Physics of the Interstellar and Intergalactic Medium: Problems for Students

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Preface

I have assembled this collection of problems to accompany *Physics of the Interstellar and Intergalactic Medium*. Although these problems do not cover all topics in the text, I hope that they will prove useful to both students and instructors.

From time to time the problem collection will be updated with new problems, and with corrections as needed. The up-to-date collection is available on-line at

http://www.astro.princeton.edu/∼draine/book/problems.pdf

Solutions to odd-numbered problems are available on-line at

http://www.astro.princeton.edu/∼draine/book/solutions_odd.pdf

If you detect errors in the problems or solutions, please notify the author at draine@astro.princeton.edu .

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Chapter 1. Introduction

1.1 The total mass of neutral gas in the Galaxy is $\sim 4\times 10^9\,M_\odot$. Assume that it is uniformly distributed in a disk of radius $R_{\rm disk} = 15\,{\rm kpc}$ and thickness $H = 200\,{\rm pc}$, and that it is a mixture of H and He with He/H=0.1 (by number). Assume ionized hydrogen to be negligible in this problem. [Note: even though the assumptions in this problem are very approximate, please carry out calculations to **two** significant digits.]

(a) What is the average number density of hydrogen nuclei within the disk?

(b) If 0.7% of the interstellar mass is in the form of dust in spherical particles of radius $a = 1000\,\text{Å} = 0.1\,\mu{\rm m}$ and density $2\,{\rm g\,cm^{-3}}$, what is the mean number density of dust grains in interstellar space?

(c) Let $Q_{\rm ext}$ be the ratio of the visual (V band, $\lambda = 0.55\,\mu{\rm m}$) extinction cross section to the geometric cross section πa^2. Suppose that $Q_{\rm ext} \approx 1$. What would be the visual extinction A_V (in magnitudes!) between the Sun and the Galactic Center (assumed to be $8.5\,{\rm kpc}$ away)?

(d) Now assume that 30% of the gas and dust mass is in spherical molecular clouds of radius 15 pc and mean density $n({\rm H_2}) = 100\,{\rm cm^{-3}}$.
What would be the mass of one such cloud?
How many such molecular clouds would there be in the Galaxy?

(e) With 30% of the gas and dust mass in molecular clouds as in (d), what is the expectation value for the visual extinction A_V to the Galactic Center?

(f) With 30% of the material in molecular clouds as in (d), what is the expectation value for the number of molecular clouds that will be intersected by the line of sight to the Galactic center?
What is the probability that zero molecular clouds will be intersected? [Hint: the number of molecular clouds in the Galaxy is large, and they occupy a small fraction of the volume, so think of this as a "Poisson process", where the presence or absence of each molecular cloud on the line-of-sight is treated as an independent event (like the number of radioactive decays in a fixed time interval).]

(g) If the line of sight to the Galactic center happens not to intersect any molecular clouds, and if the atomic hydrogen and associated dust are distributed uniformly throughout the disk volume, what will be the visual extinction to the Galactic center?

1.2 Suppose that we approximate hydrogen atoms as hard spheres with radii $a = 1.5\,\text{Å}$. In a neutral atomic hydrogen cloud with density $n_{\rm H} = 30\,{\rm cm^{-3}}$, what is the mean free path for an H atom against scattering by other H atoms (assuming the other H atoms to be at rest)?

1.3 The "very local" interstellar medium has $n_{\rm H} \approx 0.22\,{\rm cm^{-3}}$ (Lallement et al. 2004: Astr. & Astrophys. 426, 875; Slavin & Frisch 2007: Sp. Sci. Revs. 130, 409). The Sun is moving at $v_W = 26\pm 1\,{\rm km\,s^{-1}}$ relative to this local gas (Möbius et al. 2004: Astr. & Astrophys. 426, 897).

Suppose that this gas has He/H=0.1, and contains dust particles with total mass equal to 0.5% of the mass of the gas. Suppose these particles are of radius $a = 0.1\,\mu{\rm m}$ and density $\rho = 2\,{\rm g\,cm^{-3}}$, and we wish to design a spacecraft to collect them for study.

How large a collecting area A should this spacecraft have in order to have an expected collection rate of 1 interstellar grain per hour? Neglect the motion of the spacecraft relative to the Sun, and assume that the interstellar grains are unaffected by solar gravity, radiation pressure, and the solar wind (and interplanetary magnetic field).

1.4 The distance to the nearby star Proxima Centauri is $D = 1.30\,{\rm pc}$. The ISM between the Sun and Proxima Cen has a mean density of H nucleons $n_{\rm H} = 0.22\,{\rm cm^{-3}}$. Suppose that the mass in dust grains is 0.7% of the mass in H, and that the dust grains are spheres with radii $a = 0.15\,\mu{\rm m}$ and internal density $\rho = 2\,{\rm g\,cm^{-3}}$.
A chip with a forward-facing cross-sectional area $A = 2\,{\rm cm^2}$ is to travel from the Sun to Proxima Cen.
If the chip travels at $v = 0.2c$, what is the expected number $N_{\rm impact}$ of dust grain impacts on the forward-facing side of the chip?

1.5 Suppose that large rocky objects from interstellar space pass within $1\,\mathrm{AU}$ of the Sun at a rate of 1 per year, with mean speed (at infinity) $v_{rock} = 20\,\mathrm{km\,s^{-1}}$. The objects are irregular, but suppose that they have solid volumes equal (on average) to spheres with radius $50\,\mathrm{m}$.

(a) If the rock itself has a mass density of $3\,\mathrm{g\,cm^{-3}}$, and 75% of the mass in the rock is contributed by the elements Mg, Si, and Fe, estimate $\langle\rho_{\mathrm{rocks}}^{\mathrm{MgSiFe}}\rangle$, the mean mass density in the ISM of Mg, Si, and Fe contained in such rocky objects. For simplicity, neglect effects of gravitational focusing by the Sun.

(b) If the mean density of H in the ISM is $\langle n_{\mathrm{H}}\rangle = 1\,\mathrm{cm^{-3}}$, and Mg, Si, Fe together contribute a mass equal to 0.4% of the H mass, estimate the fraction f of the interstellar Mg, Si, and Fe that is contained in these large rocky objects.

1.6 Consider a cloud with density $n_{\mathrm{H}} = 30\,\mathrm{cm^{-3}}$. Suppose that it contains two types of dust grains:

- "Large" grains with radii $a_{\mathrm{large}} = 0.1\,\mu\mathrm{m}$, with total mass equal to 0.006 of the mass in hydrogen, and
- "Small" grains with radii $a_{\mathrm{small}} = 0.001\,\mu\mathrm{m}$, with total mass equal to 0.001 of the total hydrogen mass.

Suppose that both large and small grains are spheres with internal density $\rho = 2\,\mathrm{g\,cm^{-3}}$.

Suppose that the large grains are moving with rms speed $\langle v_{\mathrm{large}}^2\rangle^{1/2} = 1\,\mathrm{km\,s^{-1}}$ in random directions. Suppose that the small grains are moving much more slowly, with rms speed $v_{\mathrm{small}} = 10^{-3}\,\mathrm{km\,s^{-1}}$ in random directions.

For purposes of this problem, assume that for collisions between particles from species 1 and 2, each moving with random velocities, the particle-particle velocity difference $\Delta\mathbf{v}$ has

$$\langle|\Delta\mathbf{v}_{12}|\rangle \approx \left(\langle v_1^2\rangle + \langle v_2^2\rangle\right)^{1/2} \quad ,$$

Assume that there is no interaction between grains until they actually come into contact.

(a) Calculate the number per unit volume n_{large} and n_{small} of large and small grains.

(b) For a given large grain, what is the probability per unit time of undergoing a grain-grain collision with another large grain?

(c) For a given small grain, what is the probability per unit time of undergoing a collision with another small grain?

(d) For a given small grain, what is the probability per unit time of undergoing a collision with a large grain?

(e) For a given large grain, what is the probability per unit time of collisions with small grains?

1.7 The "very local" interstellar medium has $n_{\mathrm{H}} \approx 0.22\,\mathrm{cm^{-3}}$ (Lallement et al. 2004: Astr. & Astrophys. 426, 875; Slavin & Frisch 2007: Sp. Sci. Revs. 130, 409). The Sun is moving at $v_W \approx 26\,\mathrm{km\,s^{-1}}$ relative to this local gas (Möbius et al. 2004: Astr. & Astrophys. 426, 897).

(a) Suppose that this gas has He/H=0.1, and contains dust particles with total mass equal to 0.5% of the mass of the gas, in the form of dust grains of radius $a = 0.1\,\mu\mathrm{m}$ and density $\rho = 2\,\mathrm{g\,cm^{-3}}$.

(b) We wish to design a spacecraft to collect interstellar dust for study. The spacecraft will travel outwards from the Sun with at a speed sufficient to escape from the Solar system. Suppose that the heliocentric velocity of the spacecraft will be $v_{\mathrm{spacecraft}} = 30\,\mathrm{km\,s^{-1}}$ when it is far enough from the Sun that Solar gravity can be neglected. It will be traveling ***toward*** the incoming "interstellar wind".

How large a collecting area A is required to have an expected collection rate of 1 interstellar grain per 24 hours when it is far from the Sun?

Chapter 2. Collisional Processes

2.1 Consider an electron-proton plasma at temperature T. Let $t_s(e-e)$ be the time scale for 90 degree scattering of one electron with kinetic energy $\sim kT$ by encounters with other electrons.
The electron-proton mass ratio $m_p/m_e = 1836$. The following time scales t_x will differ from $t_s(e-e)$ by factors $(m_p/m_e)^\alpha$ and factors of order unity; ignore the latter, so that $t_x \approx (m_p/m_e)^\alpha \times t_s(e-e)$.

Identify the exponent α for each of the following processes; in each case, assume the process to be acting alone. It is not necessary to do any derivations – just give a one-sentence justification for each answer.

(a) 90 degree scattering of one electron by encounters with protons.

(b) 90 degree scattering of one proton by encounters with electrons.

(c) 90 degree scattering of one proton by encounters with other protons.

(d) exchange of energy from one electron to other electrons.

(e) exchange of energy from one electron to protons.

(f) exchange of energy from one proton to electrons.

(g) exchange of energy from one proton to other protons.

2.2 Consider a hydrogen atom in a highly-excited state with quantum number $n \gg 1$, immersed in an electron-proton plasma at temperature T.

(a) In a gas of temperature $T = 10^4 T_4\,\mathrm{K}$, for what quantum number n_c is the orbital velocity of the bound electron equal to the rms velocity of a thermal proton?

(b) For quantum number $n \gg n_c$, use the impact approximation to estimate the collisional rate coefficient for ionization by proton impact: $\mathrm{H}(n) + \mathrm{H}^+ \rightarrow 2\mathrm{H}^+ + e^-$.

(c) Compare this rate coefficient to the rate coefficient for ionization by electron impact [Eq. (2.12)].

2.3 Consider a dust grain of radius a, and mass $M \gg m_\mathrm{H}$, where m_H is the mass of an H atom. Suppose that the grain is initially at rest in a gas of H atoms with number density n_H and temperature T. Assume the grain is large compared to the radius of an H atom. Suppose that the H atoms "stick" to the grain when they collide with it, so that all of their momentum is transferred to the grain, and that they subsequently "evaporate" from the grain with no change in the grain velocity during the evaporation.

(a) What is the mean *speed* $\langle v_\mathrm{H} \rangle$ of the H atoms (in terms of m_H, T, and Boltzmann's constant k_B)?

(b) Calculate the time τ_M for the grain to be hit by its own mass M in gas atoms. Express τ_M in terms of M, a, n_H, and $\langle v_\mathrm{H} \rangle$.

(c) Evaluate $\langle v_\mathrm{H} \rangle$ and τ_M for a grain of radius $a = 10^{-5}\,\mathrm{cm}$ and density $\rho = 3\,\mathrm{g\,cm^{-3}}$, in a gas with $n_\mathrm{H} = 30\,\mathrm{cm^{-3}}$ and $T = 10^2\,\mathrm{K}$.

(d) If the collisions are random, the grain velocity will undergo a random walk. Estimate the *initial* rate of increase $(dE/dt)_0$ of the grain kinetic energy E due to these random collisions. Express $(dE/dt)_0$ in terms of n_H, m_H, $k_\mathrm{B}T$, a, and M. [Hint: think of the random walk that the grain momentum $\vec{p}$ undergoes, starting from the initial state $\vec{p} = 0$. What is the rate at which $\langle p^2 \rangle$ increases?]

(e) Eventually the grain motion will be "thermalized", with time-averaged kinetic energy $\langle E \rangle = (3/2)k_\mathrm{B}T$. Calculate the timescale

$$\tau_E \equiv \frac{(3/2)k_\mathrm{B}T}{(dE/dt)_0}$$

for thermalization of the grain speed. Compare to τ_M calculated in (b).

2.4 Consider a molecule with a permanent dipole moment $\vec{p}_0$ and mass m_1. Suppose $\vec{p}_0$ is in the $\hat{\mathbf{z}}$ direction, and consider the simple case of a neutral atom or molecule (with no permanent dipole moment) with mass m_2 and polarizability α approaching along a trajectory in the $\hat{\mathbf{x}}$–$\hat{\mathbf{y}}$ plane with velocity (at infinity) v_0 and impact parameter b. The electric field in the $z = 0$ plane due to the dipole $\vec{p}_0 = p_0\hat{\mathbf{z}}$ is

$$\vec{E} = -\frac{p_0}{r^3}\hat{\mathbf{z}}$$

(a) For an induced dipole moment $\vec{p} \propto \vec{E}$, the interaction energy is $U = -(1/2)\vec{p}\cdot\vec{E}$. For an atom in the $z = 0$ plane, what is the potential $U(r)$ describing its interaction with the fixed dipole $\vec{p} = p_0\hat{\mathbf{z}}$ at $r = 0$?

(b) For motion in the $\hat{\mathbf{x}} - \hat{\mathbf{y}}$ plane with incident velocity v_0, calculate a "critical" impact parameter b_0 such that the interaction energy at separation b_0 is equal to $1/4$ of the initial kinetic energy. (Why 1/4? Because previous study of the r^{-4} potential has shown us that for $U \propto r^{-4}$, we get orbiting collisions for b less than the distance where the interaction energy is equal to 1/4 of the initial kinetic energy E_0. The present interaction has a different dependence on r, but $U(b_0) = (1/4)E_0$ will probably be a good guide to the impact parameter separating "orbiting" from "non-orbiting" collisions.)

