}_2) \approx 6.4 \times 10^5\,\mathrm{cm}^{-3}$.

Therefore, if we are at densities $n_e \lesssim 10^3\,\mathrm{cm}^{-3}$, *we can assume that nearly all of the* O III *is in the ground state, and that all of the excitations to* ${}^1\mathrm{D}_2$ *decay radiatively. Then*

$$\begin{aligned}
n_0 n_e k_{03} &= n_3(A_{31} + A_{32}) \quad \rightarrow \quad n_3 = \frac{n_e k_{03}}{A_{31} + A_{32}} n_0 \\
k_{03} &= 8.629 \times 10^{-8} \frac{\Omega_{03}}{g_0} T_4^{-0.5}\, \mathrm{e}^{-E_{03}/k_{\mathrm{B}}T}\,\mathrm{cm}^3\,\mathrm{s}^{-1} \\
&= 8.629 \times 10^{-8} \times \frac{0.243}{1} T_4^{-0.380+0.031\ln T_4}\, \mathrm{e}^{-E_{03}/k_{\mathrm{B}}T}\,\mathrm{cm}^3\,\mathrm{s}^{-1} \quad . \\
4\pi j(5008.2\,\text{Å}) &= n_3 A_{32} E_{32} = n_e k_{03} \frac{A_{32}}{(A_{31} + A_{32})} E_{32}\, n(\mathrm{O\,III}) \quad .
\end{aligned}$$

Hβ *is recombination radiation. The effective* Hβ *production coefficient is*

$\alpha_{\mathrm{eff,H}\beta} \approx 3.03\times10^{-14}T_4^{-0.874-0.058\ln T_4}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ *[from Eq. (14.9)].*

$$
\begin{aligned}
4\pi j(\mathrm{H}\beta) &= n_e\alpha_{\mathrm{eff,H}\beta}E(\mathrm{H}\beta)n(\mathrm{H}^+) \\
\frac{n(\mathrm{O\,III})}{n(\mathrm{H}^+)} &= \frac{\alpha_{\mathrm{eff,H}\beta}}{k_{03}}\frac{(A_{31}+A_{32})}{A_{32}}\frac{E(\mathrm{H}\beta)}{E_{32}}\left[\frac{j(5008.2\,\text{Å})}{j(\mathrm{H}\beta)}\right] \\
&= \frac{3.03\times10^{-14}T_4^{-0.874-0.058\ln T_4}}{8.629\times10^{-8}\times0.243T_4^{-0.380+0.031\ln T_4}\,\mathrm{e}^{-2.917/T_4}}\times \\
&\quad \frac{(6.21\times10^{-3}+1.81\times10^{-2})}{1.81\times10^{-2}}\frac{5008.2}{4862.7}\left[\frac{j(5008.2\,\text{Å})}{j(\mathrm{H}\beta)}\right] \\
&= 2.00\times10^{-6}T_4^{-0.494-0.089\ln T_4}\,\mathrm{e}^{2.917/T_4}\left[\frac{j(5008.2\,\text{Å})}{j(\mathrm{H}\beta)}\right] \\
C &= 2.00\times10^{-6}\quad .
\end{aligned}
$$

Chapter 19. Radiative Trapping

19.1 By approximating the sum by an integral, evaluate the partition function for a rigid rotor,

$$Z_{\rm rot} = \sum_{J=0}^{\infty}(2J+1)e^{-B_0 J(J+1)/k_{\rm B}T_{\rm exc}} \quad ,$$

in the high temperature limit $k_{\rm B}T_{\rm exc}/B_0 \gg 1$.

Solution:

$$\begin{aligned}
Z_{\rm rot} &\equiv \sum_{J=0}^{\infty}(2J+1)\,{\rm e}^{-B_0 J(J+1)/k_{\rm B}T_{\rm exc}} \\
&\approx \int_{-1/2}^{\infty} dJ\;(2J+1)\,{\rm e}^{-B_0 J(J+1)/k_{\rm B}T_{\rm exc}} \\
&= \int_0^{\infty} dx\;2x\;{\rm e}^{-B_0(x^2-1/4)/k_{\rm B}T_{\rm exc}} \qquad x \equiv J+1/2 \\
&= {\rm e}^{B_0/4k_{\rm B}T_{\rm exc}} \int_0^{\infty} du\;{\rm e}^{-B_0 u/k_{\rm B}T_{\rm exc}} \qquad u \equiv x^2 \\
&= {\rm e}^{B_0/4k_{\rm B}T_{\rm exc}}\frac{k_{\rm B}T_{\rm exc}}{B_0} \approx \left(1+\frac{B_0}{4k_{\rm B}T_{\rm exc}}\right)\left(\frac{k_{\rm B}T_{\rm exc}}{B_0}\right) = \frac{k_{\rm B}T_{\rm exc}}{B_0}+\frac{1}{4} \\
&\to \frac{k_{\rm B}T_{\rm exc}}{B_0} \qquad \text{for}\;\; k_{\rm B}T_{\rm exc} \gg B_0 \quad .
\end{aligned}$$

19.3 Recall that $X_{\rm CO} \equiv N({\rm H_2})/\int T_A dv$ gives the relation between $N({\rm H_2})$ and the "antenna temperature" T_A integrated over radial velocity v of the CO 1–0 line in a resolved source.

(a) Suppose that we observe CO 1–0 line emission from an unresolved galaxy at distance D, with an integrated flux in the 1-0 line $W_{\rm CO} \equiv \int F_\nu dv$, where F_ν is the flux density, v is radial velocity, and the integral extends over the full range of radial velocities in the galaxy.

Derive an expression giving the mass $M({\rm H_2})$ of $\rm H_2$ in terms of $W_{\rm CO}$, $X_{\rm CO}$, λ, and D (and fundamental constants).

(b) NGC 7331, at a distance $D = 14.7\,{\rm Mpc}$, has $W_{\rm CO} = 4090\,{\rm Jy\,km\,s^{-1}}$. Calculate $M({\rm H_2})$. Assume that $X_{\rm CO} = 4\times10^{20}\,{\rm cm^{-2}}(\,{\rm K\,km\,s^{-1}})^{-1}$.

Solution:

(a)

$$
\begin{aligned}
W_{\rm CO} &\equiv \int dv F_\nu = \int dv \int d\Omega\, I_\nu = \int dv \int d\Omega\, \frac{2k_{\rm B}\nu^2}{c^2}\, T_A \\
&= \int d\Omega \frac{2k_{\rm B}}{\lambda^2} \times \int dv\, T_A \\
&= \int d\Omega \frac{2k_{\rm B}}{\lambda^2} \times \frac{N({\rm H_2})}{X_{\rm CO}} \quad [\text{from the definition of } X_{\rm CO}] \\
&= \frac{2k_{\rm B}}{\lambda^2}\frac{1}{X_{\rm CO}} \int \frac{dA}{D^2} N({\rm H_2}) \qquad \left[\text{recall that } d\Omega = \frac{dA}{D^2}\right] \\
&= \frac{2k_{\rm B}}{\lambda^2}\frac{1}{X_{\rm CO}}\frac{1}{2m_{\rm H}D^2} \int dA\, \Sigma({\rm H_2}) \qquad \left[\Sigma({\rm H_2}) \equiv 2m_{\rm H}N({\rm H_2})\right] \\
&= \frac{k_{\rm B}}{m_{\rm H}\lambda^2}\frac{1}{X_{\rm CO}}\frac{M({\rm H_2})}{D^2}
\end{aligned}
$$

$$
M({\rm H_2}) = \frac{m_{\rm H}\lambda^2}{k_{\rm B}} X_{\rm CO}\, D^2\, W_{\rm CO} \quad .
$$

(b)

$$
\begin{aligned}
M({\rm H_2}) &= \frac{1.67\times10^{-24}\,{\rm g} \times (0.26\,{\rm cm})^2}{1.38\times10^{-16}\,{\rm erg\,K^{-1}}} \times \frac{4\times10^{20}\,{\rm cm^{-2}}}{{\rm K\,km\,s^{-1}}} \times (14.7\,{\rm Mpc})^2 \times \\
&\qquad 4090 \times 10^{-23}\,{\rm erg\,cm^{-2}\,s^{-1}\,Hz^{-1}\,km\,s^{-1}} \\
&= 2.75\times10^{43}\,{\rm g} = 1.38\times10^{10}\,M_\odot \quad .
\end{aligned}
$$

Chapter 20. Optical Pumping

20.1 Consider UV-pumping of the rotationally-excited states of para-H_2.
$H_2(v=0, J=2)$ can be pumped by UV in the 912–1108 Å wavelength range as follows:

1. Ground-electronic state $H_2(v = 0, J = 0)$ absorbs a photon (via a permitted electric-dipole transition) that excites it to a $J = 1$ state of one of the many vibrational levels of either the $B^1\Sigma_u^+$ or $C^1\Pi_u$ states (see Fig. 5.1).
2. This is followed by spontaneous emission of a UV photon in a transition back to the ground electronic state. A fraction $f_{\rm diss} \approx 0.15$ of these transitions are to the vibrational continuum of the ground electronic state, leading to immediate dissociation $H_2 \rightarrow H{+}H$.
3. A fraction $(1 - f_{\rm diss})$ of the UV decays are to one of the bound vibration-rotation levels (either $J = 0$ or $J = 2$) of the ground electronic state. A large fraction (close to 100%) of these UV decays are to excited vibrational states $v \geq 1$ (either $J = 0$ or $J = 2$), and are then followed by a "vibrational cascade" that returns the H_2 to the $v = 0$ level, typically after emission of several infrared photons.
4. Suppose that a fraction $\phi_{\rm para}$ of the vibrational cascades of para-H_2 end up in one of the rotationally-excited levels $J = 2, 4, 6, ...$ of the ground vibrational state $v = 0$. At low densities, the $J = 4, 6, ...$ rotationally-excited levels will decay by rotational quadrupole transitions ($J \rightarrow J-2$) down the rotational ladder, eventually populating the $v = 0, J = 2$ level.

Now consider a plane-parallel cloud, and suppose that each face of this cloud is illuminated by a radiation field, isotropic over 2π steradians, with $\lambda u_\lambda = 2{\times}10^{-14}\chi\,{\rm erg\,cm^{-3}}$, where χ is a dimensionless intensity scale factor. Suppose that a fraction f_{H_2} of the incident UV photons in the 1110–912 Å range are absorbed by H_2 (rather than by dust), and suppose that a fraction $h_{\rm para}$ of the H_2 absorptions are due to $H_2(v=0, J=0)$.

Suppose that an observer views the cloud with the line-of-sight making an angle θ with respect to the cloud normal.

If collisions can be neglected, and UV pumping is the only mechanism for populating the $J \geq 2$ levels of H_2, obtain a formula for the surface brightness of the cloud in the H_2 0–0S(0) line at 28.22 μm (your result should depend on χ, f_{H_2}, $f_{\rm diss}$, $h_{\rm para}$, $\phi_{\rm para}$, and the inclination angle θ).

Solution: The flux of 1110–912 Å photons incident on **one** *face of the cloud is*

$$F_0 = \int d\lambda \frac{u_\lambda}{hc/\lambda}\int_0^{\pi/2}\frac{2\pi\sin\theta d\theta}{2\pi}\times c\cos\theta \;=\; \frac{0.5}{h}\int d\lambda\;\lambda u_\lambda = \frac{1}{2h}\lambda u_\lambda\int d\lambda = \frac{\lambda u_\lambda}{2h}\Delta\lambda$$

$$= \frac{2{\times}10^{-14}\chi\,{\rm erg\,cm^{-3}}}{2{\times}6.63{\times}10^{-27}\,{\rm erg\,s}}\times(1110-912)\,\text{Å} = 3.13{\times}10^{6}\chi\,{\rm photons\,cm^{-2}\,s^{-1}} \quad .$$

A fraction $f_{H_2}h_{\rm para}(1-f_{\rm diss})\phi_{\rm para}$ *of these photons will "pump" an* H_2 *molecule into* $v = 0, J = 2$*. The pumping rate is balanced by spontaneous decay in the* H_2 $0-0S(0)$ *line. The cloud is optically thin at* $28.22\,\mu{\rm m}$*, and therefore we see emission from both the near and far* *sides of the cloud. The surface brightness is increased by a factor* $1/\cos\theta$ *due to inclination. Thus the intensity of the* $\lambda_{S(0)} = 28.22\,\mu\mathrm{m}$ *line is*

$$\begin{aligned}
I(28.22\,\mu\mathrm{m}) &= \frac{hc}{\lambda_{S(0)}} \frac{1}{4\pi\,\mathrm{sr}} \frac{2F_0}{\cos\theta} f_{\mathrm{H}_2} h_{\mathrm{para}} (1 - f_{\mathrm{diss}}) \phi_{\mathrm{para}} \\
&= \frac{hc}{\lambda_{S(0)}} \frac{1}{4\pi\,\mathrm{sr}} \frac{2}{\cos\theta} \frac{(\lambda u_\lambda)_{UV}}{2h} (\Delta\lambda)_{UV} f_{\mathrm{H}_2} h_{\mathrm{para}} (1 - f_{\mathrm{diss}}) \phi_{\mathrm{para}} \\
&= \frac{c}{\lambda_{S(0)}} \frac{(\lambda u_\lambda)_{UV}}{4\pi\cos\theta} (\Delta\lambda)_{UV} f_{\mathrm{H}_2} h_{\mathrm{para}} (1 - f_{\mathrm{diss}}) \phi_{\mathrm{para}} \\
&= \frac{3.36\times 10^{-8}\chi}{\cos\theta} f_{\mathrm{H}_2} h_{\mathrm{para}} (1 - f_{\mathrm{diss}}) \phi_{\mathrm{para}}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,sr^{-1}} \quad .
\end{aligned}$$

Chapter 21. Interstellar Dust: Observed Properties

21.1 Suppose that dust produced extinction $A(\lambda)$ directly proportional to the frequency of the light. What would be the value of $R_V \equiv A_V/E(B-V)$? Assume $\lambda_V = 0.55\,\mu\mathrm{m}$ and $\lambda_B = 0.44\,\mu\mathrm{m}$.

Solution: If $A(\lambda) \propto \lambda^{-1}$, *then*

$$R_V \equiv \frac{A_V}{A_B - A_V} = \frac{(1/0.55)}{(1/0.44) - (1/0.55)} = \frac{0.44}{0.55 - 0.44} = 4.0 \quad .$$

21.3 The interstellar extinction at $\lambda = 0.55\,\mu\mathrm{m}$ is observed to be proportional to the hydrogen column density N_{H}, with $A_{0.55\,\mu\mathrm{m}}/N_{\mathrm{H}} \approx 3.52 \times 10^{-22}\mathrm{mag\,cm}^2$. Suppose that this extinction is produced by spherical dust grains with a single radius a. Let the usual dimensionless efficiency factor $Q_{\mathrm{ext}}(\lambda) \equiv C_{\mathrm{ext}}(\lambda)/\pi a^2$, where $C_{\mathrm{ext}}(\lambda)$ is the cross section for extinction at wavelength λ.

(a) If the grains have internal density ρ, calculate $M_{\mathrm{dust}}/M_{\mathrm{H}}$, the ratio of total dust mass to total hydrogen mass, in terms of unknown $Q_{\mathrm{ext}}(0.55\,\mu\mathrm{m})$, ρ, and a.

(b) Calculate the numerical value of $M_{\mathrm{dust}}/M_{\mathrm{H}}$ if $a = 0.1\,\mu\mathrm{m}$, $Q_{\mathrm{ext}} = 1.5$, and $\rho = 3\,\mathrm{g\,cm}^{-3}$.