(c) Without working out the dynamics, we can reasonably expect that trajectories with $b \lesssim b_0$ will be strongly scattered, and may formally pass through $r = 0$ by analogy with the trajectories for a r^{-4} potential. Estimate the cross section $\sigma_0(v_0)$ for "orbiting" collisions where the projectile approaches very close to the target.

(d) How does the product $\sigma_0 v_0$ depend on v_0?

(e) Substituting a typical thermal speed for v_0, estimate the thermal rate coefficient $\langle\sigma v\rangle$ for "orbiting" collisions as a function of gas kinetic temperature T to obtain the temperature dependence.

(f) Consider scattering of H_2 by the SiO molecule, which has dipole moment $p_0 = 3.1\,\mathrm{Debye} = 3.1 \times 10^{-18}\,\mathrm{esu\,cm}$. From Table 2.1, the polarizability of H_2 is $\alpha = 7.88 \times 10^{-25}\,\mathrm{cm}^3$. Suppose that the "hard sphere" cross section for SiO-H_2 scattering is $C_{\rm hs} = 3 \times 10^{-15}\,\mathrm{cm}^2$. Estimate the temperature T_c below which the collision rate will be strongly affected by the induced-dipole interaction.

2.5 Consider a cloud of partially-ionized hydrogen with $n(\mathrm{H}^0) = 20\,\mathrm{cm}^{-3}$, $n(\mathrm{H}^+) = n_e = 0.01\,\mathrm{cm}^{-3}$, and $T = 100\,\mathrm{K}$. Consider an electron injected into the gas with kinetic energy $E_0 = 1\,\mathrm{eV}$. We will refer to it as the "fast" electron.

(a) What is the speed v_0 of the fast electron?

(b) If the electron-neutral elastic scattering cross section is given by eq. (2.40):

$$\sigma_{\rm mt} = 7.3\times10^{-16}\left(\frac{E_0}{0.01\,\mathrm{eV}}\right)^{0.18}\,\mathrm{cm}^2$$

calculate $t_{\rm scat}$, where $t_{\rm scat}^{-1}$ is the probability per unit time for elastic scattering of the fast electron by the neutral H atoms.

(c) A result from elementary mechanics:

> If a particle of mass m_1 and kinetic energy E_0 undergoes a head-on elastic collision with a particle of mass m_2 that was initially at rest, the kinetic energy E_2 of particle 2 after the collision is just $E_2 = 4m_1m_2E_0/(m_1+m_2)^2$.

Using this result, if the electron undergoes a head-on elastic scattering with a hydrogen atom that was initially at rest, what fraction $f_{\rm max}$ of the electron kinetic energy is transfered to the H atom?

(d) If elastic scattering off H atoms were the only process acting, and if the average scattering event transferred 50% as much energy as in head-on scattering, what would be the initial time scale $t_E = E_0/|dE/dt|_{E_0}$ for the electron to share its energy with the H atoms?

(e) Now consider elastic scattering of the fast electron by the other (thermal) free electrons in the gas. Equation (2.19) from the textbook gives the energy loss time for a fast particle of mass m_1, velocity v_1, charge $Z_1 e$ moving through a gas of particles of mass m_2 and charge $Z_2 e$:

$$
\begin{aligned}
t_{\rm loss} &= \frac{m_1 m_2 v_1^3}{8\pi n_2 Z_1^2 Z_2^2 e^4 \ln\Lambda} \\
\ln\Lambda &= 22.1 + \ln\left[\left(\frac{E_1}{kT}\right)\left(\frac{T}{10^4\,{\rm K}}\right)^{3/2}\left(\frac{{\rm cm}^{-3}}{n_e}\right)\right]
\end{aligned}
$$

If the only energy loss process was scattering of the fast electrons by the thermal electrons in the gas, evaluate the energy loss time $t_{\rm loss}$ for the fast electron.

Chapter 3. Statistical Mechanics and Thermodynamic Equilibrium

3.1 Consider a "$n\alpha$" radio recombination line for $n \gg 1$, and assume the emission to be optically-thin. Assume that the probability per unit time of emitting a photon is given by the Einstein $A-$coefficient

$$A_{n+1\to n} \approx \frac{6.130\times 10^9\,\mathrm{s}^{-1}}{(n+0.7)^5} \quad .$$

Relate the ratio b_{n+1}/b_n to the observed line intensity ratio $I_{n+1\to n}/I_{n\to n-1}$. Assume that $I_{\mathrm{H}}/n^2 k_{\mathrm{B}}T \ll 1$. Retain the term of leading order in the small parameter $1/n$.

3.2 Suppose that the cross section for the reaction $AB + C \;\to\; A + B + C$ is $\sigma(E) = 0$ for $E < E_0$, $\sigma(E) = \sigma_0$ for $E > E_0$, where E is the center-of-mass translational energy. Let the masses of AB and C be m_{AB} and m_C.

Theorem: If species 1 and 2, with masses m_1 and m_2, each have Maxwellian velocity distributions characterized by temperatures T_1 and T_2, then the rate per volume of collisions with center-of-mass energy in the interval $[E, E + dE]$ is the same as for collisions between two species, with the same collision cross section $\sigma(E)$, but with one species infinitely massive, and the other species with mass $\mu = m_1 m_2/(m_1 + m_2)$ and with temperature $\bar{T} = (m_1 T_2 + m_2 T_1)/(m_1 + m_2)$.

(a) Assuming the above theorem to be true (it is!) obtain the thermally-averaged rate coefficient $\langle\sigma v\rangle$ for the reaction $AB + C \;\to\; A + B + C$ as a function of temperature T.

(b) The rate/volume for the reaction $A + B + C \to AB + C$ is $Q n_A n_B n_C$, where Q is the "three-body" rate coefficient. If E_0 is the energy required to dissociate AB into $A + B$, and m_A and m_B are the masses of A and B, obtain $Q(T)$ in terms of m_A, m_B, m_C, σ_0, E_0, and T. (Assume A, B, C, and AB to be structureless and spinless particles).

3.3 Consider a path of length L with electron density n_e and gas kinetic temperature T. Let the population of the high-n levels of H be characterized by departure coefficients b_n.

If the medium is optically-thin, and the only radiative transitions are spontaneous decays, the integrated line intensity for an $n\alpha$ (i.e., $n + 1 \to n$) transition is

$$I(n\alpha) = \frac{A(n\alpha)}{4\pi} h\nu_{n\alpha} \int_0^L ds\, n[\mathrm{H}(n+1)]$$

where $A(n\alpha)$ is the Einstein A-coefficient, and $n[\mathrm{H}(n + 1)]$ is the volume density of H atoms in quantum state $n + 1$. Assume that $A(n\alpha)$ is accurately approximated by

$$A(n\alpha) \approx \frac{A_0}{(n+0.7)^5}$$

where $A_0 \equiv 6.130 \times 10^9\,\mathrm{s}^{-1}$.

(a) Obtain an expression for $I(n\alpha)$ in terms of $T_4 \equiv T/10^4\,\mathrm{K}$, quantum number n, departure coefficient b_{n+1}, and the "emission measure"

$$EM \equiv \int_0^L ds\, n(\mathrm{H}^+) n_e \quad .$$

(b) Evaluate $I(166\alpha)/b_{167}$ for $EM = 10^6\,\mathrm{cm}^{-6}\,\mathrm{pc}$, and $T_4 = 1$.

3.4 The characteristic radius of the hydrogenic orbital with radial quantum number n is $r_n = n^2 a_0$.

(a) Calculate the quantum number $n_{\max}$ for a "Rydberg atom" such that the expected number of *field* electrons within a distance r_n is 1, for electron density n_e. Evaluate $n_{\max}$ for $n_e = 1\,\mathrm{cm}^{-3}$.

(b) A back-of-the-envelope estimate of the rate coefficient for collisional ionization of H in level $n \gg 1$ is Eq. (3.41):

$$\langle \sigma v \rangle_{n \to c} \approx n^2 \frac{e^4}{I_{\mathrm{H}}} \left(\frac{8\pi}{m_e k_{\mathrm{B}} T} \right)^{1/2} e^{-I_{\mathrm{H}}/n^2 k_{\mathrm{B}} T}$$

For the Bohr model of the atom, the electron speed is $v_n = \alpha c/n$, where $\alpha = 1/137$ is the fine structure constant, and n is the principal quantum number. The orbital period is $P_n = 2\pi r_n/v_n = 2\pi a_0 n^3/\alpha c$. For an electron in orbital $n_{\max}$ from part (a), calculate the probability of collisional ionization in one orbital period, for $n_e = 1\,\mathrm{cm}^{-3}$ and $T = 5000\,\mathrm{K}$. Show how this depends on both electron density n_e and temperature T. You may take $e^{I_{\mathrm{H}}/k_{\mathrm{B}}T} \approx 1$.

3.5 Suppose that Rydberg levels of hydrogen with quantum number $100 \leq n \leq n_{\max}$ are in LTE at $T = 5000\,\mathrm{K}$ with protons and electrons, with $n(\mathrm{H}^+) = n_e = 1\,\mathrm{cm}^{-3}$. Calculate the ratio

$$\frac{1}{n(\mathrm{H}^+)} \sum_{100}^{n_{\max}} n[\mathrm{H}(n)]$$

and evaluate it for $n_{\max} = 10^3$. Make approximations as appropriate.

Chapter 4. Energy Levels of Atoms and Ions

4.1 Classify the following emission lines as either (i) *Permitted*, (ii) *Intercombination*, or (iii) *Forbidden*, and give your reason.

(a) C III : $1s^22s2p\ ^3\mathrm{P}^o_1 \rightarrow 1s^22s^2\ ^1\mathrm{S}_0$ 1908.7 Å

(b) O III : $1s^22s^22p^2\ ^1\mathrm{D}_2 \rightarrow 1s^22s^22p^2\ ^3\mathrm{P}_2$ 5008.2 Å

(c) O III : $1s^22s^22p^2\ ^1\mathrm{S}_0 \rightarrow 1s^22s^22p^2\ ^1\mathrm{D}_2$ 4364.4 Å

(d) O III : $1s^22s2p^3\ ^5\mathrm{S}^o_2 \rightarrow 1s^22s^22p^2\ ^3\mathrm{P}_1$ 1660.8 Å

(e) O III : $1s^22s^22p^2\ ^3\mathrm{P}_1 \rightarrow 1s^22s^22p^2\ ^3\mathrm{P}_0$ 88.36 μm

(f) C IV : $1s^22p\ ^2\mathrm{P}^o_{3/2} \rightarrow 1s^22s\ ^2\mathrm{S}_{1/2}$ 1550.8 Å

(g) Ne II : $1s^22s^22p^5\ ^2\mathrm{P}^o_{1/2} \rightarrow 1s^22s^22p^5\ ^2\mathrm{P}^o_{3/2}$ 12.814 μm

(h) O I : $1s^22s^22p^33s\ ^3\mathrm{S}^o_1 \rightarrow 1s^22s^22p^4\ ^3\mathrm{P}_2$ 1302.2 Å

Chapter 5. Energy Levels of Molecules

5.1 Both H_2 and HD have similar internuclear separation $r_0 \approx 0.741$ Å. Assume that the molecules can be approximated as rigid rotors.

(a) Calculate $[E(v\!=\!0, J) - E(v\!=\!0, J\!=\!0)]/k_{\rm B}$ for H_2 for $J\!=\!1$, $J\!=\!2$, and $J\!=\!3$.

(b) Calculate $[E(v\!=\!0, J) - E(v\!=\!0, J\!=\!0)]/k_{\rm B}$ for HD for $J\!=\!1$, $J\!=\!2$, and $J\!=\!3$.

(c) Because H_2 has no electric dipole moment, $\Delta J = \pm 1$ transitions are forbidden, and instead the only radiative transitions are electric quadrupole transitions with $\Delta J\!=\!0, \pm 2$. Calculate the wavelengths of the $J\!=\!2 \rightarrow 0$ and $J\!=\!3 \rightarrow 1$ transitions of H_2

(d) Because HD has a (small) electric dipole moment, it has (weak) electric dipole transitions. What is the longest-wavelength spontaneous decay for HD in the $v = 0$ vibrational level?

5.2 Why doesn't H_2 in the ground electronic state $X^1\Sigma_g^+$ have hyperfine splitting?

5.3 Most interstellar CO is $^{12}C^{16}O$. The $J = 1 \rightarrow 0$ transition is at $\nu = 115.27\,{\rm GHz}$, or $\lambda = 0.261\,{\rm cm}$, and the $v = 1 \rightarrow 0$ transition is at $\lambda = 4.61\,\mu{\rm m}$ (ignoring rotational effects).

(a) Estimate the frequencies of the $J = 1 \rightarrow 0$ transitions in $^{13}C^{16}O$ and $^{12}C^{17}O$.

(b) Estimate the wavelengths of the $v = 1 \rightarrow 0$ transitions in $^{13}C^{16}O$ and $^{12}C^{17}O$. Ignore rotational effects.

(c) Suppose that the $^{13}C^{16}O$ $J\!=\!1-0$ line were mistaken for the $^{12}C^{16}O$ $J\!=\!1-0$ line.
What would be the error in the inferred radial velocity of the emitting gas?

(d) What is $\Delta E/k_B$, where ΔE is the difference in "zero-point energy" between $^{12}C^{16}O$ and $^{13}C^{16}O$, and k_B is Boltzmann's constant?

Chapter 6. Spontaneous Emission, Stimulated Emission, and Absorption

6.1 A hydrogen atom with principal quantum number n has energy $E_n = -I_{\rm H}/n^2$ where $I_{\rm H} = 13.602\,{\rm eV}$ is the ionization energy of hydrogen. A radiative transition from level $n+1 \rightarrow n$ is referred to as "$n\alpha$"; a radiative transition from level $n+2 \rightarrow n$ is referred to as "$n\beta$". E.g., the 1α transition is the same as Lyman alpha, and the 2α transition is the same as Balmer α (also known as Hα).

(a) Show that the frequency of the $n\alpha$ transition is given by

$$\nu_{n\rightarrow n+1} = \frac{C(n+0.5)}{\left[(n+0.5)^2 - 0.25\right]^2} \quad .$$

What is the value of C (in Hz)?