Solution:

(a) Let $N_d \equiv$ column density of dust grains.

$$
\begin{aligned}
e^{\tau} &= 10^{0.4A} \\
\tau_\lambda &= 0.4\ln(10)A = 0.921A_\lambda \\
\tau_\lambda = N_d C_{\mathrm{ext}}(\lambda) &= 0.921A_\lambda \\
N_d &= \frac{0.921A_\lambda}{C_{\mathrm{ext}}} \\
&= \frac{0.921A_\lambda}{Q_{\mathrm{ext}}(\lambda)\pi a^2} \\
\frac{M_{\mathrm{dust}}}{M_{\mathrm{H}}} &= \frac{N_d \times (4/3)\pi a^3\rho}{N_{\mathrm{H}} m_{\mathrm{H}}} \\
&= \frac{0.921A_\lambda}{Q_{\mathrm{ext}}(\lambda)\,\pi a^2} \times \frac{(4/3)\pi a^3\rho}{N_{\mathrm{H}} m_{\mathrm{H}}} \\
&= \frac{0.921(4/3)}{Q_{\mathrm{ext}}(\lambda)} \frac{a\rho}{m_{\mathrm{H}}} \frac{A_{\mathrm{ext}}}{N_{\mathrm{H}}} \\
&= 260.3 \frac{a\rho}{Q_{\mathrm{ext}}(0.55\,\mu\mathrm{m})} \frac{1}{\mathrm{g\,cm}} \\
&= \frac{0.00781}{Q_{\mathrm{ext}}(0.55\,\mu\mathrm{m})} \left(\frac{a}{0.1\,\mu\mathrm{m}}\right) \left(\frac{\rho}{3\,\mathrm{g\,cm}^{-3}}\right)
\end{aligned}
$$

(b) If $a \approx 0.1\,\mu\mathrm{m}$, $Q_{\mathrm{ext}} \approx 1.5$, and $\rho \approx 3\,\mathrm{g\,cm}^{-3}$, then

$$\frac{M_{\mathrm{dust}}}{M_{\mathrm{H}}} \approx 0.0052$$

Chapter 22. Scattering and Absorption by Small Particles

22.1 Suppose that for $\lambda = 10\,\mu\text{m}$, amorphous silicate material has dielectric function $\epsilon = \epsilon_1 + i\epsilon_2$ given in the following table:

$\lambda(\,\mu\text{m})$	ϵ_1	ϵ_2
9.50	0.731	1.987
9.60	0.774	2.131
9.70	0.831	2.260
9.80	0.891	2.373
9.90	0.946	2.476
10.0	0.996	2.575
10.1	1.040	2.678
10.2	1.085	2.792
10.3	1.141	2.920
10.4	1.224	3.056
10.5	1.333	3.184

For a spherical grain of amorphous silicate with radius $a = 0.1\,\mu\text{m}$:

(a) Calculate the absorption efficiency factor Q_{abs} and the absorption cross section per volume C_{abs}/V at $\lambda = 10\,\mu\text{m}$.

(b) Calculate the scattering efficiency factor Q_{sca} at $\lambda = 10\,\mu\text{m}$.

Solution:

(a) For $a/\lambda = 0.1\,\mu\text{m}/10\,\mu\text{m} = 0.01$, we may assume that we are in the "electric dipole" or "Rayleigh" limit. From Eq. (22.19),

$$
\begin{aligned}
Q_{\text{abs}} &= \frac{C_{\text{abs}}}{\pi a^2} = \frac{18\pi}{\pi a^2}\left(\frac{\epsilon_2}{(\epsilon_1+2)^2+\epsilon_2^2}\right)\left(\frac{V}{\lambda}\right) \\
&= \frac{24\pi\epsilon_2}{(\epsilon_1+2)^2+\epsilon_2^2}\left(\frac{a}{\lambda}\right) \;=\; 0.124 \\
\frac{C_{\text{abs}}}{V} &= \frac{18\pi}{\lambda}\frac{\epsilon_2}{(\epsilon_1+2)^2+\epsilon_2^2} \;=\; 0.933\,\mu\text{m}^{-1} \;=\; 9330\,\text{cm}^{-1} \quad .
\end{aligned}
$$

(b) From Eq. (22.20),

$$
\begin{aligned}
Q_{\text{sca}} &= \frac{C_{\text{sca}}}{\pi a^2} = \frac{24\pi^3}{\pi a^2}\left|\frac{\epsilon-1}{\epsilon+2}\right|^2 \frac{(4\pi/3)^2 a^6}{\lambda^4} \\
&= \frac{128\pi^4}{3}\frac{(\epsilon_1-1)^2+\epsilon_2^2}{(\epsilon_1+2)^2+\epsilon_2^2}\left(\frac{a}{\lambda}\right)^4 \\
&= 1.77\times 10^{-5} \quad : \text{much smaller than } Q_{\text{abs}} \quad .
\end{aligned}
$$

22.3 Consider a oblate spheroidal grain with axial ratios $a:b:b::0.1\,\mu\mathrm{m}:0.15\,\mu\mathrm{m}:0.15\,\mu\mathrm{m}$. In the electric dipole limit ($\lambda \gg a$), the absorption cross section for radiation polarized with the electric field parallel to axis j is

$$C_{\mathrm{abs},j} = \frac{2\pi V}{\lambda} \frac{\epsilon_2}{\left[1+(\epsilon_1-1)L_j\right]^2 + (\epsilon_2 L_j)^2} \quad .$$

(a) Using Eq. (22.15–22.18), evaluate the shape factors L_a and L_b for applied electric fields parallel to the short axis a or a long axis b.

(b) If the complex dielectric function of amorphous silicate material at $\lambda = 10\,\mu\mathrm{m}$ is $\epsilon = 0.996 + 2.575i$, where $i \equiv \sqrt{-1}$, calculate $C_{\mathrm{abs},a}/V$, $C_{\mathrm{abs},b}/V$, and $\left[C_{\mathrm{abs},b} - C_{\mathrm{abs},a}\right]/V$ for radiation with $\lambda = 10.0\,\mu\mathrm{m}$.

Solution:

(a) This is an oblate spheroid, with

$$\begin{aligned}
e^2 &= |1-(1.5)^2| = 1.25 \quad , \quad e \;=\; 1.118 \\
L_a &= \frac{1+e^2}{e^2}\left[1 - \frac{1}{e}\arctan(e)\right] \;=\; \frac{2.25}{1.25}\left[1 - \frac{1}{1.118}\arctan(1.118)\right] \\
&= 0.4459 \\
L_b &= (1-L_a)/2 = 0.2770 \quad .
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{C_{\mathrm{abs},j}}{V} &= \frac{2\pi}{\lambda} \times \frac{\epsilon_2}{\left[1+(\epsilon_1-1)L_j\right]^2 + (\epsilon_2 L_j)^2} \\
&\approx \frac{2\pi}{10\,\mu\mathrm{m}} \times \frac{2.58}{[(1-0.004\times L_j)^2 + (2.58\times L_j)^2]} \\
\frac{C_{\mathrm{abs},a}}{V} &= 0.699\,\mu\mathrm{m}^{-1} \\
\frac{C_{\mathrm{abs},b}}{V} &= 1.075\,\mu\mathrm{m}^{-1} \\
\frac{[C_{\mathrm{abs},b} - C_{\mathrm{abs},a}]}{V} &= (1.075 - 0.699)\,\mu\mathrm{m}^{-1} = 0.376\,\mu\mathrm{m}^{-1} \quad .
\end{aligned}$$

Chapter 23. Composition of Interstellar Dust

23.1 Suppose that in the optical and near-UV the extinction efficiency can be approximated

$$Q_{\rm ext}(a,\lambda) \approx 2(\pi a/2\lambda)^\beta \quad \text{for } \pi a/\lambda < 2$$
$$\approx 2 \quad \text{for } \pi a/\lambda > 2 \quad .$$

This is imprecise, but you can see from Fig. 22.3 that for $\beta \approx 1.5$ it roughly approximates the essential behavior if $|m-1| \approx 0.5$: a rapid increase in $Q_{\rm ext}$ with increasing a/λ until it reaches $\sim$2. Note that this is *not* a good approximation for $a/\lambda \lesssim 0.05$, but in the present problem we consider only the extinction at B and V, which is dominated by the larger particles.

Suppose that the dust density is proportional to $n_{\rm H}$, with a simple power-law size distribution

$$\frac{1}{n_{\rm H}}\frac{dn}{da} = \frac{A_0}{a_0}\left(\frac{a}{a_0}\right)^{-p} \qquad 0 < a \leq a_{\rm max} \quad ,$$

where $a_0 = 0.1\,\mu{\rm m}$ is a fiducial length, A_0 is dimensionless, and $p < 4$.
The V and B bands have wavelengths $\lambda_V = 0.55\,\mu{\rm m}$ and $\lambda_B = 0.44\,\mu{\rm m}$.

Let $\sigma_{\rm ext}(\lambda)$ be the extinction cross section per H at wavelength λ.

(a) Assume that $a_{\rm max} < 0.28\,\mu{\rm m}$ (i.e., $\pi a_{\rm max}/\lambda_V < \pi a_{\rm max}/\lambda_B < 2$). Obtain an expression for

$$\frac{\sigma_{\rm ext}(\lambda)}{A_0 \pi a_0^2}$$

that would be valid for $\lambda = \lambda_V$ or λ_B. Evaluate this ratio for $\beta = 1.5$, $p = 3.5$, $a_{\rm max} = 0.25\,\mu{\rm m}$, and $\lambda = \lambda_V$.

(b) For $a_{\rm max} < 0.28\,\mu{\rm m}$, using your result from (a), obtain an expression for the ratio

$$\frac{\sigma_{\rm ext}(\lambda_B)}{\sigma_{\rm ext}(\lambda_V)} \quad ,$$

and evaluate this for $\beta = 1.5$.

(c) Assuming $a_{\rm max} < 0.28\,\mu{\rm m}$, obtain an expression for $R_V \equiv A_V/(A_B - A_V)$, and evaluate this for $\beta = 1.5$.

(d) Now suppose that $a_{\rm max} > 2\lambda/\pi$. Obtain an expression for

$$\frac{\sigma_{\rm ext}(\lambda)}{A_0 \pi a_0^2} \quad .$$

(e) If $a_{\rm max} = 0.35\,\mu{\rm m}$, $p = 3.5$, and $\beta = 2$,
(i) Evaluate $\sigma_{\rm ext}(\lambda_V)/A_0\pi a_0^2$,
(ii) Evaluate $\sigma_{\rm ext}(\lambda_B)/A_0\pi a_0^2$,
(iii) Evaluate R_V.

Solution:

(a) For $\lambda > \pi a_{\rm max}/2$*:*

$$\frac{\sigma_{\rm ext}(\lambda)}{A_0\pi a_0^2} = \frac{1}{A_0\pi a_0^2}\frac{A_0}{a_0}\int_0^{a_{\rm max}} 2\pi a^2\left(\frac{\pi a}{2\lambda_V}\right)^\beta\left(\frac{a}{a_0}\right)^{-p}da$$

$$= 2\left(\frac{\pi a_0}{2\lambda_V}\right)^\beta\int_0^{a_{\rm max}}\left(\frac{a}{a_0}\right)^{2+\beta-p}\frac{da}{a_0}$$

$$= 2\left(\frac{\pi a_0}{2\lambda_V}\right)^\beta\frac{1}{(3+\beta-p)}\left(\frac{a_{\rm max}}{a_0}\right)^{(3+\beta-p)}$$

$$= 2\left(\frac{\pi\times 0.1}{2\times 0.55}\right)^{1.5}\frac{1}{1.0}\left(\frac{0.25}{0.1}\right)^{1.0} = 0.7631 \quad .$$

(b) From part (a), it is clear that for $a_{\rm max} < 0.28\,\mu{\rm m}$*:*

$$\frac{\sigma_{\rm ext}(\lambda_B)}{\sigma_{\rm ext}(\lambda_V)} = \left(\frac{\lambda_V}{\lambda_B}\right)^\beta$$

$$= 1.3975 \qquad\qquad \text{for } \beta = 1.5 \quad .$$

(c)

$$R_V = \frac{\sigma_{\rm ext}(\lambda_V)}{\sigma_{\rm ext}(\lambda_B)-\sigma_{\rm ext}(\lambda_V)}$$

$$= \frac{1}{\sigma_{\rm ext}(\lambda_B)/\sigma_{\rm ext}(\lambda_V)-1}$$

$$= \frac{1}{(\lambda_V/\lambda_B)^\beta-1}$$

$$= \frac{1}{1.3975-1} = 2.52 \qquad\qquad \text{for } \beta = 1.5 \quad .$$

Note that for the simplifying assumptions in this problem, R_V *is* *<u>independent</u> of the power-law index* p *provided* $a_{\rm max} < 0.28\,\mu{\rm m}$*!*

(d) For $\lambda < \pi a_{\rm max}/2$*:*

$$\frac{\sigma_{\rm ext}(\lambda)}{A_0\pi a_0^2} = \frac{1}{A_0\pi a_0^2}\frac{A_0}{a_0}\left[\int_0^{2\lambda/\pi} 2\pi a^2\left(\frac{\pi a}{2\lambda}\right)^\beta\left(\frac{a}{a_0}\right)^{-p}da\right.$$

$$= \qquad \left. + \int_{2\lambda/\pi}^{a_{\rm max}} 2\pi a^2\left(\frac{a}{a_0}\right)^{-p}da\right]$$

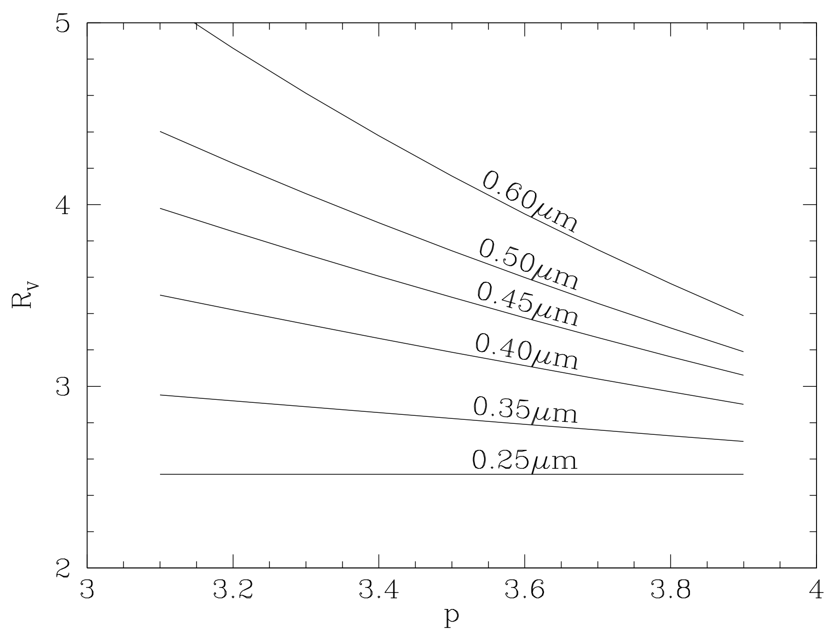
$$= 2\left(\frac{\pi a_0}{2\lambda}\right)^\beta\int_0^{2\lambda/\pi}\left(\frac{a}{a_0}\right)^{2+\beta-p}\frac{da}{a_0} + 2\int_{2\lambda/\pi}^{a_{\rm max}}\left(\frac{a}{a_0}\right)^{2-p}\frac{da}{a_0}$$

$$= \frac{2}{(3+\beta-p)}\left(\frac{2\lambda}{\pi a_0}\right)^{3-p} + \frac{2}{(3-p)}\left[\left(\frac{a_{\rm max}}{a_0}\right)^{3-p} - \left(\frac{2\lambda}{\pi a_0}\right)^{3-p}\right] .$$

(e)
$$\frac{\sigma_{\rm ext}(\lambda_V)}{A_0\pi a_0^2} = 2\left(\frac{\pi\times0.1}{2\times0.55}\right)^{1.5}\frac{1}{1.0}\left(\frac{0.35}{0.1}\right)^{1.0} = 1.0684$$
$$\frac{\sigma_{\rm ext}(\lambda_B)}{A_0\pi a_0^2} = \frac{2}{1.0}\left(\frac{\pi\times0.1}{2\times0.44}\right)^{0.5} + \frac{2}{0.5}\left[\left(\frac{\pi\times0.1}{2\times0.44}\right)^{0.5} - \left(\frac{0.1}{0.35}\right)^{0.5}\right]$$
$$= 1.1950 + 0.2519 = 1.4469$$
$$R_V = \frac{1.0684}{1.4469 - 1.0684} = 2.82 \quad .$$

Thus, for this (oversimplified!) model for the grain optics, increasing $a_{\rm max}$ from 0.25 μm to 0.35 μm increases R_V from 2.52 to 2.82.

For this model, this figure shows R_V as a function of p for several different values of $a_{\rm max}$:



From the figure we see that, for this model, smaller values of p (flatter size distributions) result in larger values of R_V provided $a_{\rm max} > 2\lambda_B/\pi = 0.28\,\mu$m.

Chapter 24. Temperatures of Interstellar Grains

24.1 Consider particles with number density n_c, mass m_c, and kinetic temperaure T_c colliding with a neutral grain. The collision rate is

$$\frac{dN}{dt} = \pi a^2 n_c \left(\frac{8k_{\rm B}T_c}{\pi m_c}\right)^{1/2} \quad .$$

Let E be the kinetic energy of an impacting particle. What is $\langle E^n \rangle$, where the average is over the impacting particles, for general n?

Evaluate the result for $n = 1$.

Solution:

$$\begin{aligned} \langle E^n \rangle &= \frac{\int dv\, v^2\, {\rm e}^{-mv^2/2k_{\rm B}T}\, v\, (mv^2/2)^n}{\int dv\, v^2\, {\rm e}^{-mv^2/2k_{\rm B}T}\, v} \\ &= \frac{(k_{\rm B}T)^n \int dx\, x^{n+1}\, {\rm e}^{-x}}{\int dx\, x\, {\rm e}^{-x}} \\ &= (k_{\rm B}T)^n \frac{\Gamma(n+2)}{\Gamma(2)} = (n+1)!(k_{\rm B}T)^n \\ &= 2k_{\rm B}T \quad \text{for } n=1 \quad . \end{aligned}$$

24.3 Suppose that interstellar dust grains have $Q_{\rm abs} \propto \lambda^{-2}$ for $\lambda > 1\mu$m. When exposed to the local interstellar radiation field (LISRF), these grains are heated to $T \approx 18\,{\rm K}$ and radiate with λI_λ peaking at $\lambda = 140\mu$m.

In a region where the starlight has the same spectrum as the LISRF but is stronger by a numerical factor U:

(a) What will be the grain temperature?

(b) If $U = 10^3$, what will be the wavelength where λI_λ peaks?

Solution:

(a) If $Q_{\rm abs} \propto \lambda^{-2}$*, then the power radiated by the grain* $P_{\rm rad} = 4\pi a^2 \sigma T^4 \langle Q_{\rm abs}\rangle_T$*, where* $\langle Q_{\rm abs}\rangle_T \propto T^2$*. Thus* $P_{\rm rad} \propto T^6$*,*

$$\frac{P_{\rm rad}(T)}{P_{\rm rad}(T_0)} = \left(\frac{T}{T_0}\right)^6$$
$$U = \left(\frac{T}{18\,{\rm K}}\right)^6$$
$$T = 18\,U^{1/6}\,{\rm K}$$

(b) $\lambda_{\rm peak} = 140\,\mu{\rm m} \times (18\,{\rm K}/T) = 140\,\mu{\rm m} \times U^{-1/6} = 44.3\,\mu{\rm m}$

Chapter 25. Grain Physics: Charging and Sputtering

25.1 Consider a grain with radius $a = 0.1\,\mu\mathrm{m}$, located in an H I cloud with $n_e = 0.01\,\mathrm{cm}^{-3}$, $T = 100\,\mathrm{K}$, and a starlight background given by the MMP83 estimate for the solar neighborhood. Assume that the "sticking efficiency" for colliding electrons $s_e = 1$.

(a) Estimate the probability per unit time t_{e0}^{-1} for electron capture by a neutral grain.

(b) For the MMP83 radiation field [see Table 12.1 and Eq. (12.7)] the number density of 10-13.6 eV photons is $n_{\rm FUV} \approx 8\times10^{-4}\,\mathrm{cm}^{-3}$ (see Problem 12.2). If the $a = 0.1\,\mu\mathrm{m}$ grain has an absorption efficiency factor $Q_{\rm abs} \approx 1$, and the mean photoelectric yield for $h\nu > 10\,\mathrm{eV}$ is $Y_{\rm pe} = 0.1$, estimate the photoelectron emission rate $t_{\rm pe}^{-1}$.

(c) As the grain becomes positively charged, Coulomb focusing will increase the rate of electron collisions. If the rate of photoelectron emission $t_{\rm pe}^{-1}$ does not change when the grain becomes positively charged, to what potential U will the grain charge?
How many unit charges does this correspond to?

Solution:

(a)

$$t_{e0}^{-1} = n_e\left(\frac{8k_{\rm B}T}{\pi m_e}\right)^{1/2}\pi a^2$$
$$= 0.01\,\mathrm{cm}^{-3}\times 6.21\times10^6\,\mathrm{cm\,s}^{-1}\times\pi\times10^{-10}\,\mathrm{cm}^2$$
$$= 1.95\times10^{-5}\,\mathrm{s}^{-1} \quad . \tag{1}$$

(b)

$$t_{\rm pe}^{-1} = n_{\rm FUV}\,c\,Q_{\rm abs}\pi a^2\,Y_{\rm pe} = 7.5\times10^{-4}\,\mathrm{s}^{-1} \quad .$$

(c) The electron impact rate will be increased by a factor $[1+(eU/k_{\rm B}T)]$ *[see Eq. (25.6)]. Thus, for capture to balance photoelectric emission, we must have:*

$$\left[1+\left(\frac{eU}{k_{\rm B}T}\right)\right]t_{e0}^{-1} \approx t_{\rm pe}^{-1}$$
$$\frac{eU}{k_{\rm B}T} \approx \frac{t_{\rm pe}^{-1}}{t_{e0}^{-1}} - 1 \;=\; 38.5 - 1$$
$$\frac{eU}{\mathrm{eV}} = 37.5\frac{k_{\rm B}T}{\mathrm{eV}} = 0.32 \;\;\rightarrow\;\; U = 0.32\,\mathrm{V}$$
$$Z = \frac{Ua}{e} = \frac{eU\times a}{e^2} = \frac{37.5k_{\rm B}T\times a}{e^2} = 22.4 \quad .$$

Thus, we expect the grain charge to fluctuate around a mean $\langle Z\rangle \approx 22.4$*. Charging of the grain to* $+0.32\,\mathrm{V}$ *will reduce the photoelectron emission rate somewhat, but the effect should be small because* $0.32\,\mathrm{eV}$ *is small compared to* $13.6\,\mathrm{eV}-W$*, where* $W \approx 5-8\,\mathrm{eV}$ *is the work function.*

25.3 Sputtering acts to erode grains at a rate $da/dt = -n_{\rm H}\beta$ independent of a. Suppose that the grain size distribution at $t = 0$ is a power-law

$$\frac{1}{n_{\rm H}}\frac{dn}{da} = \frac{A_0}{a_{\rm max}}\left(\frac{a}{a_{\rm max}}\right)^{-p} \qquad 0 \le a \le a_{\rm max} \quad .$$

(a) Let V_0 be the initial volume of grain material per H nucleon. Express V_0 in terms of A_0, $a_{\rm max}$, and p.

(b) Obtain an algebraic expression for $V(t)/V_0$ in terms of $y \equiv \Delta a/a_{\rm max} = n_{\rm H}\beta t/a_{\rm max}$ and p.

Solution:

(a)

$$\begin{aligned} V_0 &= \frac{1}{n_{\rm H}}\int_0^{a_{\rm max}} \frac{4\pi}{3}a^3\frac{dn}{da}da \\ &= \frac{4\pi}{3}A_0 a_{\rm max}^3 \int_0^1 \left(\frac{a}{a_{\rm max}}\right)^3 \left(\frac{a}{a_{\rm max}}\right)^{-p} d\left(\frac{a}{a_{\rm max}}\right) \\ &= \frac{4\pi}{(4-p)3}A_0 a_{\rm max}^3 \quad . \end{aligned}$$

(b) Every grain has $a \to \max(a - \Delta a, 0)$ where $\Delta a = |da/dt| \times t = n_{\rm H}\beta t$. Suppose that $t < a_{\rm max}/|da/dt|$ (i.e., $\Delta a < a_{\rm max}$). Then

$$\begin{aligned} V(t) &= \frac{A_0}{a_{\rm max}}\int_{\Delta a}^{a_{\rm max}} \frac{4\pi}{3}(a-\Delta a)^3\left(\frac{a}{a_{\rm max}}\right)^{-p} da \\ &= \frac{4\pi}{3}A_0 a_{\rm max}^3 \int_y^1 (x-y)^3 x^{-p} dx \qquad\qquad x \equiv \frac{a}{a_{\rm max}} \\ \frac{V(t)}{V_0} &= (4-p)\int_y^1 (x-y)^3 x^{-p} dx \\ &= (4-p)\int_y^1 \left(x^{3-p} - 3yx^{2-p} + 3y^2x^{1-p} - y^3x^{-p}\right) dx \\ &= (4-p)\left[\frac{(1-y^{4-p})}{(4-p)} - \frac{3y\,(1-y^{3-p})}{(3-p)} + \frac{3y^2\,(1-y^{2-p})}{(2-p)} - \frac{y^3\,(1-y^{1-p})}{(1-p)}\right] . \end{aligned}$$

25.5 Suppose that at $t = 0$ the dust has a size distribution

$$\frac{1}{n_{\rm H}}\frac{dn}{da} = \frac{A_0}{a_0}\left(\frac{a}{a_0}\right)^{-p} \qquad \text{for } a \leq a_{\rm max} \quad .$$

Suppose that sputtering has continued for some time t, at a sputtering rate $da/dt = -n_{\rm H}\beta$. Let $Q_{\rm ext}(a,\lambda)$ be the extinction efficiency factor at wavelength λ for a grain of radius a

Let $\sigma_{\rm ext}(\lambda)$ be the dust extinction cross section per H. Write down an integral expression for $\sigma_{\rm ext}(\lambda)$ at some fixed time $t < a_{\rm max}/|da/dt|$.

Solution: The inequality $t < a_{\rm max}/|da/dt|$ ensures that some grains will survive. Suppose that x is the initial grain size. Then $a = x - \Delta a$, where $\Delta a \equiv n_{\rm H}\beta t$. Let x be the initial grain radius:

$$\begin{aligned}\sigma_{\rm ext}(\lambda) &= \int_{\Delta a}^{a_{\rm max}} dx \frac{A_0}{a_0}\left(\frac{x}{a_0}\right)^{-p} \times \pi(x-\Delta a)^2 Q_{\rm ext}((x-\Delta a),\lambda) \\ &= \frac{A_0}{a_0}\int_0^{a_{\rm max}-\Delta a} da \left(\frac{a+\Delta a}{a_0}\right)^{-p} \pi a^2 Q_{\rm ext}(a,\lambda) \quad .\end{aligned}$$

25.7 Suppose that interstellar gas contains dust grains consisting of two populations: "large" grains of radius $a_1 = 1{\times}10^{-5}\,\mathrm{cm}$ and number density $n_1 = 2{\times}10^{-12} n_{\mathrm{H}}$, and "small" grains of radius $a_2 = 5{\times}10^{-7}\,\mathrm{cm}$ and number density $n_2 = 1 \times 10^{-9} n_{\mathrm{H}}$.

(a) Suppose that every grain is charged to a potential $U \approx +2\,\mathrm{V}$. If the gas as a whole is electrically neutral, compute $(n_e - n_I)/n_{\mathrm{H}}$, where n_e is the free electron density, and n_I is the density of free ions (where we do *not* consider the charged grains to be "ions").

(b) Discuss whether your answer to (a) is seriously affected by charge quantization.

Solution:

(a) In cgs electromagnetism, $U = Q/a$. We use the unit "eV" often enough to remember its value in cgs units ($\mathrm{eV} = 1.602{\times}10^{-12}\,\mathrm{erg}$*) and we can write*

$$
\begin{aligned}
\frac{Q}{e} &= \frac{Ua}{e} = eU \times \frac{a}{e^2} \\
&= \frac{eU}{\mathrm{eV}} \times \mathrm{eV} \times \frac{a}{e^2} \\
&= \frac{U}{\mathrm{V}} \times \frac{1.602{\times}10^{-12}}{(4.803{\times}10^{-10})^2} \times \frac{a}{\mathrm{cm}} \\
&= 6.94 \times 10^6 \times \frac{U}{\mathrm{V}} \times \frac{a}{\mathrm{cm}} \\
Z_1 = \left(\frac{Q}{e}\right)_1 &= 139 \qquad (for\ \ U = 2\,\mathrm{V},\ a_1 = 10^{-5}\,\mathrm{cm}) \\
Z_2 = \left(\frac{Q}{e}\right)_2 &= 6.94 \qquad (for\ \ U = 2\,\mathrm{V},\ a_2 = 5 \times 10^{-7}\,\mathrm{cm}) \\
\frac{n_e - n_I}{n_{\mathrm{H}}} = \frac{n_1 Z_1 + n_2 Z_2}{n_{\mathrm{H}}} &= 139 \times 2{\times}10^{-12} + 6.94 \times 1{\times}10^{-9} \\
&= 2.78{\times}10^{-10} + 6.94{\times}10^{-9} = 7.22 \times 10^{-9}
\end{aligned}
$$

(b) The small grains will presumably have $Z = 6$ or 7. Charge quantization therefore affects the answer by at most ∼15%.

25.9 Consider hot plasma with density $n_{\rm H}$ in an elliptical galaxy. Suppose that planetary nebulae and other stellar outflows are injecting dust grains with a single initial size $a = a_{\rm max}$ into the plasma with a rate $(dN_{\rm dust}/dt)_{\rm inj}$, where $N_{\rm dust}$ = number of dust grains.

Upon injection into the plasma, the grains are subject to sputtering at a rate $da/dt = -\beta n_{\rm H}$, where β is a constant.

(a) Find the steady state solution for $dN_{\rm dust}/da$, where $N_{\rm dust}(a)$ is the number of dust grains present with radii $\leq a$. Express your result in terms of β, $n_{\rm H}$, and the injection rate $(dN_{\rm dust}/dt)_{\rm inj}$.

(b) If the rate of injection of dust mass is

$$\left(\frac{dM_{\rm dust}}{dt}\right)_{\rm inj} = \frac{4\pi\rho}{3} a_{\rm max}^3 \left(\frac{dN_{\rm dust}}{dt}\right)_{\rm inj}$$

where ρ is the internal density of the dust, obtain an expression for the steady-state dust mass, $M_{\rm dust}$. Express your result in terms of $(dM_{\rm dust}/dt)_{\rm inj}$, β, $n_{\rm H}$, and $a_{\rm max}$.

(c) Obtain an expression for the characteristic "mass survival time"

$$\tau_{\rm mass\ survival} \equiv \frac{M_{\rm dust}}{(dM_{\rm dust}/dt)_{\rm inj}} \quad .$$

(d) Consider the "passive" elliptical galaxy NGC 4564 containing hot plasma $kT \approx 0.5\,{\rm keV}$ ($T \approx 6\times10^6\,{\rm K}$) and a core density $n_{\rm H} \approx 0.01\,{\rm cm}^{-3}$ (Soria et al. 2006, ApJ 640, 126). From Figure 25.4, the sputtering rate for refractory grains would be $da/dt = -\beta n_{\rm H}$, with $\beta \approx 10^{-6}\,\mu{\rm m\,cm}^3\,{\rm yr}^{-1}$.

Suppose that the injected dust has $a = 0.3\,\mu{\rm m}$. If the dust injection rate from evolved stars in the central kpc is $1.3\times10^{-4}\,M_\odot\,{\rm yr}^{-1}$ (Clemens et al. 2010: A&A 518, L50), estimate the steady-state dust mass $M_{\rm dust}$ in the central kpc.

Compare to the observed upper limit $M_{\rm dust} < 8700\,M_\odot$ from Clemens et al. (2010).

(e) Estimate the dust mass survival time $\tau_{\rm mass\ survival}$ in the core of NGC 4564.

Solution:

(a) Individual grains move downward in size, with $da/dt = const$. The "current" in "size-space" is

$$J(a) = \frac{dN_{\rm dust}}{da} \times \frac{da}{dt} \quad .$$

Note: $J(a) < 0$ because $da/dt < 0$.

In steady-state, the "downward" current in "size-space" must be equal to the rate of injection of grains:

$$J(a) = -\left(\frac{dN_{\rm dust}}{dt}\right)_{\rm inj}$$

where $(dN_{\rm dust}/dt)_{\rm inj}$ is the rate of injection of grains with radii $> a$.