(b) For $n \gg 1$, it is reasonable to neglect the term 0.25 in the denominator, so from here on approximate

$$\nu_{n\rightarrow n+1} \approx C(n+0.5)^{-3} \quad .$$

Now suppose that we want to observe 21cm radiation from gas at redshift $z = 9$, redshifted to frequency $\nu = 142.04$ MHz. Our Galaxy will also be producing hydrogen recombination radiation. What are the frequencies and n values of the $n\alpha$ transition just above, and just below, 142.04 MHz?

(c) Suppose that the high-n levels of hydrogen are found in ionized gas with an electron temperature $T = 8000\,{\rm K}$, with the hydrogen having one-dimensional velocity dispersion $\sigma_v = 10\,{\rm km\,s^{-1}}$. What will be the FWHM linewidth (in MHz) of the $n\alpha$ transitions near 142 MHz? Compare this linewidth to the frequency difference $(\nu_{n+1\rightarrow n} - \nu_{n+2\rightarrow n+1})$ between adjacent $n\alpha$ lines near 142 MHz.

(d) Find the frequency and n value for the $n\beta$ transition just below 142 MHz, and just above 142 MHz.

6.2 Neutral helium and neutral carbon will also produce $n\alpha$ transitions. For $n \gg 1$, the energies of these transitions will be almost the same as for H, the difference coming only from the reduced mass: for an atom X, the high-n levels have energies (relative to $n = \infty$)

$$E_n = -\mu\frac{(\alpha c)^2}{n^2} \quad ,$$

where $\mu = m_e m_X/(m_e + m_X)$ is the reduced mass and $\alpha = e^2/\hbar c$ is the fine-structure constant.

Can we distinguish the lines?

(a) Estimate the frequency shift $\nu_{{\rm He}n\alpha} - \nu_{{\rm H}n\alpha}$, and $\nu_{{\rm C}n\alpha} - \nu_{{\rm H}n\alpha}$, for $n\alpha$ giving a transition frequency near 142 MHz.

(b) Will the He $n\alpha$ and C $n\alpha$ lines be separated by more or less than the FWHM of He $n\alpha$ due to Doppler broadening in gas where the H $n\alpha$ line has a FWHM $= 21\,{\rm km\,s^{-1}}$? Assume that the H $n\alpha$ line width is entirely due to thermal broadening.

Chapter 7. Radiative Transfer

7.1 A local H I cloud is interposed between us and the cosmic microwave background with temperature $T_{\rm CMB} = 2.7255\,{\rm K}$. Suppose that the H I in the cloud has a spin temperature $T_{\rm spin} = 50\,{\rm K}$, and that the optical depth at line-center (of the 21 cm line) is $\tau = 0.1$. The cloud is extended. We observe the cloud with a radio telescope with a beam that is small compared to the angular extent of the cloud.

(a) What will be the (absolute) brightness temperature T_B at line-center of the 21 cm line? Express your answer in deg K. You may assume that $h\nu \ll k_{\rm B}T_B$.

(b) What will be the (absolute) intensity at line-center of the 21 cm line? Express your answer in ${\rm Jy\,sr^{-1}}$.

7.2 Consider a photon of frequency $h\nu$ entering a slab of material containing two-level atoms with excitation temperature $T_{u\ell}$. At the frequency of the photon, let the optical depth of the slab be τ.

(a) Let $P_{\rm abs}$ be the probability that the original photon will undergo absorption before exiting from the slab. Give an expression for $P_{\rm abs}$ in terms of τ and $h\nu/k_{\rm B}T_{u\ell}$.

(b) Consider a photon that crossed the slab without being absorbed. Let $P_{\rm stim.em.}$ be the probability that the incident photon will stimulate emission of one or more photons. Give an expression for $P_{\rm stim.em.}$ in terms of τ and $h\nu/k_{\rm B}T_{u\ell}$.

7.3 Suppose that we have a molecule with three energy levels – denoted 0, 1, 2 – ordered according to increasing energy, $E_0 < E_1 < E_2$. Let g_0, g_1, g_2 be the degeneracies of the levels. Supppose that there is radiation present with $h\nu = E_2 - E_0$, due to an external source plus emission in the $2 \rightarrow 0$ transition.

Let ζ_{02} be the absorption probability per unit time for a molecule in level 0, with a transition to level 2. Let A_{20}, A_{21}, and A_{10} be the Einstein A coefficients for decays 2→0, 2→1, and 1→0 by spontaneous emission of a photon. Ignore collisional processes.

(a) Ignoring possible absorption of photons in the $2 \rightarrow 1$ and $1 \rightarrow 0$ transitions, obtain an expression for the ratio n_1/n_0, where n_i is the number density of molecules in level i.

(b) How large must ζ_{02} be for this molecule to act as a maser in the 1→0 transition?

(c) Is it possible for this system to have maser emission in the 2→1 transition? If so, what conditions must be satisfied?

7.4 Consider the simple harmonic oscillator with fundamental vibrational frequency ν_0, and energy levels $E_v = (v + \frac{1}{2})h\nu_0$, where $v = 0, 1, 2, .., v_{\rm max}$ is the vibrational quantum number.

Quantum mechanics tells us that the Einstein A coefficient for a transition $v \rightarrow v - 1$ is simply related to the A coefficient for $v = 1 \rightarrow 0$: $A_{v,v-1} = vA_{1,0}$.

Suppose that we have a gas of these harmonic oscillators, with number density n_v of oscillators in level v. The total number density $n = \sum_{v=0}^{v_{\rm max}} n_v$,

Let the oscillators all have a common velocity distribution, so that they all have the same normalized line profile ϕ_ν (with $\int \phi_\nu d\nu = 1$). The level degeneracies do not depend on quantum number v: $g_{v+1}/g_v = 1$. The absorption cross section of an oscillator in level v is given by Eq. (6.18):

$$\sigma_{v\rightarrow v+1}(\nu) = (v+1)\sigma_{0\rightarrow 1} = (v+1)C_0\phi_\nu$$
$$C_0 \equiv \left(\frac{c^2}{8\pi\nu_0^2}\right) A_{1,0} \quad .$$

Thus the absorption cross section $\sigma_{v\rightarrow v+1}$ increases with increasing vibrational excitation.

Now assume that the level populations have $n_{v_{\rm max}} = 0$ (i.e., negligible population in the highest level).

With this assumption, show that the **attenuation coefficient** is simply

$$\kappa_\nu = n\sigma_{0\rightarrow 1} = nC_0\phi_\nu \quad ,$$

with **no** dependence on how the populations are distributed among the levels $v < v_{\rm max}$: **the attenuation coefficient does not depend on the degree of vibrational excitation of the gas**.

7.5 A supernova remnant (SNR) is emitting a sychrotron continuum with a brightness temperature $T_{B,\rm SNR} = 700\,\rm K$ near 21 cm. An extended H I cloud is interposed between us and the SNR. Suppose that the H I in the cloud has a spin temperature $T_{\rm spin} = 100\,\rm K$, and that the optical depth of the H I $\lambda = 21.11\,\rm cm$ line at line-center is $\tau = 0.2$.
We observe the SNR through the cloud. The radio telescope has a beam that is small compared to the angular extent of the SNR and the cloud.

(a) What will be the (absolute) brightness temperature T_B at line-center of the 21 cm line? Express your answer in deg K. You may assume that $h\nu \ll k_{\rm B}T_{\rm spin} \ll k_{\rm B}T_{B,\rm SNR}$. The cosmic background radiation can be neglected.

(b) What will be the (absolute) intensity at line-center of the 21 cm line? Express your answer in $\rm Jy\,sr^{-1}$.

Chapter 8. H I 21-cm Emission and Absorption

8.1 H I 21 cm emission observations (if optically-thin) measure the amount n_u of H I in the hyperfine excited state. In Eq. (8.3) it was assumed that exactly 75% of the HI is in the excited state, so that $n(\mathrm{H\,I}) = (4/3) \times n_u$.

(a) What is the fractional error in the assumption that $n(\mathrm{H\,I}) = (4/3) \times n_u$ if $T_{\rm spin} = 100\,\mathrm{K}$?

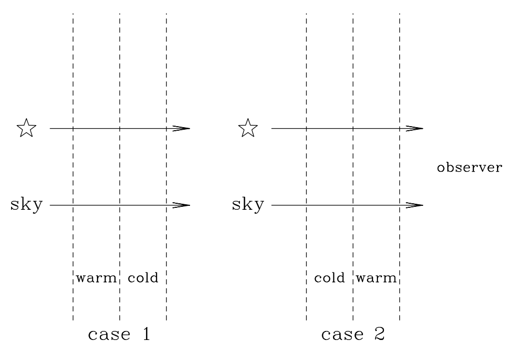
(b) What if $T_{\rm spin} = 20\,\mathrm{K}$?

8.2 For hydrogenic ions, the Lyman alpha transition has $\nu_0 \propto Z^2$, and $A_{u\ell} \propto Z^4$

Suppose that we have gas with a one-dimensional velocity $\mathrm{FWHM}_V = 100\,\mathrm{km\,s^{-1}}$. For what Z value does the intrinsic FWHM of the line equal the Doppler broadening FWHM?

8.3 Calculate the oscillator strength $f_{\ell u}$ for the H I 21 cm transition.

8.4 Interstellar H I is found with a range of temperatures, but the distribution is bimodal, leading to the concept of two distinct "phases": "cool" H I with spin temperature $T_c \approx 70\,\mathrm{K}$, and "warm" H I with spin temperature $T_w \approx 5000\,\mathrm{K}$.



Two geometries for the cool and warm H I.

Suppose that we observe an extragalactic radio source through Galactic H I consisting of a mixture of the cool and warm phases, with spin temperatures T_c and T_w. Consider two cases: case 1, where the cold material is closest to the observer, and case 2 where the warm material is closest (see above figure).

Let the cold and warm regions have H I column densities N_c and N_w. Assume that the cold and warm regions have Gaussian velocity profiles with the same central velocity and velocity dispersion σ_v.

The background sky brightness is $I_\nu^{\rm sky}$ (the CMB and background diffuse emission).

Let Ω be the beamsize of the radio telescope (defined such that a uniform intensity source I_ν gives a measured flux density $I_\nu \Omega$).

Let S_ν be the flux density from the source in the absence of any intervening absorption.

(a) What is the flux density $F_\nu^\star$ that the radio telescope will measure at the position of the source? Give the solution to the equation of radiative transfer at line-center for case 1 and for case 2. Write your answer in terms of S_ν, $I_\nu^{\rm sky}$, Ω, the temperatures T_c and T_w and the optical depths τ_c and τ_w of the two components.

(b) What is the flux density at line-center $F_\nu^{\rm off}$ that the radio telescope will measure when pointed off the source (the "sky" pointing in the figure)? Give the solution to the equation of radiative transfer for case 1 and for case 2. Write your answer in terms of $I_\nu^{\rm sky}$, Ω, the temperatures T_c and T_w and the optical depths τ_c and τ_w of the two components.

(c) If S_ν is known (by making measurements at frequencies where the H I absorption and emission are negligible), show how to determine $(\tau_c + \tau_w)$ from the observed $F_\nu^\star$ and $F_\nu^{\rm off}$. Do you need to know whether the geometry is case 1 or case 2?

(d) Suppose that $I_\nu^{\rm sky}$ is known (from measurements at frequencies where 21-cm emission and absorption are negligible). If $\tau_w \ll 1$ and $\tau_c \ll 1$, show how the total column density N can be obtained from the measurements.

(e) Suppose that the total optical depth $\tau = \tau_c + \tau_w$ is now known from part (c), and assume that the total column density $N = N_c + N_w$ is also known from part (d). If the observer thought that all the gas has a single spin temperature $T_{\rm eff}$, give a relation between $T_{\rm eff}$ and the actual temperatures (T_c and T_w) and column densities (N_c and N_w).

8.5 An extragalactic radio "point source" (unresolved by the beam of the radio telescope) is observed to have an emission feature. The observed flux density is approximately constant at $F_\nu = 0.01\,{\rm Jy}$ from $1299.9\,{\rm MHz}$ to $1300.1\,{\rm MHz}$, with a negligible continuum below 1299.9 MHz and above 1300.1 MHz.

The emission feature is interpreted as the 21 cm line of H I.

(a) What is the redshift of the galaxy?

(b) For a Hubble constant of $H = 70\,{\rm km\,s^{-1}\,Mpc^{-1}}$, estimate the "luminosity distance" D_L to the galaxy. (Assume a simple, uniform "Hubble flow" in Euclidean space – don't worry about relativistic corrections.)

(c) If self-absorption can be ignored, what is the mass of H I in the galaxy?

(d) If the galaxy is a disk of radius $R = 20\,{\rm kpc}$, what is the average H I column density $N({\rm H\,I})$ in ${\rm cm^{-2}}$?

(e) What can be said about the velocity distribution of the H I in the galaxy's rest frame?

8.6 A dwarf galaxy at a distance $D = 15\,{\rm Mpc}$ is emitting in the 21-cm line of atomic hydrogen. The observed 21-cm line flux is $F = 1 \times 10^{-18}\,{\rm erg\,cm^{-2}\,s^{-1}} = 1 \times 10^{-21}\,{\rm W\,m^{-2}}$

If the emitting gas is assumed to be optically thin, and there is no absorption by intervening gas, estimate the mass of H I in the dwarf galaxy. You may neglect relativistic corrections (the redshift is small). Express your answer in $M_\odot$.

The Einstein A coefficient and wavelength for the 21-cm line are $A_{u\ell} = 2.88 \times 10^{-15}\,{\rm s^{-1}}$ and $\lambda = 21.11\,{\rm cm}$.

Chapter 9. Absorption Lines: The Curve of Growth

9.1 Suppose that we observe a radio-bright QSO and detect absorption lines from Milky Way gas in its spectra. The 21 cm line is seen in optically-thin absorption with a profile with $\mathrm{FWHM(H\,I)} = 10\,\mathrm{km\,s^{-1}}$. We also have high-resolution observations of the Na I doublet lines referred to as "D_1" (5898 Å) and "D_2" (5892 Å) [see Table 9.3] in absorption. The Na I $\mathrm{D_2}$ λ5892 Å line width is $\mathrm{FWHM(Na\,I\,D_2)} = 5\,\mathrm{km\,s^{-1}}$. The line profiles are the result of a combination of thermal broadening plus turbulence with a Gaussian velocity distribution with one-dimensional velocity dispersion $\sigma_{v,\mathrm{turb}}$.

You will want to employ the following theorem: If the turbulence has a Gaussian velocity distribution, the overall velocity distribution function of atoms of mass M will be Gaussian, with one-dimensional velocity dispersion

$$\sigma_v^2 = \sigma_{v,\mathrm{turb}}^2 + \frac{k_\mathrm{B}T}{M} \quad .$$

(a) If the Na I $\mathrm{D_2}$ line is optically thin, estimate the kinetic temperature T and $\sigma_{v,\mathrm{turb}}$.

(b) Now suppose that the observed Na I D doublet ratio $W_2/W_1 < 2$. What can be said about T and σ_v^{turb}?

9.2 Calculate the absorption cross section per H in the pseudo-continuum when the high-n Lyman series lines blend together, and compare to the photoionization threshold value for H. [The asymptotic formula for the oscillator strength for the high-n Lyman series transitions is given in Table 9.1.]

9.3 An absorption line, assumed to be H I Lyman α, is measured to have a dimensionless equivalent width $W = (2.00 \pm 0.10) \times 10^{-4}$. Suppose that the velocity profile is a Gaussian with $b \approx 5\,\mathrm{km\,s^{-1}}$. If b is known exactly, estimate the uncertainty in $N_\ell f_{\ell u} \lambda_{\ell u}$ arising from the $\pm 5\%$ uncertainty in W.

9.4 A distant quasar at a redshift $z_Q = 2.5$ is observed on a line-of-sight which passes through the disk of an intervening galaxy. A strong absorption feature is observed in the continuum spectrum of the quasar at an observed wavelength of 3647 Å. This absorption feature is interpreted as Lyman-α absorption in the intervening galaxy, implying that the galaxy is at a redshift $z_G = 2.0$.

(a) The absorption feature at 3647 Å has an observed equivalent width $W_{\lambda,\mathrm{obs}} = 6.0$ Å. The equivalent width that would be observed by an observer in the rest-frame of the absorbing galaxy would be $W_{\lambda,G} = 6.0\,\text{Å}/(1 + z_G) = 2.0$ Å. Estimate the HI optical depth at line-center of the Lyα line which is required to produce this equivalent width. Assume the one-dimensional velocity dispersion of the HI to be $20\,\mathrm{km\,s^{-1}}$. [Hint: consider Eq. (9.15, 9.19, 9.24); by trial-and-error determine which part of the curve-of-growth you are on.]

(b) What is the column density of HI in the $n = 1$ level in the intervening galaxy? Remark on the similarity/difference between the interstellar medium of this galaxy versus the local ISM in our Galaxy.

9.5 A quasar (PKS0237-23) at a redshift $z_Q = 2.22$ is observed to have an absorption feature in its spectrum produced by Si II ions at a redshift $z_G = 1.36$ The absorption line is due to the allowed transition Si II $^2\mathrm{P}^\mathrm{o}_{1/2} \rightarrow {}^2\mathrm{S}_{1/2}$ (see the energy level diagram on p. 493) at a rest wavelength $\lambda = 1527$ Å (at an observed wavelength $\lambda_\mathrm{obs} = 3604$ Å).

The $^2\mathrm{P}^\mathrm{o}_{1/2} \rightarrow {}^2\mathrm{S}_{1/2}$ feature has an observed equivalent width $W_{\lambda,\mathrm{obs}} = 2$ Å. The conventional interpretation is that this absorption feature is produced in an intervening galaxy.

(a) What is the column density $N(\mathrm{Si\,II}\,{}^2\mathrm{P}^\mathrm{o}_{1/2})$ of Si II in the ground state? Assume the line to be optically thin (what condition does this impose on the velocity dispersion of the SiII in the intervening galaxy?). Required atomic data can be found in the text (Table 9.5).

(b) The quasar spectrum shows no trace of absorption in the $^2\mathrm{P}^\mathrm{o}_{3/2} \rightarrow {}^2\mathrm{S}_{1/2}$ transition of Si II at $\lambda = 1533$ Å. If the upper limit on the observed equivalent width is $(W_\lambda)_\mathrm{obs} < 1$ Å, what is the corresponding upper limit on the column density $N(\mathrm{Si\,II}\,{}^2\mathrm{P}^\mathrm{o}_{3/2})$ in the intervening galaxy?

(c) Given your result from (b) on the upper bound for $N(\mathrm{Si\,II}\,^2\mathrm{P}^{\mathrm{o}}_{3/2})$, what limit can be placed on the electron density n_e in the intervening galaxy if the kinetic temperature is assumed to be $10^4\,\mathrm{K}$? The Einstein A coefficient is $A(^2\mathrm{P}^{\mathrm{o}}_{3/2} \rightarrow\ ^2\mathrm{P}^{\mathrm{o}}_{1/2}) = 2.13\times10^{-4}\,\mathrm{s}^{-1}$, and the electron collision strength is $\Omega(^2\mathrm{P}^{\mathrm{o}}_{3/2},\ ^2\mathrm{P}^{\mathrm{o}}_{1/2}) = 4.45$ (see Table F1 on p. 496). (Ignore the existence of the $^2\mathrm{S}_{1/2}$ state in this and (d) below; i.e., treat the two fine-structure states as a two-level system. Assume the interstellar radiation field in the intervening galaxy to be not too wildly dissimilar to that in our Galaxy.)

(d) Can any useful limit be placed on n_e if the kinetic temperature is assumed to be 10^2 K rather than 10^4 K?

9.6 An unconventional interpretation of the observations described above (in problem 9.5) is that the Si II absorption is produced in a cloud of gas which has been shot out of the quasar with a velocity βc (relative to the quasar) which gives it a redshift (as seen from the quasar) z_{GQ} satisfying $(1 + z_G)(1 + z_{GQ}) = (1 + z_{em})$, where $z_G = 1.36$ and $z_{em} = 2.22$. Thus $(1 + z_{GQ}) = (1 + z_{em})/(1 + z_G) = 1.364$.

The velocity βc of the cloud relative to the QSO is then given by the relativistic Doppler shift formula

$$1.364 = 1 + z_{GQ} = \frac{1+\beta}{(1-\beta^2)^{1/2}} = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \quad ,$$

with the result

$$\beta = \frac{(1+z_{GQ})^2 - 1}{(1+z_{GQ})^2 + 1} = 0.301 \quad .$$

Suppose the quasar to be emitting (isotropically) a power per unit frequency (evaluated in the rest frame of the quasar) $P_\nu = (L_0/\nu_0)(\nu/\nu_0)^{-\alpha}$, where $L_0 = 10^{13}\,L_\odot$ and $\nu_0 = 10^{15}$ Hz, and the exponent α is of order unity.

At a distance D from the QSO, in a frame at rest relative to the QSO, the energy density is

$$u_\nu = \frac{P_\nu}{4\pi D^2 c} = \frac{L_0/\nu_0}{4\pi D^2 c}\left(\frac{\nu}{\nu_0}\right)^{-\alpha} \quad .$$

A little bit of special-relativistic reasoning leads to the conclusion that a "cloud" observer receding from the QSO at velocity $\beta_{GQ}c$ will find that the energy density at frequency ν_G (measured in the gas cloud frame) is given by

$$(u_\nu)_G = \frac{1}{(1+z_{GQ})^{1+\alpha}}\frac{L_0/\nu_0}{4\pi D^2 c}\left(\frac{\nu_G}{\nu_0}\right)^{-\alpha} \quad .$$

(a) For the moment consider only transitions between the $^2\mathrm{P}^{\mathrm{o}}_{1/2}$ and $^2\mathrm{P}^{\mathrm{o}}_{3/2}$ levels. What is the minimum value of D which is consistent with the observed upper limit on the ratio $N(\mathrm{Si\,II}\,^2\mathrm{P}^{\mathrm{o}}_{3/2})/N(\mathrm{Si\,II}\,^2\mathrm{P}^{\mathrm{o}}_{1/2})$? (Assume $n_e = 0$).

(b) Now consider pumping of the $^2\mathrm{P}^{\mathrm{o}}_{3/2}$ level via the $^2\mathrm{S}_{1/2}$ level. What is the probability per time for an Si II ion in the $^2\mathrm{P}^{\mathrm{o}}_{1/2}$ state to be excited to the $^2\mathrm{S}_{1/2}$ level by absorbing a UV photon? Give your answer as a function of D.

(c) What fraction of the Si II excitations to the $^2\mathrm{S}_{1/2}$ state will lead to population of the $^2\mathrm{P}^{\mathrm{o}}_{3/2}$ state?

(d) Suppose the absorbing cloud to be a spherical shell around the quasar. If the Si/H ratio in the gas does not exceed the Si/H ratio in our Galaxy (Si/H=4×10^{-5}), and the gas has He/H $= 0.1$, what is the minimum kinetic energy of this expanding shell? (This extreme energy requirement has been used in arguing against this interpretation of absorption line systems.)

9.7 An absorption line is observed in the spectrum of a quasar at an observed wavelength $\lambda = 5000.$ Å. The absorption is produced by an intergalactic cloud of gas somewhere between us and the quasar. The observer measures an equivalent width $W_\lambda = 1.0\times10^{-2}$ Å. The absorption line is resolved, with an observed $\mathrm{FWHM}_\lambda = 0.50$ Å.

The line is assumed to be H I Lyman α, with rest wavelength $\lambda_0 = 1215.7$ Å and oscillator strength $f_{\ell u} = 0.4164$.

(a) What is the redshift z of the absorber?

(b) What is the column density of H I in the absorbing cloud?

(c) In the rest frame of the cloud, the H I has a one-dimensional velocity distribution $\propto e^{-(\Delta v/b)^2}$. What is the value of b for this cloud?

9.8 The spectrum of a quasar has absorption lines at observed wavelengths $\lambda = 5000.0\,\text{Å}$ and $5008.4\,\text{Å}$, with observed equivalent widths $W_\lambda = 0.020\,\text{Å}$ and $W_\lambda = 0.010\,\text{Å}$, respectively. Both lines are resolved, each with observed $\mathrm{FWHM}_\lambda = 0.40\,\text{Å}$.

The lines are interpreted as being produced by C IV, with rest wavelengths $\lambda_0 = 1548.20\,\text{Å}$ and $1550.77\,\text{Å}$, and oscillator strengths $f_{\ell u} = 0.190$ and 0.096.

(a) What is the redshift z of the absorber?

(b) What is the column density of C IV in the absorbing cloud?

(c) In the rest frame of the cloud, the H I has a one-dimensional velocity distribution $\propto e^{-(\Delta v/b)^2}$. What is the value of b for this cloud?

9.9 The CH$^+$ molecule has an absorption line at $\lambda = 4233\,\text{Å}$ with an oscillator strength $f_{\ell u} = 0.0060$ out of the ground state ℓ. An absorption line is observed at this wavelength with an equivalent width $W_\lambda = 0.010\,\text{Å}$, and a FWHM of $10\,\mathrm{km\,s^{-1}}$. What is the column density of ground-state CH$^+$ on this line-of-sight? Single-digit accuracy is sufficient.

9.10 High-resolution spectra of a quasar show absorption by H Lyman α (rest-frame wavelength $1215.6\,\text{Å}$) at an observed wavelength $\lambda = 3890.2\,\text{Å}$ and a C IV absorption doublet (rest-frame wavelengths $1548.2, 1550.8\,\text{Å}$) at $\lambda = 4954.2\,\text{Å}$ and $\lambda = 4962.6\,\text{Å}$.
Suppose that all three lines are optically thin, with Gaussian line profiles.
The line at $3890.2\,\text{Å}$ has observed full-width-at-half-maximum $\mathrm{FWHM_H} = 0.3168\,\text{Å}$.
The line at $\lambda = 4954.2\,\text{Å}$ has observed $\mathrm{FWHM_{C\,IV}} = 0.2196\,\text{Å}$.
Recall that if a variable x has a Gaussian distribution, $\mathrm{FWHM}_x = \sqrt{8\ln 2} \times \sigma_x = 2.355\,\sigma_x$.

(a) What is the redshift z of the absorbing gas?

(b) What is the one-dimensional velocity dispersion $\sigma_{v,\mathrm{H}}$ of the hydrogen atoms (in the absorption system rest frame)? Give your answer in $\mathrm{km\,s^{-1}}$.

(c) What is the one-dimensional velocity dispersion $\sigma_{v,\mathrm{C\,IV}}$ of the C IV ions (in the absorption system rest frame)? Give your answer in $\mathrm{km\,s^{-1}}$

(d) Assume that the H and C IV are in gas with temperature T and turbulence with one-dimensional turbulent velocity dispersion σ_{turb}, so that the one-dimensional velocity dispersion of a particle of mass M is given by the sum (in quadrature) of the thermal and turbulent velocity dispersions:

$$\sigma_v^2 = \frac{k_\mathrm{B}T}{M} + \sigma_{\mathrm{turb}}^2$$

For the absorption line system, what is T (in degrees K) and σ_{turb} (in $\mathrm{km\,s^{-1}}$)?

9.11 Suppose that an H atom in the $3p$ level is at rest in an H I cloud of density $n(\mathrm{H}) = 20\,\mathrm{cm^{-3}}$ and kinetic temperature $T = 100\,\mathrm{K}$. Assume that the motions of the other H atoms in the cloud are purely thermal. Assume the cloud to be infinite in extent, and pure H (no dust, etc.).

If the H($3p$) emits a Lyman β photon, what is the mean free path of this photon before it is absorbed by another H atom? The wavelength of Lyman β is 1025.7Å. The oscillator strength for the Lyman β transition is $f_{1s,3p} = 0.0791$.

Chapter 10. Emission and Absorption by a Thermal Plasma

10.1 The brightest part of the Orion H II region has an emission measure $EM \approx 5{\times}10^6\,\mathrm{cm^{-6}\,pc}$. Assume an electron temperature $T_e = 10^4\,\mathrm{K}$.

(a) What is the optical depth τ due to free-free absorption at $\lambda = 1\,\mathrm{cm}$ ($\nu = 30\,\mathrm{GHz}$)?

(b) What is the optical depth τ due to free-free absorption at $\lambda = 21.1\,\mathrm{cm}$ ($\nu = 1420\,\mathrm{MHz}$)?