If all grains are injected with a single size $a_{\rm max}$, then

$$J(a) = -\left(\frac{dN_{\rm dust}}{dt}\right)_{\rm inj} \qquad \text{for } a < a_{\rm max}$$

and

$$\frac{dN_{\rm dust}}{da} = -\frac{J(a)}{da/dt} = \frac{1}{\beta n_{\rm H}}\left(\frac{dN_{\rm dust}}{dt}\right)_{\rm inj}$$

(b) The steady-state dust mass is just

$$\begin{aligned} M_{\rm dust} &= \int_0^{a_{\rm max}} \frac{4\pi}{3}\rho a^3 \frac{dN_{\rm dust}}{da} \\ &= \frac{\pi\rho a_{\rm max}^4}{3}\frac{1}{\beta n_{\rm H}}\left(\frac{dN_{\rm dust}}{da}\right)_{\rm inj} \\ &= \frac{\pi\rho a_{\rm max}^4}{3}\frac{1}{\beta n_{\rm H}}\frac{3}{4\pi\rho a_{\rm max}^3}\left(\frac{dM_{\rm dust}}{dt}\right)_{\rm inj} = \frac{a_{\rm max}}{4\beta n_{\rm H}}\left(\frac{dM_{\rm dust}}{dt}\right)_{\rm inj} \end{aligned}$$

(c) The mass survival time is just

$$\tau_{\rm mass\ survival} = \frac{M_{\rm dust}}{(dM_{\rm d}/dt)_{\rm inj}} = \frac{a_{\rm max}}{4\beta n_{\rm H}}$$

(d) Using the result from (b) and $\beta n_{\rm H} = 1\times 10^{-8}\,\mu{\rm m\,yr^{-1}}$ we obtain

$$M_{\rm dust} = \frac{a_{\rm max}}{4\beta n_{\rm H}}\left(\frac{dM_{\rm dust}}{dt}\right)_{\rm inj} = \frac{0.3\,\mu{\rm m}}{4\times 1\times 10^{-8}\,\mu{\rm m\,yr^{-1}}}\times 1.3\times 10^{-4}\,M_\odot\,{\rm yr^{-1}} = 975\,M_\odot$$

which is consistent with the observational upper limit $M_{\rm dust} < 8700\,M_\odot$

(e)

$$\tau_{\rm mass\ survival} \equiv \frac{M_{\rm dust}}{(dM_d/dt)_{\rm inj}} = \frac{975\,M_\odot}{1.3\times 10^{-4}\,M_\odot\,{\rm yr^{-1}}} = 7.50\times 10^6\,{\rm yr}$$

Chapter 26. Grain Dynamics

26.1 Suppose that a silicate dust grain has a radius $a = 0.1\,\mu\mathrm{m}$. Suppose that the dust grain has $Q_{abs} = 0.11$, $Q_{sca} = 0.69$, and that the scattered light has $\langle\cos\theta\rangle = 0.31$, where θ is the angle between the direction of incidence and the direction of propagation; these values have been estimated for "astronomical silicate" grains at $\lambda = 5500\,\text{Å}$ (Draine 1985: *Ap.J.Suppl.*, **57**, 587). Take $\rho = 3\,\mathrm{g\,cm^{-3}}$ for the grain density.

(a) Ignoring the wavelength dependence of these quantities, what is the value of L/M (the ratio of luminosity to mass) for a star such that the radiation pressure force on such a grain close (but not *too* close) to the star exactly balances the gravitational force due to the star? Give L/M in solar units ($L_\odot/\,M_\odot$).

(b) Now suppose that such silicate grains are mixed with gas, with the dust mass equal to 0.7% of the gas mass, and that the grains are "well-coupled" to the gas through collisions or magnetic fields. What must be the ratio of L/M for the star (in solar units) such that radiation pressure on the grains will exert a repulsion equal in magnitude to the gravitational attraction on the gas-dust mixture?

Solution:

(a)

$$
\begin{aligned}
F_{grav} &= F_{rad}\\
\frac{GM}{r^2}\frac{4\pi}{3}\rho a^3 &= Q_{pr}\pi a^2\frac{L}{4\pi r^2 c}
\end{aligned}
$$

where

$$Q_{pr} \equiv \left[Q_{abs} + (1-\langle cos\theta\rangle)\,Q_{sca}\right] = 0.11 + (1-0.31)\times 0.69 = 0.59$$

with solution

$$
\begin{aligned}
\frac{L}{M} &= \frac{16\pi cG\rho a}{3Q_{pr}} = \frac{16\pi cG\rho a}{3Q_{pr}} \times \frac{L_\odot}{3.83\times10^{33}\,\mathrm{erg\,s^{-1}}} \times \frac{1.99\times10^{33}\,\mathrm{g}}{M_\odot}\\
&= 0.89\frac{L_\odot}{M_\odot} \quad .
\end{aligned}
$$

(b) If the dust/gas mass ratio is 0.007, then to balance the gravitational force on the dust+gas mixture, must have

$$F_{rad} = \left(\frac{1.007}{0.007}\right)\frac{GM}{r^2}\frac{4\pi}{3}\rho a^3$$

thus

$$
\begin{aligned}
\frac{L/\,L_\odot}{M/\,M_\odot} &= \left(\frac{1.007}{0.007}\right)\frac{16\pi cG\,M_\odot\rho a}{3Q_{pr}\,L_\odot}\\
&= \left(\frac{1.007}{0.007}\right)\times 0.89 = 127 \quad .
\end{aligned}
$$

26.3 Consider a dust grain with the properties of the $a = 0.1\,\mu$m "astronomical silicate" grain of problem 26.1. Suppose this grain to be located in a diffuse cloud of density $n_{\rm H} = 20\,{\rm cm}^{-3}$ and temperature $T = 100\,$K, with $n({\rm He})/n({\rm H}) = 0.1$. Assume the starlight background to have an energy density of 0.5 eV ${\rm cm}^{-3}$, with 80% of the energy in an isotropic component, and 20% in a unidirectional component [cf. the "Galactic Center" contribution from problem 26.2(a)].

(a) Neglecting any forces other than gas drag and radiation pressure, what will be the "terminal" drift velocity of the grain relative to the gas if the grain is uncharged? Approximate the gas drag by the formula appropriate for subsonic motion (see Eq. 26.1-26.3):

$$F_{\rm drag} \approx C \cdot (\pi a^2) \cdot (n k_{\rm B} T) \frac{v}{\sqrt{k_{\rm B}T/\mu}} \quad ,$$

where $C = 16/3\sqrt{2\pi} \approx 2.13$, n is the gas particle density, and μ is the mass per gas particle.

(b) Approximately how long does it take the grain to reach terminal speed? Assume the grain density to be $\rho = 3\,{\rm g\,cm}^{-3}$.

(c) Moving at the terminal speed, how long would it take the grain to drift a distance of 1 pc?

Solution:

(a) Let $v_{\rm term}$ be the terminal drift speed.

$$\begin{aligned}
F_{\rm drag}(v_{\rm term}) &= F_{\rm rad} \\
C\pi a^2 (n k_{\rm B} T) \frac{v_{\rm term}}{\sqrt{k_{\rm B}T/\mu}} &= Q_{pr} \pi a^2 u_{\rm GC} \\
v_{\rm term} &= \frac{Q_{pr}}{C} \frac{u_{\rm GC}}{n k_{\rm B} T} \sqrt{k_{\rm B}T/\mu} \\
v_{\rm term} &= \frac{0.59}{2.13} \times \frac{0.1 \cdot 1.60\times10^{-12}\,{\rm erg\,cm}^{-3}}{20 \cdot 1.1 \cdot 1.38\times10^{-16} \cdot 10^2\,{\rm erg\,cm}^{-3}} \times \left(\frac{1.38\times10^{-16} \cdot 100\,{\rm erg}}{(1.4/1.1) \cdot 1.67\times10^{-24}\,{\rm g}}\right)^{1/2} \\
&= 1.18\times10^4\,{\rm cm\,s}^{-1} \quad .
\end{aligned}$$

(b) Time to approach terminal velocity:

$$\begin{aligned}
\tau_{\rm term} &= \frac{M v_{\rm term}}{F_{rad}} = \frac{(4\pi/3)\rho a^3 v_{\rm term}}{Q_{pr}\pi a^2 u_{\rm GC}} \\
&= \frac{4\rho a v_{\rm term}}{3 Q_{pr} u_{\rm GC}} = 4.98\times10^{12}\,{\rm s} = 1.58\times10^5\,{\rm yr} \quad .
\end{aligned}$$

(c) Time to travel 1 pc:

$$\Delta t = \frac{1\,{\rm pc}}{1.18\times10^4\,{\rm cm\,s}^{-1}} = 2.62\times10^{14}\,{\rm s} = 8.29\times10^6\,{\rm yr} \quad .$$

26.5 β Pictoris is an A5 ZAMS star with substantial amounts of solid matter in a circumstellar disk. An A5 ZAMS star has luminosity $L \approx 20\,L_\odot$ and mass $M \approx 2\,M_\odot$. Assume that there is no gas present in the disk – we want to consider the motion of solid particles under the influence of radiation and gravity.

(a) Estimate $\tau_{\rm PR}$ for an $a = 10\,\mu{\rm m}$ grain (with $Q_{abs} \approx 1$ and $\rho \approx 3\,{\rm g\,cm^{-3}}$) in an orbit with radius $r = 3\times10^{13}\,{\rm cm}$. (Neglect scattering).

(b) Briefly discuss the dynamics of an $a = 0.1\,\mu{\rm m}$ silicate grain in the neighborhood of this star. Assume the optical properties given in problem 26.1.

Solution:

(a) Applying Eq. (26.19) for $a = 10\,\mu{\rm m} = 10^{-3}\,{\rm cm}$*:*

$$\begin{aligned}\tau_{\rm PR} &= 8.3\times10^7\,{\rm yr} \times 10^{-3} \times 4 \times \frac{1}{20}\\ &= 1.7\times10^4\,{\rm yr} \quad .\end{aligned}$$

(b) If the grain radius is reduced from $10\,\mu{\rm m}$ *to* $0.1\,\mu{\rm m}$*, we replace* $Q_{abs} \approx 1$ *by* $Q_{pr} \approx 0.59$ *(from problem 26.1). The ratio of radiation pressure force to gravity is*

$$\begin{aligned}\epsilon \equiv \frac{F_{\rm rad}}{F_{\rm grav}} &= \frac{\langle Q_{\rm pr}\rangle \pi a^2 L_\star/4\pi R^2 c}{(4\pi a^3/3)\rho G M_\star/R^2}\\ &= \frac{3L_\star\langle Q_{\rm pr}\rangle}{16\pi G M_\star \rho a c}\\ &= 11.3 \quad .\end{aligned}$$

The radial radiation pressure force **exceeds** *the gravitational force – Poynting-Robertson drag* **does not apply!**

When $\epsilon > 1$*, the particle will accelerate outwards.*

If no gas is present to provide drag, it can easily be shown that the terminal velocity of a particle released from rest at radius r_0 *will be* $\sqrt{\epsilon - 1} \times v_{\rm esc}(r_0)$*, where* $v_{\rm esc}(r_0)$ *is the escape velocity for a massive test mass at radius* r*. For* $M = 2\,M_\odot$ *and* $r_0 = 2\,{\rm AU}$ *we have* $v_{\rm esc} = 42\,{\rm km\,s^{-1}}$*, and the terminal velocity of the particle would be* $v_{\rm esc} = 42\,{\rm km\,s^{-1}} \times \sqrt{10.3} = 135\,{\rm km\,s^{-1}}$.

26.7 The relative velocity of the Sun and the local interstellar medium is estimated to be $26\,\mathrm{km\,s^{-1}}$ (Möbius et al. 2004: A&A 426, 897): from the standpoint of the Sun there is an "interstellar wind" with a speed $v_{\rm ISW} = 26\,\mathrm{km\,s^{-1}}$. The local density of the interstellar medium can be inferred from observations of backscattered solar Lyman α and Helium resonance line radiation; if the local helium is primarily neutral, then the inferred density is $n_{\rm H} \approx 0.22\,\mathrm{cm^{-3}}$ (Lallement et al. 2004; A&A 426, 875). Suppose that the local gas contains dust grains with a mass equal to 0.01 of the hydrogen mass. Suppose that these grains are in a size distribution with $dn/da \propto a^{-3.5}$ for $.005 < a < 0.25\,\mu\mathrm{m}$ (this is the "MRN" size distribution).

(a) For this size distribution, what fraction $f_M(a > 0.1\,\mu\mathrm{m})$ of the grain mass is in particles with $a > 0.1\,\mu\mathrm{m}$?

(b) At the radius of Jupiter, estimate the mass flux ($\mathrm{g\ cm^{-2}\,s^{-1}}$) due to $a > 0.1\,\mu\mathrm{m}$ interstellar grains if they are not deflected after passing through the "heliopause" where the interstellar medium and the interplanetary medium are both shocked. The location of the heliopause is uncertain; it is estimated to be at $\sim 100\,\mathrm{AU}$.

(c) Now suppose the grains have internal densities of $\rho = 2\,\mathrm{g\,cm^{-3}}$, and suppose that sunlight charges them to a potential $U = 5\,\mathrm{V}$. Let the solar wind be in the radial direction with a speed $v_{\odot\rm W} = 450\,\mathrm{km\,s^{-1}}$. Assume that in the frame of reference where the solar wind is locally at rest, the local electric field vanishes. Further assume, for simplicity, that the interplanetary magnetic field is perpendicular to the direction of the interstellar wind, and has a strength $B = 2\,\mu\mathrm{G}$ (at $\sim$100 AU). With the above assumptions, calculate the gyroradius of an $a = 0.1\,\mu\mathrm{m}$ interstellar grain once it has entered the region containing the solar wind. How does the gyroradius depend on the grain radius a?

Solution:

(a)
$$f_M(a > 0.1\,\mu\mathrm{m}) = \frac{\int_{0.1}^{0.25} a^3 \times a^{-3.5} da}{\int_{0.005}^{0.25} a^3 \times a^{-3.5} da} = \frac{(0.25)^{1/2} - (0.10)^{1/2}}{(0.25)^{1/2} - (0.005)^{1/2}} = 0.428 \quad .$$

(b)
$$\begin{aligned}(\rho v)_{a>0.1\,\mu\mathrm{m}} &= f_M(a > 0.1\,\mu\mathrm{m}) \times 0.01 n_{\rm H} m_{\rm H} v_{\rm ISW} \\ &= 0.428 \times 0.01 \times 0.22\,\mathrm{cm^{-3}} \times 1.67\times10^{-24}\,\mathrm{g} \times 26\,\mathrm{km\,s^{-1}} \\ &= 4.09\times10^{-21}\,\mathrm{g\,cm^{-2}\,s^{-1}} \quad .\end{aligned}$$

(c)
$$\begin{aligned} q &= U a = 5\,\mathrm{V} \times a = \left(\frac{5\,\mathrm{eV}}{e}\right) \times a \\ &= \frac{5\times1.60\times10^{-12}\,\mathrm{erg}}{4.80\times10^{-10}\,\mathrm{esu}} \times 10^{-5} a_{-5}\,\mathrm{cm} = 1.76\times10^{-7} a_{-5}\,\mathrm{esu} \end{aligned}$$

$$\omega_{\rm gyro} = \frac{qB}{Mc} = \frac{1.76\times10^{-7} a_{-5} \times 2\,\mu\mathrm{G}}{(4\pi/3)\rho a^3 c} = 1.40\times10^{-9} a_{-5}^{-2}\,\mathrm{s^{-1}}$$

$$\frac{2\pi}{\omega_{\rm gyro}} = 142\, a_{-5}^2\,\mathrm{yr} \quad .$$

In a frame of reference moving with the solar wind, the grain speed is $v = v_{\odot\mathrm{W}} + v_{\mathrm{ISW}} = (450 + 26)\,\mathrm{km\,s^{-1}}$. *The gyroradius is*

$$\begin{aligned} R_{\mathrm{gyro}} &= \frac{v}{\omega_{\mathrm{gyro}}} = 3.40 \times 10^{16} a_{-5}^2\,\mathrm{cm} \\ &= 2270\, a_{-5}^2\,\mathrm{AU} \quad . \end{aligned}$$

Therefore, grains with $a \gtrsim 10^{-5}\,\mathrm{cm}$ *are able to reach the inner solar system without substantial deflection by the iterplanetary magnetic field, but small grains will undergo large magnetic deflections.*

Chapter 27. Heating and Cooling of H II Regions

27.1 Consider an H II region consisting only of hydrogen. Suppose that the source of ionizing photons is a blackbody with temperature $T = 32000\,\mathrm{K}$. Assume the nebula is in thermal and ionization equilibrium.

(a) Near the center of the nebula, at what temperature will heating by photoionization balance cooling?

(b) Estimate the mass-weighted *average* temperature of the gas in the nebula.

Solution:

(a) Heating will balance cooling when

$$\Gamma_{\rm pe} = \Lambda_{\rm rr} + \Lambda_{\rm ff}$$

$$\alpha_B n_e^2 \psi_0 k_{\rm B} T_c = \alpha_B n_e^2 \left(0.68 k_{\rm B} T + 0.54 T_4^{0.37} k_{\rm B} T\right) \quad .$$

For a $T_c = 32000\,\mathrm{K}$ *star we have (from Table 27.1)* $\psi_0 = 0.864$*, thus*

$$0.864 T_c = T\left(0.68 + 0.54 T_4^{0.37}\right) \quad .$$

This can be solved in a few iterations to obtain $T = 2.00\times10^4\,\mathrm{K}$.

(b) The average heating per recombination in the nebula is $\langle\psi\rangle T_c$*, where* $\langle\psi\rangle = 1.380$ *(Table 27.1). To estimate the average temperature* T *we solve*

$$1.380 T_c = T\left(0.68 + 0.54 T_4^{0.37}\right) \quad .$$

For $T_c = 32000\,\mathrm{K}$ *the solution is* $T = 2.97\times10^4\,\mathrm{K}$. *The outer regions of the nebula, with* $\psi > \langle\psi\rangle$*, will be even hotter than this.*

27.3 Suppose that the cosmic ray flux within the Orion Nebula corresponds to a cosmic ray ionization rate $\zeta_{\rm CR} < 10^{-15}\,{\rm s}^{-1}$ for an H atom, with the ionization dominated by $\sim 1\,{\rm GeV}$ protons. Compare the heating rate due to plasma drag on the cosmic rays with the photolectric heating rate $\Gamma_{\rm pe}$. Assume $n_{\rm H} \approx 4000\,{\rm cm}^{-3}$ and $T \approx 8000\,{\rm K}$ for the gas, and assume the H is fully ionized.