(c) Suppose that there is atomic hydrogen on the far side of the H II region with a column density $N(\mathrm{H\,I}) = 10^{21}\,\mathrm{cm^{-2}}$ and a spin temperature $T_{\rm spin} = 1000\,\mathrm{K}$. Calculate the observed strength of the 21 cm line (where "line" is the excess above the continuum), integrated over the line profile, and expressed in the usual "antenna temperature-velocity" units of $\mathrm{K\,km\,s^{-1}}$. Assume the line to be broad enough to be optically thin.

(d) A radio telescope observes the brightest part of the H II region. Calculate the dimensionless "equivalent width" W of the 21 cm line, and also calculate the "velocity" equivalent width $W_V \equiv c \times W$ and the "frequency" equivalent width $W_\nu \equiv \nu \times W$.
Note: the dimensionless equivalent width of an emission line is defined to be

$$W \equiv \int \frac{[I_\nu - I_\nu^{(c)}]}{I_\nu^{(c)}} \frac{d\nu}{\nu}$$

where $I_\nu^{(c)}$ is the "continuum" level of the free-free emission on either side of the 21 cm line.

10.2 Suppose that a slab of ionized hydrogen has emission measure EM, temperature $T = 10^4 T_4\,\mathrm{K}$, and a Gaussian velocity distribution with one-dimensional velocity dispersion $\sigma_v = \sigma_{v5}\,\mathrm{km\,s^{-1}}$.

(a) Calculate the optical depth $\tau_{n\alpha}$ at line center for $n\alpha$ radiation propagating through the slab.
You may assume that $n \gg 1$, and you should leave the departure coefficient b_n and the factor β_n [defined in Eq. (10.30)] as unknown quantities (i.e., represented by symbols b_n and β_n). The Einstein A coefficient $A_{n+1\to n}$ is given by Eq. (10.27). The number density n_n of hydrogen in level n is related to $n_e n_p$ and b_n using Eq. (3.45).
Your result for $\tau_{n\alpha}$ should be given as an expression containing only a numerical coefficient, and the variables n, b_n, β_n, T_4, σ_{v5}, and $(EM/\,\mathrm{cm^{-5}})$.

(b) Assume $T_4 = 1$, $EM = 10\,\mathrm{cm^{-6}\,pc}$, $b_n = 0.9$, and $\beta_n = -100$ for $n = 166$. Evaluate $\tau_{n\alpha}$ for $n = 166$ if there is no turbulence present.

10.3 We are hoping to observe 21 cm emission from redshift $z \approx 9$ and need to model the "Galactic foreground" produced by a slab of partially-ionized hydrogen (at redshift 0) at temperature T. Consider a H $n\alpha$ line originating in this slab.

(a) For what n will the H $n\alpha$ line be near $142\,\mathrm{MHz}$?

(b) Suppose that $\beta_{n\alpha}$ [defined by Eq. (10.30)] is negative, and suppose that the optical depth $\tau_{n\alpha}$ is a small negative number. Suppose that just beyond the slab of partially-ionized hydrogen, there is a region producing synchrotron emission with antenna temperature $T_{A,0}$ at 142 MHz.
If the hydrogen in the slab is "isothermal" (or perhaps we should say "iso-excited"), then the *exact* solution to the equation of radiative transfer is simply

$$I_\nu = I_{\nu,0} \mathrm{e}^{-\tau_\nu} + B_\nu(T_{\rm exc}) \left(1 - \mathrm{e}^{-\tau_\nu}\right) \quad ,$$

where recall that the *definition* of $T_{\rm exc}$ is such that

$$\frac{n_u}{n_\ell} \equiv \frac{g_u}{g_\ell} \mathrm{e}^{-E_{u\ell}/k_{\rm B} T_{\rm exc}} \quad .$$

Assuming that $|h\nu/k_{\rm B} T_{\rm exc}| \ll 1$ and $|\tau_\nu| \ll 1$, show that

$$T_A(\nu) \approx T_{A,0} \mathrm{e}^{-\tau_\nu} + T_{\rm exc} \tau_\nu \quad .$$

(c) Recall the definition of $\beta_{n\alpha}$:

$$\beta_{n\alpha} \equiv \frac{1-(n_u g_\ell)/(n_\ell g_u)}{1-\exp(-h\nu/k_{\rm B}T)} \quad ,$$

where $u = n+1$ and $\ell = n$. If $|h\nu/k_{\rm B}T| \ll 1$, $|h\nu/k_{\rm B}T_{\rm exc}| \ll 1$, and $|\tau_\nu| \ll 1$, show that

$$T_A(\nu) = T_{A,0}{\rm e}^{-\tau_\nu} + \beta_{n\alpha}^{-1} T\tau_\nu \quad .$$

(d) The sky-averaged synchrotron background is approximately given by Eq. (12.3). What is the sky-averaged antenna temperature T_{A0} of the synchrotron background at 142 MHz?

(e) Suppose that the difference in antenna temperature "on-line" vs. "off-line" is

$$\Delta T_A = [T_{A,0}{\rm e}^{-\tau_\nu} + \beta_{n\alpha}^{-1}T\tau_\nu] - T_{A,0} = T_{A,0}({\rm e}^{-\tau_\nu} - 1) + \beta_{n\alpha}^{-1}T\tau_\nu \quad .$$

If $\tau_\nu = -10^{-7}$, $\beta_{n\alpha} = -100$, $T = 10^4\,{\rm K}$, and $T_{A,0} = 500\,{\rm K}$, calculate the "antenna temperature" ΔT_A of the line. Which is larger – the amplification of the synchrotron emission, or the contribution $\beta_{n\alpha}^{-1}T\tau_\nu$ that is independent of the synchrotron emission? [For comparison, the redshifted 21cm line is expected to have $\Delta T_A \approx +15$ mK if the universe was reionized by radiation from massive "Pop III" stars, and the fluctuations in antenna temperature due to "minihalos" at $z \approx 9$ are expected to be of order ~ 1 mK (Furlanetto et al. 2006: Physics Reports 433, 181, Fig. 12)].

10.4 Consider an H II region with $n({\rm H}^+) = n_e = 10^3\,{\rm cm}^{-3}$, $T = 8000\,{\rm K}$, and radius $R = 1\,{\rm pc}$. Estimate the radio frequency ν at which the optical depth across the diameter of the H II region is $\tau = 1$. To make this estimate you may assume that the Gaunt factor $g_{\rm ff} \approx 6$.

10.5 Consider an ionized wind flowing outward from a point source. Suppose that the temperature $T = 10^4 T_4\,{\rm K}$ and the electron density n_e varies as

$$n(r) = n_0 \left(\frac{R_0}{r}\right)^2 \quad .$$

R_0 is simply some reference radius, with n_0 the electron density at that radius. Assume the attenuation coefficient κ_ν to have the simple power-law dependence on frequency $\nu = \nu_9\,{\rm GHz}$ and temperature given by Eq. (10.8):

$$\kappa_\nu = \frac{A}{R_0}\left(\frac{n}{n_0}\right)^2 \nu_9^{-2.12}$$
$$A \equiv 1.09\times10^{-25} n_0^2 R_0 T_4^{-1.32}\,{\rm cm}^5 \quad .$$

(a) Let $\tau(R)$ be the attenuation optical depth along a radial path from $r = R$ to $r = \infty$. Define $R_p(\nu)$ to be the radius where $\tau = 2/3$. Obtain an expression for R_p/R_0 in terms of A and the frequency $\nu_9 \equiv \nu/\,{\rm GHz}$.

(b) When viewed at frequency ν by a distant observer, the wind will have a "photosphere" at radius R_p. Suppose that the emission from this photosphere at frequency ν can be approximated as a blackbody. Assume we are in the Rayleigh-Jeans limit ($h\nu \ll k_{\rm B}T$). The observer is at distance D.
Obtain an expression for the flux density $F_\nu^{\rm (photo)}$ of the "photospheric emission".
The "spectral index" β is defined by $F_\nu^{\rm (photo)} \propto \nu^\beta$. Obtain β.

(c) In addition to the "photospheric" emission, there will be additional emission from the optically-thin wind outside the photosphere. Assume the emissivity to have the simple power-law dependence given by Eq. (10.8):

$$4\pi j_\nu = B\left(\frac{n}{n_0}\right)^2 \nu_9^{-0.12}$$
$$B = 4\pi \times 3.35\times10^{-40} T_4^{-0.32} n_0^2\,{\rm erg\,cm^3\,s^{-1}\,Hz^{-1}} \quad .$$

If the wind extends to infinity, and absorption can be entirely ignored (ignore the fact that the "photosphere" blocks radiation from some of the material on the far side), calculate the flux density $F_\nu^{\rm (outer)}$ from this extended emission in terms of B, R_0, and D. What is the spectral index of $F_\nu^{\rm (outer)}$?

(d) According to this approximate treatment, what is the ratio $F_\nu^{(\rm outer)}/F_\nu^{(\rm photo)}$?

(e) By neglecting absorption, the above treatment has overestimated $F_\nu^{(\rm outer)}$, so we now neglect $F_\nu^{(\rm photo)}$ and take $F_\nu \approx F_\nu^{(\rm outer)}$. If the wind has a mass-loss rate $\dot{M}_{\rm w}$ and velocity $v_{\rm w}$, show that

$$F_\nu^{(\rm outer)} \approx 0.013\,{\rm Jy}\left(\frac{\rm kpc}{D}\right)^2 T_4^{0.12}\left(\frac{\dot{M}}{10^{-6}\,M_\odot\,{\rm yr}^{-1}}\right)^{4/3}\left(\frac{20\,{\rm km\,s}^{-1}}{v_{\rm w}}\right)^{4/3}\nu_9^{0.59} \quad .$$

Assume H to be fully ionized but He to be neutral, so that $n = \rho/1.4m_{\rm H}$.

10.6 Consider an ionized wind flowing outward from a point source. Suppose that the electron density and temperature $T = 10^4 T_4$ K vary as

$$n(r) = n_0\left(\frac{R_0}{r}\right)^2$$
$$T(r) = T_0\left(\frac{R_0}{r}\right)^\gamma$$

with $\gamma = 0$ for a constant temperature outflow, or $\gamma = 4/3$ if the expanding wind cools adiabatically. R_0 is simply some reference radius, with n_0 and T_0 the density and temperature at that radius. Let $T_{40} \equiv T_0/10^4$ K. Assume the attenuation coefficient κ_ν to have the simple power-law dependence on frequency $\nu = \nu_9$ GHz and temperature given by Eq. (10.8):

$$\kappa_\nu = \frac{A}{R_0}\left(\frac{n}{n_0}\right)^2\left(\frac{T_4}{T_{40}}\right)^{-1.32}\nu_9^{-2.12}$$
$$A \equiv 1.09\times10^{-25} n_0^2 R_0 T_{40}^{-1.32}\,{\rm cm}^5$$

(a) Let $\tau(R)$ be the attenuation optical depth along a radial path from $r = R$ to $r = \infty$. Define $R_p(\nu)$ to be the radius where $\tau = 2/3$. Obtain an expression for R_p/R_0 in terms of A, γ, and the frequency $\nu_9 \equiv \nu/$ GHz.

(b) Let $T_p(\nu)$ be the temperature at $r = R_p(\nu)$. Obtain an expression for $T_p(\nu)/T_0$ as a function of A, γ, and ν_9.

(c) When viewed at frequency ν by a distant observer, the wind will have a "photosphere" at radius R_p. Suppose that the emission from this photosphere at frequency ν can be approximated as a blackbody with temperature $T_p(\nu)$. Assume we are in the Rayleigh-Jeans limit ($h\nu \ll k_{\rm B}T$). If the observer is at distance D, obtain an expression for the flux density F_ν of this photospheric emission as function of frequency ν. The "spectral index" β is defined by $F_\nu^{(\rm photo)} \propto \nu^\beta$. What is the range of the spectral index β if $0 \le \gamma \le 4/3$?

(d) In addition to the "photospheric" emission, there will be additional emission from the optically-thin wind outside the photosphere. Assume the emissivity to have the simple power-law dependence given by Eq. (10.8):

$$4\pi j_\nu = B\left(\frac{n}{n_0}\right)^2\left(\frac{T_4}{T_{40}}\right)^{-0.32}\nu_9^{-0.12}$$
$$B = 4\pi\times3.35\times10^{-40} T_{40}^{-0.32} n_0^2\,{\rm erg\,cm^3\,s^{-1}\,Hz^{-1}}$$

If the wind extends to infinity, and absorption can be entirely ignored (ignore the fact that the "photosphere" blocks radiation from some of the material on the far side), calculate the flux density $F_\nu^{(\rm outer)}$ from this extended emission in terms of B, R_0, D, and γ. What is the spectral index of $F_\nu^{(\rm outer)}$?

(e) According to this approximate treatment, what is the ratio $F_\nu^{(\rm outer)}/F_\nu^{(\rm photo)}$?

(f) By neglecting absorption, the above treatment has overestimated $F_\nu^{(\mathrm{outer})}$, so we now neglect $F_\nu^{(photo)}$ and take $F_\nu \approx F_\nu^{(\mathrm{outer})}$. If the wind has a mass-loss rate $\dot{M}_\mathrm{w}$ and velocity v_w, and constant temperature ($\gamma = 0$), show that

$$\frac{F_\nu^{(\mathrm{outer})}}{\mathrm{Jy}} \approx 0.013\,\mathrm{Jy}\left(\frac{\mathrm{kpc}}{D}\right)^2 T_{40}^{0.12}\left(\frac{\dot{M}}{10^{-6}\,M_\odot\,\mathrm{yr}^{-1}}\right)^{4/3}\left(\frac{20\,\mathrm{km\,s^{-1}}}{v_\mathrm{w}}\right)^{4/3}\nu_9^{0.59}$$

Assume H to be fully ionized but He to be neutral, so that $n = \rho/1.4m_\mathrm{H}$.

10.7 Consider an H I cloud of column density $N_\mathrm{H} = 10^{21}\,\mathrm{cm}^{-2}$, H nucleon density $n_\mathrm{H} = 30\,\mathrm{cm}^{-3}$, temperature $T = 100\,\mathrm{K}$, and fractional ionization $n_e/n_\mathrm{H} = 10^{-3}$.

(a) What is the emission measure $EM = \int n_e n_i ds$?

(b) What is the surface brightness (i.e., specific intensity) of free-free emission from the cloud at $\nu = 5\,\mathrm{GHz}$? Assume the cloud to be optically-thin at $5\,\mathrm{GHz}$, and take $g_\mathrm{ff} \approx 1.83$. Express your answer in $\mathrm{Jy\,sr}^{-1}$.