Solution: From Eq. (27.11):

$$\begin{aligned}
\Gamma_{\rm CR} &\approx 4\times 10^{-10}\,{\rm erg}\,\zeta_{\rm CR}\,n_e \\
&\approx 4\times 10^{-10} \times 10^{-15} \times 4400\,{\rm erg\,cm^{-3}\,s^{-1}} \\
&= 1.8\times 10^{-21}\,{\rm erg\,cm^{-3}\,s^{-1}}. \\
\Gamma_{\rm pe} &\approx \alpha_B n_e^2 \langle\psi\rangle k_{\rm B} T_c \\
&\approx 3\times 10^{-13}(4400)^2 \times 1.45 \times k_{\rm B} \times 4\times 10^4 \\
&= 4.7\times 10^{-17}\,{\rm erg\,cm^{-3}\,s^{-1}} \quad .
\end{aligned}$$

Thus $\Gamma_{\rm pe}/\Gamma_{\rm CR} \approx 2.6\times 10^4 (10^{-15}\,{\rm s}^{-1}/\zeta_{\rm CR}) \gg 1$ *in Orion.*

Chapter 28. The Orion H II Region

28.1 The free-free emission from the Orion Nebula has been measured with radio telescopes. At $\nu = 1.4\,\mathrm{GHz}$ the integrated flux density from M42 is $F_\nu = 495\,\mathrm{Jy}$. Assuming a distance $D = 414\,\mathrm{pc}$, estimate the hydrogen photoionization rate $\dot{N}_L$ required to keep this gas ionized, if the gas temperature is $T = 9000\,\mathrm{K}$. Assume helium to be singly-ionized, with $n_{\mathrm{He}}/n_{\mathrm{H}} = 0.10$.

Solution: First, we use the free-free emissivity from Eq. (10.9) to determine the quantity $\int n_e n_i dV \equiv n_e n_i V$*:*

$$\begin{aligned}
F_\nu &= \frac{1}{4\pi D^2} L_\nu = \frac{1}{4\pi D^2}\, V\, 4\pi j_{\mathrm{ff},\nu} \\
&= \frac{1}{D^2} n_e n_i V \times 3.34\times10^{-40} \nu_9^{-0.118} T_4^{-0.323}\,\mathrm{erg\,cm^3\,s^{-1}\,Hz^{-1}} \\
n_e n_i V &= \frac{F_\nu D^2}{3.34\times10^{-40}\nu_9^{-0.118}T_4^{-0.323}\,\mathrm{erg\,cm^3\,s^{-1}\,Hz^{-1}}} \\
&= \frac{495\times10^{-23}\times(414\times3.086\times10^{18})^2}{3.34\times10^{-40}\times(1.4)^{-0.118}\times(0.9)^{-0.323}}\,\mathrm{cm^{-3}} \\
&= 2.43\times10^{61}\,\mathrm{cm^{-3}} \quad .
\end{aligned}$$

If $n(\mathrm{He}^+) = 0.1 n(\mathrm{H}^+)$, *and every ionizing photon leads to one* H *ionization, the* H *recombination rate is*

$$\begin{aligned}
\dot{N}_L &= \alpha_B\, n_e n_p V = \alpha_B\, n_e \frac{n_i}{1.1} V \\
&= 2.54\times10^{-13}\times(0.9)^{-0.8163-0.0208\ln(0.9)}\times\frac{2.43\times10^{61}\,\mathrm{s^{-1}}}{1.1} \\
&= 6.11\times10^{48}\,\mathrm{s^{-1}}.
\end{aligned}$$

Chapter 29. H I Clouds: Observations

29.1 Suppose the H I gas to be in a plane-parallel slab geometry, with full thickness $6 \times 10^{20}\,\mathrm{cm}^{-2}$, and take the velocity distribution be Gaussian with a one-dimensional velocity dispersion $\sigma_V = 10\,\mathrm{km\,s}^{-1}$. Neglect the effects of Galactic rotation.

(a) If the spin temperature is $T_{\rm spin} = 100\,\mathrm{K}$, for what galactic latitudes is the line-center optical depth $\tau < 0.5$, as seen from a point in the mid-plane?

(b) If the full-thickness of the H I disk is 300 pc, out to what radius (in the plane) can it be observed with line-center optical depth $\tau < 0.5$?

(c) What is the maximum $N(\mathrm{H\,I})$ that can be observed with $\tau < 0.5$ at all radial velocities?

Solution:

(a) The observed $N(\mathrm{H\,I})$ will be $N(\mathrm{H\,I}) = 3 \times 10^{20}\,\mathrm{cm}^{-2}/|\sin b|$. For a Gaussian velocity distribution, the optical depth in the 21-cm line is given by Eq. (8.11):

$$\tau(u) = 2.19 \frac{N(\mathrm{H\,I})}{10^{21}\,\mathrm{cm}^{-2}} \frac{100\,\mathrm{K}}{T_{\rm spin}} \frac{\mathrm{km\,s}^{-1}}{\sigma_V} e^{-u^2/2\sigma_V^2}$$

$$\tau(u=0) = 2.19 \times \frac{0.3}{|\sin b|} \times 1 \times \frac{1}{10} \;=\; 0.0657 \frac{1}{|\sin b|} \quad .$$

Therefore to have $\tau(0) < 0.5$, we require $|\sin b| > 2 \times 0.0657 = 0.131$, or $|b| > 7.55°$.

(b) From (a) we have $|b| > 7.55°$. The height $z = r \tan b$. At $b = 7.55°$, a sightline will reach a height $z = 150\,\mathrm{pc}$ at a distance $r = 150\,\mathrm{pc}/\tan 7.55° = 1.13\,\mathrm{kpc}$.

(c) $N(\mathrm{H\,I}) = 3 \times 10^{20}\,\mathrm{cm}^{-2}/\sin 7.55° = 2.28 \times 10^{21}\,\mathrm{cm}^{-2}$.

29.3 Suppose we observe a background radio continuum point source through a layer of "foreground" H I with $dN(\mathrm{H\,I})/du = 3 \times 10^{20}\,\mathrm{cm}^{-2}/(20\,\mathrm{km\,s}^{-1})$, where u is the radial velocity. If the measured flux density of the background continuum source changes by less than 1% on-line to off-line, what can be said about the spin temperature of the H I? Assume the beamsize is very small. You may use the result from problem 29.2:

$$\tau = 0.552 \left(\frac{100\,\mathrm{K}}{T_{\mathrm{spin}}}\right) \frac{dN(\mathrm{H\,I})/du}{10^{20}\,\mathrm{cm}^{-2}/(\,\mathrm{km\,s}^{-1})} \quad .$$

Solution: The flux $F_\nu = F_\nu(0)e^{-\tau_\nu}$.
If $(F_\nu(0) - F_\nu) < 0.01F_\nu(0)$, *then* $e^{-\tau_\nu} > 0.99$, *or* $\tau_\nu < -\ln(0.99) = 0.0101$.
Using the result from problem 29.2:

$$T_{\mathrm{spin}} = 55.2\,\mathrm{K} \times \frac{dN(\mathrm{H\,I})/du}{10^{20}\,\mathrm{cm}^{-2}/(\,\mathrm{km\,s}^{-1})} \times \frac{1}{\tau}$$

$$T_{\mathrm{spin}} > 55.2\,\mathrm{K} \times \frac{3}{20} \times \frac{1}{0.0101} = 820\,\mathrm{K} \quad .$$

Chapter 30. H I Clouds: Heating and Cooling

30.1 The local X-ray background (see Figure 12.1) can be approximated by

$$\nu u_\nu \approx 1\times10^{-18}\left(\frac{h\nu}{400\,\mathrm{eV}}\right)^{\beta}\ \mathrm{erg\,cm^{-3}}$$

for $400 \lesssim h\nu \lesssim 1\,\mathrm{keV}$.

(a) Using the photoionization cross section from eq. (13.3), obtain an expression for the rate for photoionization of H by the 0.4–1 keV X-ray background, showing explicitly the dependence on β. To keep the algebra simple, define $u_0 \equiv 10^{-18}\,\mathrm{erg\,cm^{-3}}$ and $\sigma_0 \equiv 6.3\times10^{-18}\,\mathrm{cm^2}$, and leave your result in terms of u_0, σ_0, and c.

(b) Evaluate the rate for $\beta = 2$. Is the photoionization rate dominated by the low-energy X-rays or the high-energy X-rays?

(c) What is the mean energy of the absorbed photons for the above X-ray spectrum?

(d) The photoelectrons resulting from X-ray ionization of H and He have sufficient energy to produce secondary ionizations. If the fractional ionization $x_e \approx 4\times10^{-4}$, use Eq. (13.6) to estimate the number of secondary ionizations per photoelectron.

Solution:

(a)

$$\begin{aligned}\zeta &= \int \frac{u_\nu d\nu}{h\nu}\, c\,\sigma \\ &\approx u_0\, c\,\sigma_0 \int_{0.4\,\mathrm{keV}}^{1\,\mathrm{keV}} dE\, E^{-2}\left(\frac{E}{0.4\,\mathrm{keV}}\right)^{\beta}\left(\frac{I_H}{E}\right)^3 \\ &= u_0\, c\,\sigma_0 \frac{1}{\mathrm{keV}}\int_{0.4}^{1}\frac{dx}{x^2}\left(\frac{x}{0.4}\right)^{\beta}\left(\frac{0.0136}{x}\right)^3 \\ &= \frac{u_0\, c\,\sigma_0\times(0.0136)^3}{\mathrm{keV}\times(0.4)^{\beta}}\frac{1}{(4-\beta)}\left(0.4^{\beta-4}-1\right) \quad .\end{aligned}$$

(b)

$$\zeta = 4.9\times10^{-21}\,\mathrm{s^{-1}}$$

For $\beta < 4$, the photoionization rate ζ is dominated by the lower-energy X-rays.

(c)

$$\begin{aligned}\langle h\nu\rangle &= \frac{\int d\nu\, u_\nu\,\sigma}{\int d\nu\,(u_\nu/h\nu)\,\sigma} = \mathrm{keV}\frac{\int_{0.4}^{1} dx\, x^{\beta-4}}{\int_{0.4}^{1} dx\, x^{\beta-5}} \\ &= \frac{(4-\beta)}{(3-\beta)}\frac{(0.4^{\beta-3}-1)}{(0.4^{\beta-4}-1)}\,\mathrm{keV} = 0.57\,\mathrm{keV} \quad \text{for } \beta = 2\end{aligned}$$

(d) For absorption by H and He the mean ionization energy will be $\sim 20\,\mathrm{eV}$, so the mean photoelectron energy will be $\sim 550\,\mathrm{eV}$. For $E_e = 550\,\mathrm{eV}$ and $x_e = 4\times10^{-4}$, Eq. (13.6) gives $\phi_s \approx 15.1$. Therefore the total ionization rate resulting from the X-rays would be $\sim 16.1\times4.9\times10^{-21}\,\mathrm{s^{-1}} \approx 7.9\times10^{-20}\,\mathrm{s^{-1}}$, which is small compared to the cosmic ray ionization rate.

Chapter 31. Molecular Hydrogen

31.1 The radiative attachment reaction

$$\mathrm{H} + e^- \rightarrow \mathrm{H}^- + h\nu$$

has a rate coefficient $k_{\rm ra} = 1.9\times10^{-16}T_2^{0.67}\,{\rm cm}^3\,{\rm s}^{-1}$. The associative detachment reaction

$$\mathrm{H}^- + \mathrm{H} \rightarrow \mathrm{H}_2 + e^-$$

is a fast ion-molecule reaction with rate coefficient $k_{\rm ad} = 1.3\times10^{-9}\,{\rm cm}^3\,{\rm s}^{-1}$, but H^- also undergoes photodetachment

$$\mathrm{H}^- + h\nu \rightarrow \mathrm{H} + e^-$$

with a rate $\zeta_{\rm pd} = 2.4\times10^{-7}\,{\rm s}^{-1}$ in the interstellar radiation field (rates are from Le Teuff et al. 2000: A&A Suppl., 146, 157).

Consider an H I cloud with density $n_{\rm H} = 30\,{\rm cm}^{-3}$ and electron density $n_e = 0.02\,{\rm cm}^{-3}$. The temperature is $T = 10^2 T_2\,{\rm K}$ (show the dependence of your results on T_2).

(a) What is the steady-state ratio $n(\mathrm{H}^-)/n_{\rm H}$?

(b) What fraction of the H^- ions undergo the reaction $\mathrm{H}^- + \mathrm{H} \rightarrow \mathrm{H}_2 + e^-$?

(c) Evaluate the quantity

$$R_{\mathrm{H}^-} \equiv \frac{k_{\rm ad}n(\mathrm{H}^-)n(\mathrm{H})}{n_{\rm H}n(\mathrm{H})}$$

Compare this to the empirical “rate coefficient” for formation of H_2 by dust grain catalysis.

Solution:

(a)

$$k_{\rm ra}n(\mathrm{H})n_e = n(\mathrm{H}^-)\left[k_{\rm ad}n(\mathrm{H}) + \zeta_{\rm pd}\right]$$

$$\frac{n(\mathrm{H}^-)}{n_{\rm H}} = \frac{k_{\rm ra}n(\mathrm{H})n_e}{n_{\rm H}\times\left[k_{\rm ad}n(\mathrm{H}) + \zeta_{pd}\right]} = \frac{1.9\times10^{-16}T_2^{0.67}\times 30\times 0.02}{30\times\left[1.3\times10^{-9}\times 30 + 2.4\times10^{-7}\right]} = 1.36\times10^{-11}T_2^{0.67} \quad .$$

(b)

$$\frac{k_{\rm ad}n(\mathrm{H})}{k_{\rm ad}n(\mathrm{H}) + \zeta_{pd}} = \frac{1.3\times10^{-9}\times 30}{1.3\times10^{-9}\times 30 + 2.4\times10^{-7}} = 0.14 \quad .$$

(c)

$$R_{\mathrm{H}^-} = \frac{k_{\rm ad}n(\mathrm{H})n(\mathrm{H}^-)}{n(\mathrm{H})n_{\rm H}} = k_{\rm ad}\frac{n(\mathrm{H}^-)}{n_{\rm H}} = 1.3\times10^{-9}\times 1.36\times10^{-11}T_2^{0.67} = 1.8\times10^{-20}T_2^{0.67}\,{\rm cm}^3\,{\rm s}^{-1} \quad .$$

This is smaller (by a factor ~1700) than the empirical rate coefficient $R \approx 3\times10^{-17}\,{\rm cm}^3\,{\rm s}^{-1}$ *for formation on grains.*

31.3 In the early universe, near redshift $z \approx 100$, the H nucleon density

$$n_{\rm H} \approx 0.20\,{\rm cm}^{-3} \quad ,$$

and the fractional ionization of hydrogen has dropped to

$$\frac{n_e}{n_{\rm H}} \approx 3 \times 10^{-4} \quad .$$

The CMB temperature is $T \approx 275\,{\rm K}$; the gas temperature is close to the CMB temperature. The radiative attachment reaction

$${\rm H} + e^- \rightarrow {\rm H}^- + h\nu$$

has a rate coefficient

$$k_{\rm ra} = 3.7\times10^{-16}\left(\frac{T}{275\,{\rm K}}\right)^{0.67} {\rm cm}^3\,{\rm s}^{-1} \quad .$$

The associative detachment reaction

$${\rm H}^- + {\rm H} \rightarrow {\rm H}_2 + e^-$$

is a fast ion-neutral reation with a rate coefficient

$$k_{\rm ad} = 1.3 \times 10^{-9}\,{\rm cm}^3\,{\rm s}^{-1} \quad ,$$

and $\rm H^-$ is also destroyed by

$${\rm H}^- + {\rm H}^+ \rightarrow {\rm H} + {\rm H} \quad ,$$

with a rate coefficient

$$k_{\rm n} \approx 7.8 \times 10^{-8}\left(\frac{T}{275\,{\rm K}}\right)^{-1/2} {\rm cm}^3\,{\rm s}^{-1} \quad .$$

(a) If the Universe were not continuing to expand and recombine, what would be the steady-state density $n({\rm H}^-)$ for $n_{\rm H} = 0.20\,{\rm cm}^{-3}$, $T = 275\,{\rm K}$, and $n_e/n_{\rm H} = 3\times10^{-4}$?

(b) Assuming this steady-state abundance of $\rm H^-$, calculate the rate per volume of $\rm H_2$ formation ($\rm H_2\,cm^{-3}\,s^{-1}$) at this time.