Chapter 11. Propagation of Radio Waves through the ISM

11.1 The pulses from a pulsar arrive later at low frequencies than at high frequencies. Suppose that the arrival times at $1420\,\mathrm{MHz}$ and $1610\,\mathrm{MHz}$ differ by $\Delta t(1420\,\mathrm{MHz}, 1610\,\mathrm{MHz}) = 0.0913\,\mathrm{s}$.

(a) What is the "dispersion measure" for this pulsar?

(b) If the pulsar is assumed to be at a distance $D = 6\,\mathrm{kpc}$, what is the mean electron density $\langle n_e \rangle$ along the path to the pulsar?

11.2 A pulsar is observed at 1610 and 1660 MHz. The plane of polarization at these two frequencies differs by $57.5°$.

(a) What is the *minimum* possible magnitude of the rotation measure $|RM|$ toward this source? Why is it a minimum? What would be the next-largest possible value of $|RM|$?

(b) If the source has a dispersion measure $DM = 200\,\mathrm{cm}^{-3}\,\mathrm{pc}$, and using the minimum $|RM|$ from (a), what is the electron-density-weighted component of the magnetic field along the line-of-sight?

11.3 A fast radio burst (FRB) occurs in a galaxy at redshift z_{FRB}. The pulse arrival is delayed at low frequencies because of dispersion contributed by electrons along the path [including electrons in the Milky Way, the intergalactic medium (IGM), and the host galaxy of the FRB]. The observed DM is

$$DM_{\mathrm{obs}} = -\frac{\pi m_e c}{e^2} \nu_{\mathrm{obs}}^3 \frac{dt_{\mathrm{arrival}}}{d\nu_{\mathrm{obs}}} \quad .$$

For $z \lesssim 7$ (i.e., after reionization), assume the electron density in the IGM to be

$$\begin{aligned} n_e &= n_0 (1+z)^3 \\ n_0 &\approx 1.1 \times 10^{-7}\,\mathrm{cm}^{-3} \quad , \end{aligned}$$

(corresponding to an IGM containing ∼50% of the baryons in the Universe). Assume a Hubble constant $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

If the redshift is not too large, we can assume a simple Hubble flow, with redshift proportional to distance, $cdz = H_0 dr$.

(a) At low redshift $z \ll 1$, show that the contribution of the IGM to the observed dispersion measure is

$$DM_{\mathrm{IGM}} = 471 \left(\frac{n_0}{1.1 \times 10^{-7}\,\mathrm{cm}^{-3}} \right) z_{\mathrm{FRB}}\,\mathrm{cm}^{-3}\,\mathrm{pc}$$

(b) At larger redshifts, one needs to take into account both the change in density of the universe and redshifting of the radiation in the pulse as it travels from the FRB to us. Show that the contribution of the IGM to DM is

$$DM_{\mathrm{IGM}} = 471 \left(\frac{n_0}{1.1 \times 10^{-7}\,\mathrm{cm}^{-3}} \right) \frac{\left[(1+z_{\mathrm{FRB}})^2 - 1\right]}{2}\,\mathrm{cm}^{-3}\,\mathrm{pc}$$

[To keep things simple, continue to assume a simple Hubble flow, $cdz = H_0 dr$.]

11.4 A fast radio burst (FRB) occurs in a galaxy at redshift z_{FRB}. The pulse arrival is delayed at low frequencies because of dispersion contributed by electrons along the path [including electrons in the Milky Way, the intergalactic medium (IGM), and the host galaxy of the FRB] with observed dispersion measure

$$DM_{\mathrm{obs}} = -\frac{\pi m_e c}{e^2} \nu_{\mathrm{obs}}^3 \frac{dt_{\mathrm{arrival}}}{d\nu_{\mathrm{obs}}} \quad .$$

In addition, the polarization varies with wavelength with observed rotation measure

$$[RM]_{\rm obs} \equiv \frac{d\Psi}{d\lambda^2_{\rm obs}}$$

Suppose that the source is at redshift $z_{\rm FRB}$, and that the intergalactic medium (IGM) has electron density

$$n_e = n_0(1+z)^3$$
$$n_0 \approx 1.1 \times 10^{-7}\,{\rm cm}^{-3}$$

(corresponding to an IGM containing ∼50% of the baryons in the Universe). Assume a Hubble constant $H_0 = 70\,{\rm km\,s^{-1}\,Mpc^{-1}}$. Suppose also that there is a magnetic field parallel to the direction of propagation with

$$B_\parallel = B_{\parallel 0}(1+z)^2 \quad .$$

(If for some reason there were a large-scale primordial magnetic field, this is how it would vary in an expanding Universe.)

If the redshift $z \lesssim 1$, we can assume a simple Hubble flow, with redshift proportional to distance: $cdz = H_0 dr$.

(a) Show that the contribution of the IGM to the RM is

$$[RM]_{\rm IGM} = \frac{e^3}{2\pi m_e^2 c^3}\frac{n_0 B_{0\parallel}}{H_0}\frac{\left[(1+z)^4 - 1\right]}{4} \quad .$$

(b) FRB 150807 was observed to have $DM = 266\,{\rm cm^{-3}\,pc}$ and $RM = 12 \pm 7\,{\rm rad\,m^{-2}}$. Assume that the DM is given by (see Problem 11.3).

$$[DM]_{\rm IGM} = 471\left(\frac{n_0}{1.1\times10^{-7}\,{\rm cm^{-3}}}\right)\frac{\left[(1+z_{\rm FRB})^2 - 1\right]}{2}\,{\rm cm^{-3}\,pc}$$

Suppose that *all* of the DM and RM come from the IGM, and $n_0 = 1.1\times10^{-7}\,{\rm cm^{-3}}$. Estimate the redshift $z_{\rm FRB}$ and $B_{0\parallel}$.

Chapter 12. Interstellar Radiation Fields

12.1 After the Sun, Sirius (α Canis Majoris) is the brightest star in our sky. It is actually a binary; Sirius A and Sirius B. Sirius A is spectral type A1V, with mass $2.1\,M_\odot$; Sirius B is a (much fainter) white dwarf, with mass $0.98\,M_\odot$.

The Sirius system has luminosity $L = 25\,L_\odot$, and is at a distance $D = 2.6\,\mathrm{pc}$. What is the energy density u due to radiation from Sirius alone at the location of the Sun? What fraction of the local starlight background energy density is contributed by Sirius alone?

12.2 The MMP83 radiation field (see Table 12.1) has an energy density $u(912\,\text{Å} < \lambda < 2460\,\text{Å}) = 7.1{\times}10^{-14}\,\mathrm{erg\,cm^{-3}}$ of photons in the energy range 5.04–13.6 eV. Eq. (12.7) describes the spectrum of this radiation. Calculate the number density of $10.0 < h\nu < 13.6\,\mathrm{eV}$ photons.

Chapter 13. Ionization Processes

13.1 Define E_x to be the energy at which the photoelectric cross section (13.4) for a hydrogenic ion is equal to the Compton scattering cross section, $\sigma_T = (8\pi/3)(e^2/m_e c^2)^2$.

(a) Express $E_x/I_{\rm H}$ in terms of Z and the fine structure constant $\alpha \equiv e^2/\hbar c = 1/137.04$

(b) For hydrogen, calculate E_x in eV.

13.2 Recall the definition of oscillator strength [Eq. (6.19)]:

$$f_{\ell u} \equiv \frac{m_e c}{\pi e^2} \int \sigma_{\ell u}(\nu) d\nu \quad .$$

Let us now apply this with lower level ℓ being an initial bound state $n\ell$ (don't get confused about the two usages of ℓ here!), and u being all the free electron states.

Suppose that the electrons in some atomic shell $n\ell$ have photoionization cross section

$$\sigma_{\rm pi} = \sigma_{t,n\ell} \left(\frac{h\nu}{I_{n\ell}}\right)^{-3} \quad \text{for } h\nu > I_{n\ell} \quad ,$$

with oscillator strength $f_{{\rm pi},n\ell}$ associated with photoionization transitions from the initial bound state $n\ell$ to all of the free electron states.

(a) With these assumptions, express the photoionization cross section at threshold $\sigma_{t,n\ell}$ in terms of πa_0^2 and the dimensionless numbers $f_{{\rm pi},n\ell}$, $(I_{n\ell}/I_{\rm H})$, and $\alpha \equiv e^2/\hbar c$.

(b) For hydrogen, the oscillator strength associated with photoionization is $f_{\rm pi,1s} = 0.4359$. Using this power-law approximation for the photoelectric absorption, estimate the value of the photoionization cross section at threshold, and compare to the value of the exact result Eq. (13.2).

(c) C has 2 electrons in the $n = 1$ shell (the "K shell"). The photoionization threshold from the $n = 1$ shell is $I_{\rm K} = 285.4\,{\rm eV}$. Suppose that $f_{{\rm pi},1s} \approx 2 \times 0.5 = 1$. Estimate the cross section just above threshold for photoionization out of the $1s$ shell.

(d) C has 4 electrons in the $n = 2$ shell. Approximate the photoionization threshold from the $n = 2$ shell as $I_{2s2p} = 12\,{\rm eV}$. Suppose that $f_{{\rm pi},1s} \approx 2 \times 0.5 = 1$, and $f_{{\rm pi},2s2p} \approx 4 \times 0.5 = 2$. With the above assumptions about the energy dependence, estimate the ratio of the C photoionization cross section just above $I_{\rm K}$ to the value just below $I_{\rm K}$.

13.3 From Figure 13.2, one sees that the photoionization cross section for neutral Si can be approximated by

$$\sigma(h\nu) \approx 7 \times 10^{-17} \left(\frac{h\nu}{8.15\,{\rm eV}}\right)^{-3.5} \,{\rm cm}^2$$

for $8.15\,{\rm eV} < h\nu < 13.6\,{\rm eV}$. Suppose that the energy density of starlight in an H I cloud (see Fig. 12.2) can be approximated by

$$\nu u_\nu \approx 9 \times 10^{-14} \left(\frac{h\nu}{8.15\,{\rm eV}}\right)^{-1} \,{\rm erg\,cm^{-3}}$$

for $8.15\,{\rm eV} < h\nu < 13.6\,{\rm eV}$, and $u_\nu \approx 0$ for $h\nu > 13.6\,{\rm eV}$.

Calculate the photoionization rate ζ for an Si atom.

Chapter 14. Recombination of Ions with Electrons

14.1 Suppose that an electron recombines into the $n = 5$, $\ell = 4$ (also known as $5g$) level of hydrogen. What is the probability that an Hα photon will be emitted during the radiative cascade starting from $(n, \ell) = (5, 4)$?

14.2 The Einstein A coefficients for all of the allowed transitions of hydrogen from levels $n \leq 3$ are given in the table below:

u	ℓ	$A_{u\ell}(\mathrm{s}^{-1})$	$\lambda_{u\ell}(\text{Å})$	
$3d$	$2p$	6.465×10^7	6564.6	Hα
$3p$	$2s$	2.245×10^7	6564.6	Hα
$3s$	$2p$	6.313×10^6	6564.6	Hα
$3p$	$1s$	1.672×10^8	1025.7	Lyβ
$2p$	$1s$	6.265×10^8	1215.7	Lyα

(a) Consider a hydrogen atom in the $3p$ state as the result of radiative recombination: $p + e^- \rightarrow \mathrm{H}(3p)$. What is the probability p_β that this atom will emit a Lyman β photon?

(b) In an H I region where hydrogen is the only important opacity source, and averaged over many atoms "prepared" in the $3p$ state, what is the mean number $\langle n \rangle$ of times a Lyman β photon is "scattered" (that is, absorbed and then re-emitted) before an H α photon is emitted?
Hint: you may want to use the result

$$\sum_{n=1}^{\infty} n q^n = q \sum_{n=1}^{\infty} n q^{n-1} = q \frac{d}{dq} \sum_{n=1}^{\infty} q^n = q \frac{d}{dq}\left[\frac{q}{1-q}\right] = \frac{q}{(1-q)^2} \quad .$$

14.3 For case B recombination at $T = 10^4\,\mathrm{K}$, estimate $j(\mathrm{Ly}\,\alpha)/j(\mathrm{H}\alpha)$ for $n_e = 10^2\,\mathrm{cm}^{-3}$, $10^3\,\mathrm{cm}^{-3}$, and $10^4\,\mathrm{cm}^{-3}$. Here j is the power radiated per unit volume, where "radiated" is interpreted as creation of "new" photons (i.e., scattering is not included). Note that j is a *local* property – it does not take into account whether or not the photons will "escape" the emitting region.

14.4 Consider an H I cloud with $n_\mathrm{H} = 30\,\mathrm{cm}^{-3}$, $T = 100\,\mathrm{K}$, $n(\mathrm{H}^+) = 0.005\,\mathrm{cm}^{-3}$, $n(\mathrm{C}^+) = 0.005\,\mathrm{cm}^{-3}$, and $n_e = 0.01\,\mathrm{cm}^{-3}$. The ultraviolet starlight intensity is characterized by $G_0 = 1$. You may wish to refer to Table 14.6.

(a) What is the probability per unit time for a given proton to radiatively recombine with an electron?

(b) What is the probability per unit time for a given C^+ to radiatively recombine with an electron?

(c) Using Eq. (14.37) and Table 14.9, estimate the effective rate coefficients α_gr for neutralization of a proton as a result of a collision with a grain.

(d) Evaluate $\alpha_\mathrm{gr}(\mathrm{C}^+)$ for neutralization of a C^+ as the result of a collision with a grain.

(e) What fraction of proton recombinations with electrons are due to grains? What fraction of C^+ recombinations with electrons are due to grains?

14.5 In the standard Big Bang model, H and He were nearly fully ionized at redshifts $z \gtrsim 2000$. As the expanding Universe cooled, the rates for photoionization and collisional ionization dropped and the gas began to recombine. According to current estimates of the baryon density, the hydrogen fractional ionization was $x = 0.5$ at a redshift and temperature

$$z_{0.5} \approx 1250 \quad , \quad T_{0.5} \approx 3410\,\mathrm{K} \quad .$$

At this temperature, collisional ionization and photoionization are still important, and the Saha equation provides a good approximation to the ionization fraction. As the temperature and density continue to drop, the Universe is expanding too rapidly to maintain thermodynamic equilibrium, and a kinetic calculation is necessary. According to detailed calculations (Grin & Hirata 2010; Phys. Rev. D 81, 083005), the fractional ionization has dropped to $x = 0.01$ at redshift

$$z_{0.01} = 880 \quad , \quad T_{0.01} = 2400\,\mathrm{K} \ .$$

From this point on, let us assume that photoionization and collisional ionization can be neglected, and consider only radiative recombination.