Solution:

(a) We have $n({\rm H}) = 0.20\,{\rm cm}^{-3}$, $n_e = n({\rm H}^+) = 3\times10^{-4} n_{\rm H} = 6\times10^{-5}\,{\rm cm}^{-3}$.
In steady state statistical equilibrium,

$$\begin{aligned}
{\rm H}^-\ \text{formation} &= {\rm H}^-\ \text{destruction} \\
k_{\rm ra} n({\rm H}) n_e &\approx \left[k_{\rm ad} n({\rm H}) + k_{\rm n} n({\rm H}^+)\right] n({\rm H}^-) \\
n({\rm H}^-) &= \frac{k_{\rm ra} n({\rm H}) n_e}{k_{\rm ad} n({\rm H}) + k_{\rm n} n({\rm H}^+)} \\
&\approx \frac{3.7\times10^{-16} \times 0.20 \times 6\times10^{-5}\,{\rm cm}^{-3}\,{\rm s}^{-1}}{(1.3\times10^{-9} \times 0.20 + 7.8\times10^{-8} \times 6\times10^{-5})\,{\rm s}^{-1}} \\
&\approx 1.7\times10^{-11}\,{\rm cm}^{-3} \quad \text{at} \quad z = 100 \quad .
\end{aligned}$$

(b) The rate/volume of H_2 *formation is*

$$\begin{aligned}\left[\frac{dn(\mathrm{H}_2)}{dt}\right]_{\mathrm{form}} &= k_{\mathrm{ad}} n(\mathrm{H}) n(\mathrm{H}^-) \\ &= 1.3\times10^{-9} \times 0.20 \times 1.7\times10^{-11}\,\mathrm{cm}^{-3}\,\mathrm{s}^{-1} \\ &= 4.4\times10^{-21}\,\mathrm{cm}^{-3}\,\mathrm{s}^{-1} \quad .\end{aligned}$$

Chapter 32. Molecular Clouds: Observations

32.1 The mass distribution of GMCs in the Galaxy is given by [eq. (32.1) in the textbook]:

$$\frac{dN_{\rm GMC}}{d\ln M_{\rm GMC}} \approx N_u \left(\frac{M_{\rm GMC}}{M_u}\right)^{-\alpha} \qquad 10^3\,M_\odot \lesssim M_{\rm GMC} < M_u$$

with $M_u \approx 6\times10^6\,M_\odot$, $N_u \approx 63$, and $\alpha \approx 0.6$ (Williams & McKee 1997, Astrophys. J. 476, 166).

(a) Calculate the total mass in GMCs in the Galaxy.

(b) Calculate the number of GMCs in the Galaxy with $M > 10^6\,M_\odot$.

Solution:

(a)

$$\begin{aligned} M_{\rm tot} &= \int_{10^3 M_\odot}^{M_u} M\frac{dN}{dM}dM = \int_{10^3 M_\odot}^{M_u} \frac{dN}{d\ln M}\,dM \\ &= N_u \int_{10^3 M_\odot}^{M_u} \left(\frac{M}{M_u}\right)^{-\alpha} dM = N_u M_u \int_{10^3\,M_\odot/M_u}^{1} x^{-\alpha}dx \qquad\qquad x \equiv \frac{M}{M_u} \\ &= \frac{N_u M_u}{1-\alpha}\left[1-\left(\frac{10^3\,M_\odot}{M_u}\right)^{1-\alpha}\right] \\ &= 9.2\times10^8\,M_\odot \quad . \end{aligned}$$

(b)

$$\begin{aligned} N(>10^6\,M_\odot) &= \int_{10^6 M_\odot}^{M_u} \frac{dN}{dM}dM = \int_{10^6 M_\odot}^{M_u} \frac{dN}{M\,d\ln M}dM = \int_{10^6 M_\odot/M_u}^{1} N_u x^{-\alpha}\frac{dx}{x} \\ &= \frac{N_u}{\alpha}\left[\left(\frac{10^6\,M_\odot}{M_u}\right)^{-\alpha} - 1\right] \approx 203 \quad . \end{aligned}$$

Chapter 33. Molecular Clouds: Chemistry and Ionization

33.1 Consider a diffuse molecular cloud with $n_{\rm H} = 10^2\,{\rm cm}^{-3}$. The hydrogen is predominantly molecular, with $n({\rm H}_2) = 50\,{\rm cm}^{-3}$. Assume that 30% of the total C (250 ppm) abundance is in C$^+$: $n({\rm C}^+) \approx 7.5\times10^{-5} n_{\rm H} = 7.5\times10^{-3}\,{\rm cm}^{-3}$. Assume that $n_e \approx 10^{-4} n_{\rm H} = 0.01\,{\rm cm}^{-3}$. Assume that $n({\rm O})/n_{\rm H} \approx 4\times10^{-4} = 0.04\,{\rm cm}^{-3}$. Treat $T_2 \equiv T/10^2\,{\rm K}$ as a free parameter.

Consider the reactions in the reaction network (33.6-33.13).

(a) Calculate the steady-state abundance of ${\rm CH}_2^+$.

(b) Calculate the steady-state abundance of ${\rm CH}$.

(c) Calculate the steady-state abundance of CO, leaving $f_{\rm shield}({\rm CO})$ as a free parameter. What fraction of all of the carbon is in CO?

Solution:

(a) Balance formation and destruction of ${\rm CH}_2^+$*:*

$$
\begin{aligned}
k_{33.6} n({\rm C}^+) n({\rm H}_2) &= k_{33.7} n({\rm CH}_2^+) n_e \\
\frac{n({\rm CH}_2^+)}{n_{\rm H}} &= \frac{k_{33.6} n({\rm C}^+) n({\rm H}_2)}{k_{33.7} n_e n_{\rm H}} \\
&= \frac{5\times10^{-16} T_2^{-0.2} \times 7.5\times10^{-3} \times 50}{1.24\times10^{-6} T_2^{-0.6} \times 10^{-2} \times 10^2} \\
&= 1.51\times10^{-10} T_2^{0.4} \quad .
\end{aligned}
$$

(b) Balance formation and destruction of ${\rm CH}$*:*

$$
\begin{aligned}
0.25\, k_{33.7}\, n_e n({\rm CH}_2^+) &= (k_{33.8} n({\rm O}) + k_{33.13}) n({\rm CH}) \\
\frac{n({\rm CH})}{n_{\rm H}} &= \left[\frac{0.25 k_{33.7} n_e}{k_{33.8} n({\rm O}) + k_{33.13}}\right] \frac{n({\rm CH}_2^+)}{n_{\rm H}} \\
&= \left[\frac{0.25 \times 1.24\times10^{-6} T_2^{-0.6} \times 0.01}{6.6\times10^{-11} \times 0.04 + 1.62\times10^{-9}}\right] \times 1.51\times10^{-10} T_2^{0.4} \\
&= 2.88\times10^{-10} T_2^{-0.4} \quad .
\end{aligned}
$$

(c) Balance formation and destruction of CO*:*

$$
\begin{aligned}
k_{33.8}n(\mathrm{CH})n(\mathrm{O}) &= k_{33.9}n(\mathrm{CO}) \\
\frac{n(\mathrm{CO})}{n_\mathrm{H}} &= \frac{k_{33.8}n(\mathrm{O})}{k_{33.9}}\frac{n(\mathrm{CH})}{n_\mathrm{H}} \\
&= \frac{6.6\times10^{-11}\times 0.04}{2.3\times10^{-10}f_\mathrm{shield}(\mathrm{CO})}\times 2.88\times10^{-10}T_2^{-0.4} \\
&= 3.31\times10^{-12}\,\frac{1}{f_\mathrm{shield}(\mathrm{CO})}\,T_2^{-0.4} \\
\frac{n(\mathrm{CO})}{2.5\times10^{-4}n_\mathrm{H}} &= 1.32\times10^{-8}\,\frac{1}{f_\mathrm{shield}(\mathrm{CO})}\,T_2^{-0.4} \quad .
\end{aligned}
$$

Therefore, unless $f_\mathrm{shield}(\mathrm{CO}) \ll 1$*, the* CO *abundance will be very low. If* $f_\mathrm{shield}(\mathrm{CO}) \approx 1$*, then the* CO *will contain only* $\sim10^{-8}$ *of the carbon.*

33.3 Consider a hypothetical molecule XH^+. Suppose that the principal channel for its formation in a diffuse cloud is the radiative association reaction

$$\mathrm{X}^+ + \mathrm{H} \rightarrow \mathrm{XH}^+ + h\nu$$

with a rate coefficient $k_{ra} = 5\times10^{-17}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$. Suppose that the two principal reactions for destroying XH^+ are dissociative recombination

$$\mathrm{XH}^+ + e^- \rightarrow \mathrm{X} + \mathrm{H}$$

with a rate coefficient $k_{dr} = 2\times10^{-7}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ and photodissociation

$$\mathrm{XH}^+ + h\nu \rightarrow \mathrm{X}^+ + \mathrm{H}$$

with a rate $\beta = 5\times10^{-10}\,\mathrm{s}^{-1}$ due to the ambient starlight background.

(a) If only these processes act, compute the steady-state density n_s of XH^+ in a diffuse cloud with $n(\mathrm{H}) = 20\,\mathrm{cm}^{-3}$, $n(\mathrm{X}^+) = 5\times10^{-3}\,\mathrm{cm}^{-3}$, and $n_e = 0.01\,\mathrm{cm}^{-3}$.

(b) Suppose that at time $t = 0$ we have $n(\mathrm{XH}^+) = n_s + \Delta_0$. Assume that $n(\mathrm{H})$, $n(\mathrm{X}^+)$, and n_e can all be approximated as constant. It is easy to show that for $t > 0$, $n(\mathrm{XH}^+) = n_s + \Delta_0 e^{-t/\tau}$. Calculate the value of τ.

Solution:

(a) Balance formation and destruction of XH^+*:*

$$\begin{aligned}
k_{ra}n(\mathrm{X}^+)n(\mathrm{H}) &= k_{dr}n(\mathrm{XH}^+)n_e + \beta n(\mathrm{XH}^+) \\
n(\mathrm{XH}^+) &= \frac{k_{ra}n(\mathrm{X}^+)n(\mathrm{H})}{k_{dr}n_e + \beta} \\
&= \frac{5\times10^{-17} \times 5\times10^{-3} \times 20\,\mathrm{cm}^{-3}\,\mathrm{s}^{-1}}{2\times10^{-7} \times 0.01\,\mathrm{s}^{-1} + 5\times10^{-10}\,\mathrm{s}^{-1}} \\
&= \frac{5\times10^{-18}\,\mathrm{cm}^{-3}\,\mathrm{s}^{-1}}{2.5\times10^{-9}\,\mathrm{s}^{-1}} = 2\times10^{-9}\,\mathrm{cm}^{-3}
\end{aligned}$$

(b) Let $n(\mathrm{XH}^+) = n_s + \Delta(t)$.

$$\begin{aligned}
\frac{d}{dt}(n_s + \Delta) &= k_{ra}n(\mathrm{X}^+) - n(\mathrm{XH}^+)(k_{dr}n_e + \beta) \\
\frac{d}{dt}\Delta &= k_{ra}n(\mathrm{X}^+) - (k_{dr}n_e + \beta)(n_s + \Delta) \\
&= -(k_{dr}n_e + \beta)\Delta \\
\Delta &= \Delta_0 e^{-t/\tau} \\
\text{where}\quad \tau &= \frac{1}{k_{dr}n_e + \beta} = \frac{1}{2.5\times10^{-9}\,\mathrm{s}^{-1}} = 4\times10^{8}\,\mathrm{s}
\end{aligned}$$

Chapter 34. Physical Processes in Hot Gas

34.1 Consider a spherical cloud of radius R_c immersed in hot gas with temperature and density (far from the cloud) T_h and n_h. In the regime where classical evaporation applies, and the evaporative mass loss rate is $\dot{M} = 16\pi\mu R_c \kappa_h / 25 k_{\rm B}$, estimate the velocity $v(r)$ of the evaporative flow. Express your answer for $v(r)$ in terms of n_h, T_h, κ_h, R_c, and (r/R_c).

Solution: The conductive energy flux and the advective energy flux must be equal in magnitude (opposite in direction). The thermal profile is (from Eq. 34.11)

$$\begin{aligned}
T(r) &= T_h\,(1 - R_c/r)^{2/5} \\
The\ flow\ has\ nT \approx const,\ hence: \qquad n(r) &= n_h\,(1 - R_c/r)^{-2/5} \\
n(r)v(r)\frac{5}{2}k_{\rm B}T(r)4\pi r^2 &= \kappa_h (T/T_h)^{5/2}\left[\frac{d}{dr}T_h(1 - R_c/r)^{2/5}\right]4\pi r^2 \\
v(r) &= \frac{\kappa_h T_h (1 - R_c/r)\left[\frac{d}{dr}(1 - R_c/r)^{2/5}\right]}{n_h(1 - R_c/r)^{-2/5}(5/2)k_{\rm B}T_h(1 - R_c/r)^{2/5}} \\
&= \frac{\kappa_h (2/5)(1 - R_c/r)^{2/5}(R_c/r^2)}{(5/2)n_h k_{\rm B}} \\
&= \left(\frac{4\kappa_h}{25 n_h k_{\rm B} R_c}\right)\frac{(1 - R_c/r)^{2/5}}{(r/R_c)^2} \quad .
\end{aligned}$$

34.3 Suppose that hot interstellar gas contains dust grains of radius $a = 1 \times 10^{-5}\,\mathrm{cm}$ and number density $n_{\rm gr} = 2 \times 10^{-12} n_{\rm H}$. Suppose that the grains are uncharged, and that every ion or electron that collides with the grain surface transfers a fraction α of its original kinetic energy to the grain, which then cools radiatively.

Estimate $\Lambda =$ the rate per volume at which the gas loses thermal energy due to this process, for density $n_{\rm H} = n_0\,\mathrm{cm}^{-3}$ and temperature $T = 1 \times 10^7 T_7\,\mathrm{K}$. Assume the H and He to be fully ionized, and He/H=0.1. Give your answer in terms of α, n_0 and T_7.

Solution: The mean speed of particles is $(8kT/\pi m_i)^{1/2}$. The mean kinetic energy per impacting particle is $2kT$. Thus the rate per volume at which the plasma loses energy is

$$\begin{aligned}
\Lambda &= \sum_i n_i \left(\frac{8kT}{\pi m_i}\right)^{1/2} 2kT \times \alpha \times n_{\rm gr} \pi a^2 \\
&= \left(1 + 0.1 \times \frac{1}{\sqrt{4}} + 1.2 \times \sqrt{1836}\right) n_{\rm H} \left(\frac{8kT}{\pi m_{\rm H}}\right)^{1/2} 2kT\,\alpha\, n_{\rm H} \left(\frac{n_{\rm gr}}{n_{\rm H}}\right) \pi a^2 \\
&= 52.5\, {n_{\rm H}}^2 \left(\frac{8kT}{\pi m_{\rm H}}\right)^{1/2} 2\alpha kT \left(\frac{n_{\rm gr}}{n_{\rm H}}\right) \pi a^2 \\
&= 4.17 \times 10^{-21}\, \alpha\, n_0^2\, T_7^{3/2}\,\mathrm{erg\,cm^{-3}\,s^{-1}}
\end{aligned}$$

Chapter 35. Fluid Dynamics

35.1 Show that the term $(c/4\pi\sigma)\nabla \times \partial\mathbf{D}/\partial t$ that has been omitted in Eq. (35.46) is smaller than $(c^2/4\pi\sigma)\nabla^2\mathbf{B}$ by a factor $\sim (v/c)^2$, where v is a characteristic velocity in the flow.

Solution: If L is a characteristic length scale, and v is a characteristic velocity, then the characteristic time scale will be L/v. If the E field vanishes in the frame moving with the fluid, then in our frame $E \approx (v/c)B$. The magnitude of the term that has been omitted is

$$\begin{aligned}\frac{c}{4\pi\sigma}\nabla \times \frac{\partial D}{\partial t} &\sim \frac{c}{4\pi\sigma}\frac{1}{L}\frac{(v/c)B}{L/v} \\ &\sim \frac{v^2}{c^2} \times \frac{c^2}{4\pi\sigma}\frac{B}{L^2} \\ &\sim \frac{v^2}{c^2} \times \frac{c^2}{4\pi\sigma}\nabla^2 B \quad .\end{aligned}$$

35.3 The "cooling time" $\tau_{\rm cool} \equiv |d\ln T/dt|^{-1}$. Suppose the power radiated per unit volume Λ can be approximated by

$$\Lambda \approx A n_{\rm H} n_e \left[T_6^{-0.7} + 0.021 T_6^{1/2}\right]$$

for gas of cosmic abundances, where $A = 1.1\times10^{-22}\,{\rm erg\,cm^3\,s^{-1}}$, and $T_6 \equiv T/10^6\,{\rm K}$. Assume the gas to have $n_{\rm He} = 0.1 n_{\rm H}$, with both H and He fully ionized.

Compute the cooling time (at constant pressure) due to radiative cooling

(a) in a supernova remnant at $T = 10^7\,{\rm K}$, $n_{\rm H} = 10^{-2}\,{\rm cm^{-3}}$.

(b) for intergalactic gas within a dense galaxy cluster (the "intracluster medium") with $T = 10^8\,{\rm K}$, $n_{\rm H} = 10^{-3}\,{\rm cm^{-3}}$.

Solution:

(a) $n_{\rm H} = 0.01\,{\rm cm^{-3}}$, $n_e = 0.012\,{\rm cm^{-3}}$, $n = 0.023\,{\rm cm^{-3}}$

$$\begin{aligned}\tau_{\rm cool} &= \frac{5nkT}{2\Lambda} = \frac{5\times0.023\times1.38\times10^{-16}\times10^7\,{\rm erg\,cm^{-3}}}{2\times0.01\times0.012\times1.1\times10^{-22}(10^{-0.7}+0.021\times10^{0.5})\,{\rm erg\,cm^{-3}\,s^{-1}}}\\ &= 2.26\times10^{16}\,{\rm s} = 7.16\times10^8\,{\rm yr} \quad .\end{aligned}$$

(b) $n_{\rm H} = 0.001\,{\rm cm^{-3}}$, $n_e = 0.0012\,{\rm cm^{-3}}$, $n = 0.0023\,{\rm cm^{-3}}$.

$$\begin{aligned}\tau_{\rm cool} &= \frac{5nkT}{2\Lambda} = \frac{5\times0.0023\times1.38\times10^{-16}\times10^8\,{\rm erg\,cm^{-3}}}{2\times0.001\times0.0012\times1.1\times10^{-22}(10^{-1.4}+0.21)\,{\rm erg\,cm^{-3}\,s^{-1}}}\\ &= 2.41\times10^{18}\,{\rm s} = 7.62\times10^{10}\,{\rm yr} \gg \text{Hubble time.}\end{aligned}$$

Chapter 36. Shock Waves

36.1 Consider a strong shock wave propagating into a medium that was initially at rest. Assume the gas to be monatomic ($\gamma = 5/3$). Consider the material just behind the shock front. The gas has an energy density $u_{\rm thermal}$ from random thermal motions, and an energy density $u_{\rm flow}$ from the bulk motion of the shocked gas. If cooling is negligible, calculate the ratio $u_{\rm flow}/u_{\rm thermal}$ in the frame of reference where the shock front is stationary.

Solution: For a strong shock with $\gamma = 5/3$, the temperature of the shocked gas is

$$kT_s = \frac{3\mu v_s^2}{16}$$

In the frame of reference where the shock is stationary, the postshock flow velocity is

$$v = \frac{v_s}{4}$$

and

$$\frac{u_{\rm flow}}{u_{\rm thermal}} = \frac{n\mu v^2/2}{(3/2)nkT} = \frac{n\mu(v_s/4)^2/2}{(3/2)n \times (3/16)\mu v_s^2} = \frac{1}{9} \quad .$$

In the shock frame, the bulk kinetic energy is only 10% of the total fluid energy.

For comparison, in the frame of reference where the preshock gas is at rest,

$$v = \frac{3v_s}{4}$$

$$\frac{u_{\rm flow}}{u_{\rm thermal}} = \frac{n\mu v^2/2}{(3/2)nkT} = \frac{n\mu(3v_s/4)^2/2}{(3/2)n \times (3/16)\mu v_s^2} = 1 \quad .$$

In this frame the bulk kinetic energy of the flow is equal to the thermal kinetic energy within the flow.

36.3 Suppose that a shock wave propagates at velocity v_s through a fluid with preshock number density n_0, preshock temperature $T_0 = 0$, and preshock magnetic field $B_0 = 0$. Take the fluid to be a monatomic ideal gas of molecular weight μ.

(a) What is the density n_s just behind the shock?

(b) What is the temperature T_s just behind the shock?

(c) What is the ratio of the thermal pressure $n_s kT_s$ to the preshock "ram pressure" $n_0 \mu v_s^2$?

(d) Suppose that the postshock gas is subject to radiative cooling with a loss rate per unit volume $\Lambda = An^2T^\alpha$, where A and α are constants. Assume that the shock is steady and plane-parallel, neglect heat conduction, and make the simplifying assumption that the postshock cooling occurs at constant pressure, i.e., $nT = n_sT_s$.
For what values of α does a fluid element cool to $T = 0$ in a finite time $t_{\rm cool}$ after being shocked? Obtain a formula for $t_{\rm cool}$ as a function of n_s, T_s, A, and α. Would this hold true for bremsstrahlung cooling, in particular?

(e) With the same assumptions as in (c), for what values of α does the fluid element cool to $T = 0$ within a finite distance $x_{\rm cool}$ of the shock front?
Hint: Remember that the distance x traveled from the shock and the time t elapsed since passing through the shock are related by $dx = v\,dt$, where v is related to the shock speed v_s through mass conservation, $nv = n_0v_s$. Thus $dx = (n_0/n)v_s\,dt$.

Solution:

(a) Strong shock with $\gamma = 5/3$: $n_s = 4n_0$.

(b) Strong shock with $\gamma = 5/3$: $T_s = (3/16)\mu v_s^2/k$.

(c)
$$\frac{n_skT_s}{n_0\mu v_s^2} = \frac{4n_0(3/16)\mu v_s^2}{n_0\mu v_s^2} = \frac{3}{4} \quad .$$

(d) Cooling at constant pressure (from Eq. 35.29):

$$\begin{aligned}
\frac{dT}{dt} &= \frac{-\Lambda}{(5/2)nk} = \frac{-An^2T^\alpha}{(5/2)nk} = \frac{-AnT^\alpha}{(5/2)k} \\
&= \frac{-A(n_sT_s/T)T^\alpha}{(5/2)k} \qquad \text{(since } nT = n_sT_s\text{)} \\
&= \frac{-An_sT_sT^{\alpha-1}}{(5/2)k} \\
-T^{1-\alpha}dT &= \frac{An_sT_s}{(5/2)k}dt \\
\frac{T_s^{2-\alpha} - T^{2-\alpha}}{(2-\alpha)} &= \frac{An_sT_s}{(5/2)k}(t - t_s) \\
t_{\rm cool} &= \frac{(5/2)k}{An_sT_s}\frac{T_s^{2-\alpha}}{(2-\alpha)} = \frac{5k}{8An_0}\frac{T_s^{1-\alpha}}{(2-\alpha)} \quad .
\end{aligned}$$

Thus the gas will cool to $T = 0$ *in a finite time* $t_{\rm cool}$ *provided* $\alpha < 2$. *Bremsstrahlung cooling has* $\Lambda \propto n^2 T^{1/2}$, *hence* $\alpha = 1/2$, *satisfying the condition* $\alpha > -2$ *for cooling to* $T = 0$ *in a finite time.*

(e) Cooling at constant pressure:

$$
\begin{aligned}
\frac{dT}{dx} &= \frac{1}{v}\frac{dT}{dt} = \frac{1}{v} \times \frac{-An^2T^\alpha}{(5/2)nk} \\
&= \frac{-An^2T^\alpha}{(5/2)n_0 v_s k} \qquad \text{since } nv = n_0 v_s \\
&= \frac{-An_s^2 T_s^2 T^{\alpha-2}}{(5/2)n_0 v_s k} \qquad \text{since } n = n_s T_s/T \\
-T^{2-\alpha} dT &= \frac{An_s^2 T_s^2}{(5/2)n_0 v_s k} dx \\
\frac{T_s^{3-\alpha} - T^{3-\alpha}}{(3-\alpha)} &= \frac{An_s^2 T_s^2}{(5/2)n_0 v_s k}(x - x_s) \\
x_{\rm cool} &= \frac{(5/2)n_0 v_s k}{An_s^2 T_s^2}\frac{T_s^{3-\alpha}}{(3-\alpha)} = \frac{5 v_s k}{32 A n_0}\frac{T_s^{1-\alpha}}{(3-\alpha)} \quad .
\end{aligned}
$$

Thus the gas will (formally) cool to $T = 0$ *in a finite distance* $x_{\rm cool}$ *provided* $\alpha < 3$. *However, we have seen above that the cooling will only occur in a finite time if* $\alpha < 2$. *Hence* $\alpha < 2$ *is the necessary condition for a physical solution.*

36.5 Consider a strong shock with velocity v_s propagating into a monatomic ($\gamma = 5/3$) gas. The preshock gas contains dust grains that are at rest relative to the gas. Immediately after passage of the shock front, the grains still have their original velocity. What is the velocity of the grains relative to the shocked gas?

Solution: For a strong shock in $\gamma = 5/3$ gas, the compression ratio is $(\gamma + 1)/(\gamma - 1) = 4$.
In the frame where the shock is at rest, the preshock gas is moving at velocity v_s, and the postshock gas is moving at velocity $v_s/4$.
The postshock grains are moving at velocity v_s.
Hence the velocity of the grains relative to the shocked gas is $v_s - v_s/4 = (3/4)v_s$.

Chapter 37. Ionization/Dissociation Fronts

37.1 Eq. (37.15) gives the velocity V_i of the ionization front propagating outward from a source of ionizing photons that turned on at $t = 0$ in a uniform, initially neutral, medium.

Consider early times when $t/\tau \ll 1$. For $Q_0 = 10^{48}\,\mathrm{s}^{-1}$ and $n_0 = 10^3\,\mathrm{cm}^{-3}$, evaluate V_i for $t/\tau = 10^{-4}$. Discuss the physical significance of the result, and comment on the validity of the analysis leading to this result.

Solution: Eq. (37.16):

$$\begin{aligned} V_i &= 842\, Q_{0,48}^{1/3}\, n_3^{1/3} \frac{\mathrm{e}^{-t/\tau}}{(1 - \mathrm{e}^{-t/\tau})^{2/3}} \\ &\approx 842\, Q_{0,48}^{1/3}\, n_3^{1/3} \left(\frac{t}{\tau}\right)^{-2/3} \mathrm{km\,s^{-1}} \qquad \text{for } t/\tau \ll 1 \\ &= 3.91\times 10^5\, Q_{0,48}^{1/3}\, n_3^{1/3} \left(\frac{t/\tau}{10^{-4}}\right)^{-2/3} \mathrm{km\,s^{-1}} \quad , \end{aligned}$$

resulting in $V_i > c$ for $Q_{0,48} = 1$, $n_3 = 1$, and $t/\tau = 10^{-4}$. This is clearly unphysical – the ionization front cannot propagate faster than light.

Eq. (37.15) is an exact solution to Eq. (37.12), so why did we obtain an unphysical result?

The reason is that Eq. (37.12) assumes that the photoionization rate is exactly equal to the rate of emission of ionizing photons Q_0 – in effect, the speed of light is being taken to be infinite. However, this is not the case: it takes time for photons to travel from the source to the point of absorption. Eq. (37.12) should be modified to include a term representing the time derivative of the number of not-yet-absorbed photons "in transit" within the H II region. This modification would eliminate the unphysical behavior in the solution. However, if we ignore these very early times and restrict attention to times when $V_i \ll c$, the present analysis provides satisfactory accuracy.

Chapter 38. Stellar Winds

38.1 Suppose that a star has spent 10^6 yr as a red supergiant with a mass loss rate $\dot{M}_{rg} = 10^{-6}\,M_\odot\,\mathrm{yr}^{-1}$ and a wind velocity $v_{rg} = 10\,\mathrm{km\,s^{-1}}$. At time $t = 0$ the star suddenly begins producing a fast wind with $\dot{M}_{fw} = 10^{-7}\,M_\odot\,\mathrm{yr}^{-1}$ and $v_{fw} = 10^3\,\mathrm{km\,s^{-1}}$. Assume that radiative cooling and heat conduction are negligible. The resulting structure will contain four zones:

1. unshocked fast wind;
2. shocked fast wind;
3. shocked slow wind;
4. unshocked slow wind.

So long as the shock has not reached the outer boundary of the slow wind, the radius of the (outer) shock wave propagating into the unshocked slow wind material will vary as some power of t: $R_{sw} \propto t^\alpha$. You can use simple dimensionless analysis to obtain the value of α.

Proceed by assuming that the radius $R_{sw}(t)$ of the shock wave propagating into the slow wind material varies as some power of time: $R_{sw}(t) \propto t^\alpha$. If $M_{sw}(t)$ is the mass of shocked slow wind material (i.e., slow wind material that has been overtaken by the shock front), this will also vary as some power of time; similarly, the kinetic energy of the shocked slow wind material will increase as a power of time. Since we have assumed that there are no radiative losses, the total energy (kinetic energy of the ordered motion plus thermal kinetic energy) $E_{sw}(t)$ of the shocked slow wind material must be some (constant) fraction of the energy input from the fast wind up to time t; use this to determine the value of α. Let $E(t) = (1/2)\dot{M}_{fw}v_{fw}^2 t$ be the total energy input from the fast wind up to time t. If you now assume that $E_{sw}(t)$ is some (as yet unknown, but constant) fraction β of $E(t)$ [i.e., $E_{sw}(t) = \beta E(t)$], you can now obtain an estimate of $R_{sw}(t)$.

(a) Use simple "dimensional analysis" to determine the value of the power-law index α.

(b) Estimate the radius R_{sw} of the region of shocked slow wind at $t = 10^4$ yr.

(c) Estimate the temperature of the shocked slow wind material. (Assume the gas to be fully-ionized with He/H=0.1).

(d) Estimate the temperature of the shocked fast wind material.

Solution:

(a) Let R_{sw} be the radius of the shock front propagating into the (slow) red giant (rg) wind. Assume

$$R_{sw} = At^\alpha \quad .$$

The speed of the outer shock front relative to the star is

$$v_{sw} \equiv \frac{dR_{sw}}{dt} = \alpha A t^{\alpha-1} = \alpha\frac{R_{sw}}{t} \quad .$$

The velocity of the outer shock front relative to the rg wind (the "shock speed") is

$$v_{sw} - v_{rg} = \alpha\frac{R_{sw}}{t} - v_{rg} \quad .$$

In the frame where the shock front is locally stationary, the rg wind flows toward the shock at speed $v_{sw} - v_{rg}$, is compressed by a factor 4, and the shocked gas flows away from the shock at speed $(v_{sw} - v_{rg})/4$. The radial velocity of the shocked rg wind just interior to the outer shock front is

$$V = \alpha \frac{R_{sw}}{t} - \frac{1}{4}\left[\alpha \frac{R_{sw}}{t} - v_{rg}\right] = \frac{3\alpha R_{sw}}{4t} - \frac{1}{4} v_{rg} \quad .$$

The swept-up mass from the rg wind has mass

$$M_{sw} = \dot{M}_{rg}\left[\frac{R_{sw}}{v_{rg}} - t\right] \quad .$$

Assume

- *A fraction β of the energy from the fast wind is transferred to the shocked red giant wind.*
- *The shocked rg wind has energy divided equally between thermal and ordered kinetic energy.*
- $v_{sw} \equiv dR_{sw}/dt \gg v_{rg}$.
- *The rms radial velocity V of the shocked red giant wind material is equal to the radial velocity of the gas just interior to the shock front: $V \approx 0.75 v_{sw}$.*

Then

$$M_{sw} \approx \dot{M}_{rg} \frac{R_{sw}}{v_{rg}}$$

$$\begin{aligned}
\beta \frac{1}{2} \dot{M}_{fw} v_{fw}^2 t &\approx 2 \times \frac{1}{2} M_{sw} V^2 \\
&\approx M_{sw}\left(\frac{3\alpha R_{sw}}{4t}\right)^2 \\
&\approx \frac{\dot{M}_{rg}}{v_{rg}} \frac{9\alpha^2}{16t^2} R_{sw}^3 \\
&\approx \frac{\dot{M}_{rg}}{v_{rg}} \frac{9\alpha^2}{16} A^3\, t^{3\alpha-2}
\end{aligned}$$

Matching powers of t: $1 = 3\alpha - 2 \;\rightarrow\; \alpha = 1$.
With $\alpha = 1$, we estimate

$$\begin{aligned}
v_{sw} = A &\approx \left(\frac{8\beta \dot{M}_{fw} v_{fw}^2 v_{rg}}{9\dot{M}_{rg}}\right)^{1/3} \\
&= \left(\frac{8\beta}{9}\right)^{1/3} \left(\frac{\dot{M}_{fw}}{\dot{M}_{rg}}\right)^{1/3} \left(v_{fw}^2 v_{rg}\right)^{1/3} \\
&= 76.3(\beta/0.5)^{1/3}\,\mathrm{km\,s^{-1}} \quad .
\end{aligned}$$

$$R_{sw} = At \;\approx\; 2.4 \times 10^{18} \left(\frac{\beta}{0.5}\right)^{1/3} \left(\frac{t}{10^4\,\mathrm{yr}}\right)\ \mathrm{cm} \quad .$$

(b) *If the shock surface is expanding outward at 76.3* $\mathrm{km\,s^{-1}}$, *and the red giant wind speed is 10* $\mathrm{km\,s^{-1}}$, *then the shock speed* $v_s = 66.3\,\mathrm{km\,s^{-1}}$, *and the shock temperature is*

$$T_s = \frac{3\mu v_s^2}{16k} = 6.1 \times 10^4\,\mathrm{K} \quad (setting\ \mu = \frac{1.4 m_{\mathrm{H}}}{2.3}\ )$$

(c) *Let* R_{fws} *be the radius of the surface where the fast wind is shocked. A self-similar solution will have* $R_{fws}/R_{sw} = \xi = constant$. *We might guess* $\xi \approx 1/3$. *The radial velocity of the fast wind shock front is then* $(\mathrm{d}/\mathrm{d}t)R_{fws} = \xi v_{sw}$, *and the fast wind shock speed (the velocity of the fast wind relative to the shock surface) is* $v_{fw} - \xi v_{sw} = (1 - \xi v_{sw}/v_{fw}) \times v_{fw}$.