According to current estimates of cosmological parameters ($H_0 = 70.2\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $\Omega_{\rm baryon} = 0.0458$, $\Omega_{\rm dark\ matter} = 0.229$, $\Omega_\Lambda = 0.725$: Komatsu et al. 2010; arXiv1001.4538), the age of the Universe at redshift $z \gtrsim 10$ is

$$t(z) \approx \frac{17\,\mathrm{Gyr}}{(1+z)^{3/2}} \ ,$$

and the age of the Universe when $x = 0.01$ was

$$t_{0.01} \approx \frac{17\,\mathrm{Gyr}}{(1+z_{0.01})^{3/2}} = 6.5\times10^5\,\mathrm{yr} \ .$$

According to current estimates of the baryon density based on nucleosynthetic constraints, the hydrogen density $n_{\rm H} = n(\mathrm{H}^0) + n(\mathrm{H}^+)$ in the expanding Universe is

$$n_{\rm H}(t) = n_{\rm H,0.01}\left(\frac{1+z}{1+z_{0.1}}\right)^3 = n_{\rm H,0.01}\left(\frac{t}{t_{0.01}}\right)^{-2} \quad ,$$

where the H density at $z_{0.01} = 880$ was

$$n_{\rm H,0.01} = 130\ \mathrm{cm}^{-3} \quad .$$

Because of Compton scattering, the temperature of the free electrons remains coupled to the radiation field until quite late times. If we assume that this coupling persists throughout the main phase of recombination, then the electron temperature evolves as

$$T_e(z) = T_{0.01}\left(\frac{1+z}{1+z_{0.01}}\right) = T_{0.01}\left(\frac{t}{t_{0.01}}\right)^{-2/3} \quad .$$

Suppose that for $t > t_{0.01}$ (i.e., $z < z_{0.01}$), no further ionization takes place, and the ionized fraction x continues to drop due to radiative recombination. Suppose that the rate coefficient for radiative recombination for $T < 2400\,\mathrm{K}$ can be written

$$\alpha_B = 7.8\times10^{-13}(T_e/2400\,\mathrm{K})^{-0.75}\,\mathrm{cm^3\,s^{-1}} \ .$$

Note: Ignore helium in this problem.

(a) Obtain an equation for $dx/d\tau$, where $x \equiv n(\mathrm{H}^+)/n_{\rm H}$ is the hydrogen fractional ionization, and $\tau \equiv t/t_{0.01}$ is time in units of $t_{0.01}$. (Hint: do not let the expansion of the Universe confuse you. Remember that if there were no recombination, the fractional ionization would remain constant even as the Universe expands.)

(b) Solve the differential equation from part (a) to find the solution $x(t)$ for $t > t_{0.01}$.

(c) Assuming that photoionization and collisional ionization remain negligible, evaluate the fractional ionization x at redshift $z = 50$, $z = 100$ and $z = 15$. The nonzero ionization remaining at $z \lesssim 100$ is sometimes referred to as "ionization freezout".

(d) *WMAP* observations of polarization in the CMB appear to require partial reionization of the Universe at $z \approx 12$. Suppose that a large region is reionized by photoionization at $z = 12$. The photoionized gas will initially be at $T \approx 2\times10^4\,\mathrm{K}$, with case B recombination coefficient $\alpha_B \approx 1.7\times10^{-13}\,\mathrm{cm^3\,s^{-1}}$. Compare the timescale for recombination at $z = 12$ to $t(z = 12)$, the age of the Universe at $z = 12$.

14.6 Absorption line observations of an interstellar cloud measure column densities $N(\mathrm{Ca\,I}) = 1.00 \times 10^{12}\,\mathrm{cm}^{-2}$ and $N(\mathrm{Ca\,II}) = 3.08 \times 10^{14}\,\mathrm{cm}^{-2}$. The gas temperature is estimated to be $T = 50\,\mathrm{K}$. At this temperature the radiative recombination coefficient for $\mathrm{Ca\,II}+e^- \to \mathrm{Ca\,I}+h\nu$ is $\alpha = 1.3\times10^{-11}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$. The starlight present within the cloud can photoionize $\mathrm{Ca\,I} + h\nu \to \mathrm{Ca\,II} + e^-$ with a photoionization rate $\zeta = 1.2 \times 10^{-10}\,\mathrm{s}^{-1}$. Assume that grain-assisted recombination can be neglected.

Estimate the electron density n_e in the cloud.

14.7 From observation of the K I absorption line at 7667 Å, an H I cloud is determined to have a column density $N(\mathrm{K\ I}) = 1.0\times10^{13}\,\mathrm{cm}^{-2}$. Starlight ionizes the K I at a rate (from Table 13.1) $\zeta(\mathrm{K} + h\nu \to \mathrm{K}^+ + e^-) = 6.85\times10^{-12}\,\mathrm{s}^{-1}$. Assume that the electron density in the cloud is $n_e = 0.03\,\mathrm{cm}^{-3}$, the gas temperature is $T = 100\,\mathrm{K}$, and the radiative recombination rate coefficient $\alpha_{\mathrm{rr}} \equiv \alpha(\mathrm{K}^+ + e^- \to \mathrm{K} + h\nu) = 1.11\times10^{-11}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$. Assume that radiative recombination is the only process removing K II and producing K I (i.e., neglect grain-assisted recombination). Assume that higher ion stages (K III, K IV, ...) can be neglected.

(a) Estimate the total column density of gas-phase K (both K I and K II) on this sightline.

(b) Given that grain-assisted recombination has been neglected, is the above estimate for the total column density of gas-phase K a lower bound or an upper bound?

14.8 Absorption at HeI10833 Å has been observed during some exoplanet transits (e.g., HAT-P-11n and WASP-107b), and is thought to be produced by an extended atmosphere (or wind) from the planet as it transits in front of the star.

Consider a slab with H nucleon density n_{H}, and He nucleon density $n_{\mathrm{He}} = 0.1 n_{\mathrm{H}}$. Suppose that radiation from a star is partially ionizing both H and He, maintaining fractional ionizations $x_{\mathrm{H}} = n(\mathrm{H}^+)/n_{\mathrm{H}}$ and $x_{\mathrm{He}} = n(\mathrm{He}^+)/n_{\mathrm{He}}$.

Let α_{H} and α_{He} be the rate coefficients for radiative recombination of H and He. Let $f_s \approx 0.3$ be the fraction of H radiative recombinations that populate the H $2s$ state, and let $f_{\mathrm{trip}} \approx 0.75$ be the fraction of He radiative recombinations that populate the triplet states. Let $A(2s)$ be the probability/time that H $2s$ will undergo 2-photon decay. Let $A(2^3S_1)$ be the radiative decay rate for the metastable state $\mathrm{He}(2^3S_1)$.

(a) In the low density limit, obtain an expression for $n(\mathrm{H}\,2s)$ in terms of n_{H}, x_{H}, x_{He}.

(b) In the low density limit, obtain an expression for $n(\mathrm{He}\,2\,^3S_0)$ in terms of n_{H}, x_{H}, and x_{He}.

(c) At higher densities, metastable levels can be depopulated by collisions with electrons (and protons, but ignore them for simplicity). Let q_{2s} be the rate coefficient for depopulation of H $2s$ by electron collisions, and $q_{\mathrm{He2^3S_1}}$ be the rate coefficient for dopopulation of $\mathrm{He2^3S_1}$ by electron collisions.

Obtain expressions for $n(\mathrm{H}\,2s)$ and $n(\mathrm{He}\,2^3S_1)$ including electron collisions.

(d) $\mathrm{He(^3S_1)}$ has an absorption line triplet $\mathrm{He^3S_1} \to ^3\mathrm{P}_{0,1,2}$ at $\lambda = 10833$ Å. Treat this as a single line with oscillator strength $f = 0.539$. Suppose that the slab has density $n_{\mathrm{H}} = 10^8\,\mathrm{cm}^{-3}$, $T = 2000\,\mathrm{K}$ (so that $\alpha_{\mathrm{He}} \approx 1\times10^{-12}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ and $q_{\mathrm{He2^3S_1}} \approx 2\times10^{-9}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$), $x_{\mathrm{H}} = 0.1$, $x_{\mathrm{He}} = 0.1$, and thickness $L = 5\times10^9\,\mathrm{cm}$.

Calculate the equivalent width $(W_\lambda)_{\mathrm{He10833}}$ through the slab.

(e) H atoms in the $n = 2$ levels produce Hα absorption, at $\lambda = 6565$ Å. The $2p$ state spontaneously decays in 1.6 ns, and therefore is negligibly populated. However the metastable $2s$ state will have a larger population. Obtain an expression for the ratio

$$\frac{(W_\lambda/\lambda)_{\mathrm{H}\alpha}}{(W_\lambda/\lambda)_{\mathrm{He\,I\,10833}}}$$

and evaluate it for the above conditions. Take $\alpha_{\mathrm{H}} = 9\times10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ and $q_{2s} = 1.2\times10^{-3}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ for $T = 2000\,\mathrm{K}$. The Hα line has an oscillator strength $f_{\ell u} = 0.641$.

14.9 Observation of He I 10833 Å absorption during transit of some exoplanets has been interpreted as absorption by metastable He I 2^3S_1 produced by recombination of He^+ (see textbook Figure 14.3). A simple toy model that attempts to explain this postulates that the exoplanet has a an extended atmosphere with $n_\mathrm{H} = 10^8\,\mathrm{cm}^{-3}$, He/H=0.1, $n(\mathrm{H}^+)/n_\mathrm{H} = 0.1$, $n(\mathrm{He}^+)/n_\mathrm{He} = 0.1$, and $T = 2000\,\mathrm{K}$, and thickness $L \approx 5\times10^9\,\mathrm{cm}$.

The He ionization in this extended atmosphere is assumed to be maintained by $h\nu > 24.6\,\mathrm{eV}$ radiation from the star. Suppose that the $h\nu > 24.6\,\mathrm{eV}$ stellar photons have a typical energy $35\,\mathrm{eV}$, and suppose that the He photoionization cross section is $\sigma = 4\times10^{-18}\,\mathrm{cm}^2$ (see Figure 13.1a).

If the exoplanet has orbital radius $R = 0.05\,\mathrm{AU}$, what must be the stellar luminosity in $h\nu > 24.6\,\mathrm{eV}$ photons? Neglect attenuation of the ionizing radiation in the extended atmosphere.

[For comparison, the average spectrum of the Sun has a luminosity $\sim 10^{-6}L_\odot$ in $h\nu > 24.6\,\mathrm{eV}$ photons.]

14.10 Suppose that spectra of distant galaxies routinely showed a weak absorption feature at $\lambda = 6565\,\text{Å}$ that is interpreted as Hα absorption by hydrogen in a "veil' of excited hydrogen surrounding the Galaxy. Suppose that the Hα absorption feature has an equivalent width $W_\lambda = 0.17\,\text{Å}$. The feature is broad, with velocity $\mathrm{FWHM}_v \approx 1000\,\mathrm{km\,s}^{-1}$.

(a) What is the implied column density $N(\mathrm{H}(n=2))$ of hydrogen in the $n = 2$ level? The oscillator strength $f_{\ell u} = 0.641$ for either $2s \rightarrow 3p$ absorption, or $2p \rightarrow 3s, 3d$ absorption.

(b) Suppose that this hydrogen is all in the $2s$ state, with the column density $N(n = 2)$ from part (a), and is in a shell of radius $R = 15\,\mathrm{kpc}$. What would be the radiated power L of this shell in $2s \rightarrow 1s$ 2-photon emission? The decay rate of the 2s level is $A_{2s\rightarrow1s} = 8.23\,\mathrm{s}^{-1}$. Give your result in units of $L_\odot = 3.826\times10^{33}\,\mathrm{s}^{-1}$.

Chapter 15. Photoionized Gas

15.1 A O9V star has luminosity $L = 10^{4.77}\,L_\odot$, emits $h\nu > 13.6\,\mathrm{eV}$ photons at a rate $Q_0 = 10^{48.06}\,\mathrm{s}^{-1}$, and emits $h\nu > 24.6\,\mathrm{eV}$ photons at a rate $Q_1 = 0.0145 Q_0$ (see Table 15.1). The star is surrounded by a steady-state H II region.

(a) If the ionized region has a uniform density $n_\mathrm{H} = 10^2\,\mathrm{cm}^{-3}$ and temperature $T = 10^4\,\mathrm{K}$, estimate the neutral fraction $n(\mathrm{H}^0)/n_\mathrm{H}$ at a distance $r = 0.9 R_\mathrm{H\,II}$ from the star, where $R_\mathrm{H\,II}$ is the radius of the zone where H is ionized. Assume that the gas is pure hydrogen, and that dust is negligible.

(b) Now assume that the gas has He/H$\approx$0.1 by number. What will be the ratio $R_\mathrm{He\,II}/R_\mathrm{H\,II}$, where $R_\mathrm{He\,II}$ is the radius of the zone where helium is ionized? An answer accurate to 10% is OK – don't worry over details. State your assumptions.

15.2 Hydrogen 166α (i.e., $167\ell \rightarrow 166\ell'$) and He 166α (i.e., $1s167\ell \rightarrow 1s166\ell'$) recombination lines are observed from an H II region. Assume that the telescope beamwidth is much larger than the nebula. The strengths of the lines are in the ratio $T(\mathrm{He})/T(\mathrm{H}) = 0.032$, i.e.,

$$\int d\Omega \int_{\mathrm{He}\,166\alpha} I_\nu d\nu = 0.032 \int d\Omega \int_{\mathrm{H}\,166\alpha} I_\nu d\nu \quad .$$

(a) Using Table 15.1, estimate the temperature of the exciting star for the H II region, assuming it to be of luminosity class V. Assume that all $h\nu > 24.6\,\mathrm{eV}$ photons are absorbed by He. Assume $\alpha_B(\mathrm{H}) \approx 2.54 \times 10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ for HII and $\alpha_B(\mathrm{He}) \approx 2.72 \times 10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ for HeII.