Thus with $v_{sw}/v_{fw} = 76/1000$, *and guessing* $\xi \approx 1.3$, *we estimate the fast wind shock speed* $[1 - (1/3) \times 0.076]v_{fw} = 975\,\mathrm{km\,s^{-1}}$.

The temperature of the shocked fast wind is then

$$T_{fws} \approx \frac{3\mu}{16k} \times (975\,\mathrm{km\,s^{-1}})^2 = 1.31 \times 10^7\,\mathrm{K} \quad .$$

Chapter 39. Effects of Supernovae on the ISM

39.1 Obtain an estimate of the dimensionless factor A in eq. (39.8) by assuming that 50% of the total energy will be in ordered kinetic energy, and that the ordered kinetic energy is $\approx (1/2)Mv_s^2$, where M is the swept-up mass. Compare the resulting estimate for A with the exact solution.

Solution:

$$
\begin{aligned}
R &= AE^{1/5}\rho_0^{-1/5}t^{2/5} \\
\frac{1}{2}E &= \frac{1}{2}Mv_s^2 = \frac{1}{2}\frac{4\pi}{3}\rho_0 R^3 \left(\frac{2}{5}\right)^2 \left(\frac{R}{t}\right)^2 \\
&= \frac{1}{2}\frac{16\pi}{75}\rho_0\frac{R^5}{t^2} \\
&= \frac{1}{2}\frac{16\pi}{75}\rho_0\frac{(A^5 E\rho_0^{-1}t^2)}{t^2} \;=\; \frac{1}{2}E \times \frac{16\pi}{75}A^5 \\
A &= (75/16\pi)^{1/5} = 1.0833 \quad .
\end{aligned}
$$

which differs by only 6% from the exact result 1.15167.

Chapter 40. Cosmic Rays and Gamma Rays

40.1 Observations of 1.809 MeV γ rays resulting from the decay of ^{26}Al indicate that the ISM of the Milky Way contains $\sim 2.7 \pm 0.7\, M_\odot$ of ^{26}Al. The total mass of H in the ISM today is $4.9 \times 10^9\, M_\odot$ (see Table 1.2). What is the ratio of $^{26}\mathrm{Al}/^{27}\mathrm{Al}$ in the ISM today?

Solution:

$$
\begin{aligned}
N(^{26}\mathrm{Al})/N(\mathrm{H}) &= \frac{2.7\, M_\odot/26\,\mathrm{amu}}{4.9 \times 10^9\, M_\odot/1\,\mathrm{amu}} = 2.1 \times 10^{-11} \\
N(^{27}\mathrm{Al})/N(\mathrm{H}) &= 2.95 \times 10^{-6} \qquad\qquad \text{from Table 1.4} \\
N(^{27}\mathrm{Al})/N(^{26}\mathrm{Al}) &= \frac{2.1 \times 10^{-11}}{2.95 \times 10^{-6}} = 7.2 \times 10^{-6}
\end{aligned}
$$

This is $\sim 1/7$ *of the highest* $^{26}\mathrm{Mg}/^{27}\mathrm{Al}$ *ratios found in* Ca-Al *rich inclusions (CAIs) in the Allende meteorite.*

Chapter 41. Gravitational Collapse and Star Formation: Theory

41.1 Consider a plane-parallel slab of gas. At $t = 0$, suppose that the slab is of uniform density ρ, with half-thickness H. Let z be a coordinate perpendicular to the slab, with the center of the slab at $z = 0$. Suppose that the gas is at zero temperature, but supported (against its own self-gravity) entirely by magnetic pressure.

(a) The gravitational potential Φ satisfies Poisson's equation $\nabla^2\Phi = 4\pi G\rho$. What is the gravitational acceleration $g = -\nabla\Phi$ as a function of z?

(b) If the magnetic field strength at the slab surface is B_0, what must be the magnetic field $B(z)$ within the slab (i.e., $-H < z < H$) in order to provide the necessary support against gravity for the overall fluid (neutrals + ions)?

(c) Assume that at $t = 0$ the ionization fraction is uniform throughout the slab: $n_i = x_i\rho/m_n$, where $x_i \ll 1$ is the ionization fraction and m_n is the molecular mass of the neutrals (which are assumed to provide essentially all of the mass density ρ). If $\langle\sigma v\rangle_{\rm mt}$ is the "momentum transfer rate coefficient" for ion-neutral scattering (i.e., the force per volume exerted on the neutrals by the ions is $n_i n_n\langle\sigma v\rangle_{\rm mt}[m_n m_i/(m_n + m_i)](\vec{v}_i - \vec{v}_n)$), obtain an expression for the ambipolar diffusion drift velocity v_{in} as a function of z.

(d) Obtain an expression for the ambipolar diffusion timescale z/v_{in}. Evaluate this timescale for $m_n = 2m_{\rm H}$, $x_i = 10^{-6}$, $m_i = 9m_n$, and $\langle\sigma v\rangle_{\rm mt} = 1.9\times10^{-9}\,{\rm cm^3\,s^{-1}}$.

Solution:

(a) Hydrostatic equilibrium with gas pressure $p = 0$: find $\nabla\Phi$:

$$0 = -\frac{d}{dz}\left(\frac{B^2}{8\pi}\right) - \rho\nabla\Phi$$

$$\nabla^2\Phi = 4\pi G\rho \quad\rightarrow\quad \nabla\Phi = 4\pi G\rho z \quad \text{for } |z| < H \quad (\text{since } \nabla\Phi = 0 \text{ at } z = 0)$$

$$\begin{aligned} g &= -\nabla\Phi = -4\pi G\rho z \quad \text{for } |z| \le H \\ &= -4\pi G\rho H \,\mathrm{sgn}(z) \quad \text{for } |z| > H \quad . \end{aligned}$$

(b) Find the magnetic field $B(z)$:

$$\frac{d}{dz}\left(\frac{B^2}{8\pi}\right) = -\rho\nabla\Phi = -4\pi G\rho^2 z \quad \text{for } |z| < H$$

$$\rightarrow \frac{(B(H))^2}{8\pi} - \frac{(B(z))^2}{8\pi} = -4\pi G\rho^2\frac{(H^2 - z^2)}{2} \quad \text{for } |z| < H$$

$$\begin{aligned} B(z) &= \left[B_0^2 + 16\pi^2 G\rho^2(H^2 - z^2)\right]^{1/2} \quad \text{for } |z| \le H \\ &= B_0 \quad \text{for } |z| > H \end{aligned}$$

(c) *Consider force balance on the ions:*

$$
\begin{aligned}
-n_i n_n \langle\sigma v\rangle_{\rm mt} \frac{m_n m_i}{(m_n+m_i)} v_{in} - \nabla\left(\frac{B^2}{8\pi}\right) &= 0 \\
\frac{x_i\rho}{m_n}\frac{\rho}{m_n}\langle\sigma v\rangle_{\rm mt}\frac{m_n m_i}{(m_n+m_i)} v_{in} &= 4\pi G\rho^2 z \\
v_{in} &= \frac{4\pi G(m_n+m_i)m_n}{x_i\langle\sigma v\rangle_{\rm mt} m_i} z \quad .
\end{aligned}
$$

(d) *Ambipolar diffusion timescale* $\tau_{\rm ad} \equiv z/v_{in}$*:*

$$
\begin{aligned}
\tau_{\rm ad} &\equiv \frac{z}{v_{in}} \\
&= \frac{x_i\langle\sigma v\rangle_{\rm mt} m_i}{4\pi G m_n(m_n+m_i)} = \left(\frac{x_i}{10^{-6}}\right)\frac{10^{-6}\times 1.9\times 10^{-9}\,{\rm cm}^3\,{\rm s}^{-1}}{4\pi\, G\times 2\times 1.67\times 10^{-24}\,{\rm g}}\frac{9}{10} \\
&= 6.11\times 10^{14}\left(\frac{x_i}{10^{-6}}\right)\,{\rm s} = 1.94\times 10^{7}\left(\frac{x_i}{10^{-6}}\right)\,{\rm yr} \quad .
\end{aligned}
$$

Note that $\tau_{\rm ad}$ *depends only on the fractional ionization* x_i *and fundamental constants (*G*,* m_n*,* m_i*, and* $\langle\sigma v\rangle_{\rm mt}$*).*

41.3 The Taurus Molecular Cloud has regions with H nucleon density $n_{\rm H} = 1 \times 10^3\,{\rm cm}^{-3}$, temperature $T = 12\,{\rm K}$. The hydrogen is almost entirely molecular. Assume the gas remains isothermal. If the magnetic field can be neglected, calculate the maximum mass of a self-gravitating non-rotating density peak in such gas.

Solution: The gas has $n({\rm H}_2) = 500\,{\rm cm}^{-3}$ *and* $n({\rm He}) = 100\,{\rm cm}^{-3}$. *Hence* $p_0/k = 600\,{\rm cm}^{-3}\times 12\,{\rm K} = 7.2\times10^3\,{\rm cm}^{-3}\,{\rm K}$. *From eq. (41.44), the Bonnor-Ebert mass is:*

$$
\begin{aligned}
M_{\rm BE} &= 0.26\left(\frac{T}{10\,{\rm K}}\right)^2\left(\frac{10^6\,{\rm cm}^{-3}\,{\rm K}}{p_0/k}\right)^{1/2} M_\odot \\
&= 0.26\times(1.2)^2\times\left(\frac{10^6}{7.2\times10^3}\right)^{1/2} = 4.4\,M_\odot
\end{aligned}
$$

Chapter 42. Star Formation: Observations

42.1 Star formation with a specified IMF implies steady production of massive stars which, although short-lived, emit large numbers of ionizing photons. Using the stellar models in the Starburst99 code (Leitherer et al. 1999, ApJS, 123, 3), the time-averaged emission of $h\nu > 13.6\,\mathrm{eV}$ photons from steady star formation is found to be

$$Q_0 = 1.37\times 10^{53}\left(\frac{\mathrm{SFR}}{M_\odot\,\mathrm{yr}^{-1}}\right)\,\mathrm{s}^{-1} \quad .$$

(e.g., Murphy et al. 2011, ApJ, 737, 67).

(a) Suppose that we observe a galaxy at a distance D, and measure an integrated $\mathrm{H}\alpha$ energy flux $F(\mathrm{H}\alpha)$. If dust is not important, and the H II regions in the galaxy have an electron temperature $10^4 T_4\,\mathrm{K}$, show that star formation rate SFR can be obtained from the observed $F(\mathrm{H}\alpha)$:

$$\frac{\mathrm{SFR}}{M_\odot\,\mathrm{yr}^{-1}} = \frac{4\pi D^2 F(\mathrm{H}\alpha)}{1.91\times 10^{41}\,\mathrm{erg\,s}^{-1}} \times T_4^{0.126+0.010\ln T_4} \quad .$$

State any important assumptions.

(b) Suppose that the thermal radio free-free emission from a galaxy at distance D, is observed to have a flux density F_ν at frequency $\nu = \nu_9\,\mathrm{GHz}$. Show that the star formation rate can be deduced from the observed F_ν using

$$\frac{\mathrm{SFR}}{M_\odot\,\mathrm{yr}^{-1}} = 5.53\times 10^{-27}\nu_9^{0.118}T_4^{-0.493}\times\frac{D^2 F_\nu}{\mathrm{erg\,s}^{-1}\,\mathrm{Hz}^{-1}} \quad .$$

State any important assumptions.

Solution:

(a) Assume that

(1) all $h\nu > 13.6\,\mathrm{eV}$ *photons are absorbed by* H*;*

(2) photoionizations are balanced by radiative recombinations;

(3) recombinations occur under "Case B" conditions;

(4) the $\mathrm{H}\alpha$ *does not suffer extinction by dust;*

(5) there are no other sources of $\mathrm{H}\alpha$*.*

Then, using α_B *from Eq. (14.6) and* $\alpha_{\rm eff,H\alpha}$ *from Eq. (14.8):*

$$
\begin{aligned}
F(\mathrm{H}\alpha) &= \frac{Q_0}{4\pi D^2}\times\frac{\alpha_{\rm eff,H\alpha}}{\alpha_B}\times\frac{hc}{\lambda_{\rm H\alpha}}\\
Q_0 &= 4\pi D^2 F(\mathrm{H}\alpha)\times\frac{\alpha_B}{\alpha_{\rm eff,H\alpha}}\times\frac{\lambda_{\rm H\alpha}}{hc}\\
\frac{\rm SFR}{M_\odot\,\mathrm{yr}^{-1}} &= \frac{4\pi D^2 F(\mathrm{H}\alpha)}{1.37\times10^{53}\,\mathrm{s}^{-1}}\times\frac{\alpha_B}{\alpha_{\rm eff,H\alpha}}\times\frac{\lambda_{\rm H\alpha}}{hc}\\
&= \frac{4\pi D^2 F(\mathrm{H}\alpha)}{1.37\times10^{53}\,\mathrm{s}^{-1}}\times\frac{2.54\times10^{-13}T_4^{-0.816-0.021\ln T_4}}{1.17\times10^{-13}T_4^{-0.942-0.031\ln T_4}}\times\frac{0.6565\,\mu\mathrm{m}}{hc}\\
&= \frac{4\pi D^2 F(\mathrm{H}\alpha)}{1.91\times10^{41}\,\mathrm{erg\,s}^{-1}}\times T_4^{0.126+0.010\ln T_4} \quad .
\end{aligned}
$$

(b) *Assume that*

(1) *all* $h\nu > 13.6\,\mathrm{eV}$ *photons are absorbed by* H*;*

(2) *photoionizations are balanced by radiative recombinations;*

(3) *recombinations occur under "Case B" conditions;*

(4) *other sources of 33 GHz emission (synchrotron emission and spinning dust, in particular) are either negligible or can be corrected for.*

Then, using α_B *from Eq. (14.6) and* j_ν *from Eq. (10.8) (which requires that* $0.2 < \nu_9 < 200$ *for ∼10% accuracy):*

$$
\begin{aligned}
F_\nu &= \frac{Q_0}{4\pi D^2}\times\frac{4\pi j_\nu}{n(\mathrm{H}^+)n_e\alpha_B}\\
Q_0 &\approx \frac{D^2F_\nu}{\mathrm{erg\,s^{-1}\,Hz^{-1}}}\times\frac{n(\mathrm{H}^+)}{n_i}\times\frac{2.54\times10^{-13}T_4^{-0.816-0.021\ln T_4}}{3.35\times10^{-40}\nu_9^{-0.118}T_4^{-0.323}}\,\mathrm{s}^{-1}\\
\frac{\rm SFR}{M_\odot\,\mathrm{yr}^{-1}} &\approx \frac{1}{1.37\times10^{53}}\times7.58\times10^{26}\nu_9^{0.118}T_4^{-0.493}\times\frac{D^2F_\nu}{\mathrm{erg\,s^{-1}\,Hz^{-1}}}\\
&\approx 5.53\times10^{-27}\nu_9^{0.118}T_4^{-0.493}\times\frac{D^2F_\nu}{\mathrm{erg\,s^{-1}\,Hz^{-1}}} \quad .
\end{aligned}
$$

where we have dropped the $\ln T_4$ *term in the exponent of* T_4 *because the temperature dependence of the free-free emissivity* j_ν *is not being represented to this accuracy, and have assumed that* $n_i/n(\mathrm{H}^+)\approx 1$ *in the* $\mathrm{H\,II}$ *that dominates the free-free emission.*