(b) The observed recombination lines have full widths at half-maximum (FWHM) of 23.5 and $15.3\,\mathrm{km\,s}^{-1}$ for H and He respectively, as observed with a receiver with an instrumental line width (FWHM) of $5\,\mathrm{km\,s}^{-1}$. Assume that the only motions are from thermal motions plus turbulence with an unknown velocity dispersion.

- What is the kinetic temperature T in the nebula?
- What is the one-dimensional velocity dispersion σ_turb of the turbulence?

[You may assume that both the instrumental response function and the thermal and turbulent velocity distribution functions are gaussians. The convolution of a gaussian with a gaussian yields a gaussian with variance equal to the sum of the variances of the original two gaussians.]

15.3 Consider a spherically-symmetric stellar wind with mass-loss rate $\dot{M}_w = 10^{-4} M_\odot\,\mathrm{yr}^{-1}$. and wind speed $v_w = 20\,\mathrm{km\,s}^{-1}$. Suppose the mass-loss continues steadily for $t_w = 10^3\,\mathrm{yr}$ and then stops, with the wind continuing to "coast" outwards. Suppose that after a time t, the central star suddenly becomes an ionizing source emitting hydrogen-ionizing photons at a rate Q_0, creating a "protoplanetary nebula".

(a) After time t, the outflowing wind has a spherical outer surface and a spherical inner "hole". What is the density just inside the outer surface?

(b) What is the density just outside the inner hole?

(c) Ignoring expansion of the nebula during the ionization process, what is the minimum value of Q_0 required to ionize the H throughout the nebula?

(d) What is the recombination time just inside the outer surface? Compare this to the $10^3\,\mathrm{yr}$ dynamical age.

15.4 Consider a runaway O star, of spectral type O8V, traveling through a diffuse region with $n_\mathrm{H} \approx 0.2\,\mathrm{cm}^{-3}$.

(a) What is the Strömgren radius R_{S0} if the photoionized gas has $T = 10^4\,\mathrm{K}$?

(b) If the star is traveling at $v_\star = 100\,\mathrm{km\,s}^{-1}$, compare the time required for the star to travel a distance equal to the Strömgren radius to the recombination time.

(c) Very briefly discuss the implications of the comparison in item (b).

15.5 Consider an H II region powered by a star emitting ionizing photons at a rate Q_0. Assume a pure hydrogen nebula (no He, no dust), and approximate the H II region by a Strömgren sphere with uniform density $n_{\rm H}$. Let α_B be the case B recombination rate coefficient, T be the gas temperature, and assume the ionizing stellar photons to be monoenergetic with energy $h\nu$. Assume that the hydrogen is nearly fully ionized. ($n_e \approx n_{\rm H}$).

(a) The star exerts radiation pressure on the gas – each ionizing photon, when absorbed, deposits a momentum $h\nu/c$. Assume that the only interaction of the photons with the gas is through photoionization (i.e., electron scattering and free-free absorption are neglected). Define a function $p_{\rm rad}$ by the differential equation $-dp_{\rm rad}/dr$ = radial force/volume exerted by the absorbed stellar radiation. If the density in the H II region is uniform, calculate $\Delta p_{\rm rad} \equiv p_{\rm rad}(0) - p_{\rm rad}(R_S)$, where R_S is the Strömgren radius. Write your result in terms of α_B, $n_{\rm H}$, Q_0, and $h\nu$.

(b) Obtain the ratio $\Delta p_{\rm rad}/2n_{\rm H}k_{\rm B}T$ in terms of Q_0, $n_{\rm H}$, α_B, $h\nu$, c, and $k_{\rm B}T$.

(c) Evaluate the ratio $\Delta p_{\rm rad}/2n_{\rm H}k_{\rm B}T$ for Orion Nebula-like conditions: $T = 10^4\,{\rm K}$, $n_{\rm H} = 3200\,{\rm cm}^{-3}$, $Q_0 \approx 6.5\times10^{48}\,{\rm s}^{-1}$, and $h\nu \approx 18\,{\rm eV}$.

(d) When this radiation pressure is taken into consideration, it is clear that a uniform density isothermal nebula would not be in dynamical equilibrium. If the nebula needs to be in dynamical equilibrium, will the gas pressure at the edge adjust to be larger or smaller than the gas pressure at the center?

15.6 An O8V star radiates $h\nu > 13.6\,{\rm eV}$ photons at a rate $Q_0 = 10^{48.44}\,{\rm s}^{-1}$. The total luminosity of the star is $L = 10^{4.96}\,L_\odot$.

(a) If the average energy of the $h\nu > 13.6\,{\rm eV}$ photons is $18\,{\rm eV}$, what fraction $f_{\rm ioniz}$ of the total power L is radiated in $h\nu > 13.6\,{\rm eV}$ photons?

(b) If the star is surrounded by pure hydrogen gas with H nucleon density $n_{\rm H} = 10^2\,{\rm cm}^{-3}$, estimate the radius of the volume around the star where the hydrogen is predominantly ionized. Assume that the ionized gas has temperature $T \approx 10^4\,{\rm K}$, and the case B recombination coefficient $\alpha_B = 2.54\times10^{-13}\,{\rm cm}^3\,{\rm s}^{-1}$.

(c) 45% of case B recombinations generate an Hα photon. What will be the Hα luminosity of the ionized gas?

15.7 Consider an H II region with uniform electron density n_e, powered by a star emitting ionizing photons at a rate Q_0. Neglect helium and neglect dust. Let α_B be the case B recombination rate coefficient, and let f_{2s} be the fraction of case B recombinations that populate the $2s$ level.

Suppose that the only processes depopulating the $n = 2$ levels are spontaneous decays and collisions with electrons. Let A_{2s} be the rate for spontaneous decay of the $2s$ level, and let $n_e q_{2s\rightarrow2p}$ be the rate for $2s \rightarrow 2p$ collisional transitions.

(a) Obtain an expression for $N({\rm H}\,2s)$, the column density from center to edge of H in the $2s$ level, as a function of Q_0, n_e, α_B, f_{2s}, A_{2s}, and $q_{2s\rightarrow2p}$.

(b) Evaluate $N({\rm H}\,2s)$ for $f_{2s} = 0.325$, $\alpha_B = 2.59\times10^{-13}\,{\rm cm}^3\,{\rm s}^{-1}$, $A_{2s} = 8.21\,{\rm s}^{-1}$, $q_{2s\rightarrow2p} = 5.31\times10^{-4}\,{\rm cm}^3\,{\rm s}^{-1}$, $Q_0 = 10^{48}\,{\rm s}^{-1}$, and $n_e = 10^4\,{\rm cm}^{-3}$.

15.8 An O7V star radiates $h\nu > 13.6\,{\rm eV}$ photons at a rate $Q_0 = 10^{48.75}\,{\rm s}^{-1}$. The total luminosity of the star is $L = 10^{5.14}\,L_\odot$.

(a) If the average energy of the $h\nu > 13.6\,{\rm eV}$ photons is $20\,{\rm eV}$, what fraction $f_{\rm ioniz}$ of the total power L is radiated in $h\nu > 13.6\,{\rm eV}$ photons?

(b) If the star is surrounded by pure hydrogen gas with H nucleon density $n_{\rm H} = 10^3\,{\rm cm}^{-3}$, estimate the radius of the volume around the star where the ionized fraction will be close to 1. Assume that the ionized gas has temperature $T \approx 10^4\,{\rm K}$, and the case B recombination coefficient $\alpha_B = 2.54\times10^{-13}\,{\rm cm}^3\,{\rm s}^{-1}$.

(c) 11.7% of case B recombinations generate an Hβ photon. What will be the Hβ luminosity of the ionized gas?

15.9 An O7V star radiates $h\nu > 13.6\,\mathrm{eV}$ photons at a rate $Q_0 = 10^{48.75}\,\mathrm{s}^{-1}$. The star is surrounded by pure hydrogen gas (no He, no dust) with uniform H nucleon density $n_\mathrm{H} = 10^2\,\mathrm{cm}^{-3}$. Assume that the star has been shining long enough to achieve "steady-state" conditions. Assume that the ionized gas has temperature $T \approx 10^4\,\mathrm{K}$, and the case B recombination coefficient $\alpha_B = 2.54 \times 10^{-13}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$.

(a) Estimate the radius R of the volume around the star where the ionized fraction will be close to 1.

(b) 45% of case B recombinations generate an Hα photon. What will be the Hα luminosity of the ionized gas?

(c) Estimate $n(\mathrm{H}^0)/n_\mathrm{H}$ at a distance $r = 0.8R$ from the star. Assume the ionizing radiation at this location to have a typical photon energy $h\nu \approx 15\,\mathrm{eV}$ and a typical photoabsorption cross section $\sigma_\mathrm{pe} \approx 5\times10^{-18}\,\mathrm{cm}^2$.

15.10 Sirius A, at a distance $d = 2.6\,\mathrm{pc}$, is the brightest star (other than the Sun) in the sky at visual wavelengths. Its hot white dwarf companion, Sirius B, outshines Sirius A at short wavelengths. Sirius B has an effective temperature $T_\mathrm{eff} = 25200\,\mathrm{K}$ and a radius $R = 0.0081R_\odot$.

(a) Calculate the luminosity of Sirius B. Give your answer in $L_\odot$.

(b) A blackbody radiates photons at a rate

$$\dot{N} = \frac{L}{\langle h\nu \rangle}$$

where the mean photon energy

$$\langle h\nu \rangle = 3\frac{\zeta(4)}{\zeta(3)}k_\mathrm{B}T = 2.701k_\mathrm{B}T$$

[$\zeta(x)$ is the Riemann ζ-function]. For $I_\mathrm{H}/k_\mathrm{B}T = 13.6\,\mathrm{eV}/2.17\,\mathrm{eV} = 6.26$, it turns out that 42.7% of the radiated photons have $h\nu > I_\mathrm{H}$.

If Sirius B radiates like a blackbody, what is $Q_0 =$ the rate of emission of H-ionizing photons?

(c) Suppose the local ISM density were $n_\mathrm{H} = 0.05\,\mathrm{cm}^{-3}$: calculate the Strömgren radius R_S for Sirius B, and compare to our distance to Sirius B. Assume an electron temperature $T_e = 7000\,\mathrm{K}$.

(d) Compare the value of R_S that you obtained with the expected thickness $\Delta R = 1/(n_\mathrm{H}\sigma_\mathrm{pi})$ of the transition from nearly-fully-ionized to nearly-fully-neutral for a Strömgren sphere (here σ_pi is a representative photoioinization cross section for the ionizing photons.) Discuss what you expect for the ionization of the ISM around Sirius B.

(e) Suppose that Sirius is moving at a speed $\sim 20\,\mathrm{km\,s}^{-1}$ relative to the local ISM. Does the steady-state assumption make sense?

Chapter 16. Ionization in Predominantly Neutral Regions

16.1 The diffuse molecular cloud toward the bright star ζ Persei has $N(\mathrm{H}_3^+) = 8\times10^{13}\,\mathrm{cm}^{-2}$ and $N(\mathrm{H}_2) = 5\times10^{20}\,\mathrm{cm}^{-2}$. Estimate the abundance $n(\mathrm{OH}^+)/n_\mathrm{H}$ in the molecular region of this cloud if the gas-phase abundance $n(\mathrm{O})/n_\mathrm{H} \approx 4\times10^{-4}$.

Assume $n_\mathrm{H} \approx 10^2\,\mathrm{cm}^{-3}$ and $T \approx 60\,\mathrm{K}$. Assume that most of the hydrogen is molecular, so that $n(\mathrm{H}_2) \approx n_\mathrm{H}/2$. The rate coefficient for $\mathrm{OH}^+ + e^-$ is given in Table 14.8. Assume that the free electrons come primarily from photoionization of "metals" with $n(M+)/n_\mathrm{H} = x_M \approx 1.07\times10^{-4}$ (as per Eq. 16.3).

16.2 **[Note: This problem is appropriate for Chapter 17, not 16.]**
Consider a two-level system. Suppose that there is only one collision partner. If the critical density [as defined in Eq. (17.7)] is n_crit, and the actual density of the collision partner is n, what fraction of collisional excitations will be followed by a radiative decay back to the ground state?

16.3 The ion H_3^+ can react with electrons ($\mathrm{H}_3^+ + e^- \rightarrow \mathrm{H}_2 + \mathrm{H}$ and $\mathrm{H}_3^+ + e^- \rightarrow 3\mathrm{H}$) or neutral atoms or molecules M ($\mathrm{H}_3^+ + M \rightarrow M\mathrm{H}^+ + \mathrm{H}_2$). If the eligible species M (e.g., O, C, S) have abundance $n(M)/n_\mathrm{H} = 3\times10^{-4}$, what is the fractional ionization x_e below which the destruction of H_3^+ is dominated by

$$\mathrm{H}_3^+ + M \rightarrow M\mathrm{H}^+ + \mathrm{H}_2 \ ?$$

Assume that this reaction proceeds with a typical ion-neutral rate coefficient $k \approx 2\times10^{-9}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$, and that the gas temperature $T = 30\,\mathrm{K}$. The rate coefficient for $\mathrm{H}_3^+ + e^- \rightarrow \mathrm{H}_2 + \mathrm{H}$ is $5.0\times10^{-8}T_2^{-0.48}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ and the rate coefficient for $\mathrm{H}_3^+ + e^- \rightarrow 3\mathrm{H}$ is $8.9\times10^{-8}T_2^{-0.48}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$.

16.4 In a dark cloud with density $n_\mathrm{H} = 10^4\,\mathrm{cm}^{-3}$ and fractional ionization $x_e \approx 10^{-7}$ (see Figure 16.3), the hydrogen is mostly H_2. Assume that $k(\mathrm{H_2}^+ + \mathrm{H}_2 \rightarrow \mathrm{H}_3^+ + \mathrm{H}) \approx 2\times10^{-9}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ The cosmic ray flux is such that an H atom would have a primary ionization rate $\zeta_\mathrm{CR} \approx 10^{-16}\,\mathrm{s}^{-1}$. Ignore helium. Assume that H_3^+ is destroyed primarily by $\mathrm{H}_3^+ + M \rightarrow \mathrm{H}_2 + M\mathrm{H}^+$, with $n(M)/n_\mathrm{H} \approx 3\times10^{-4}$ and $k_{16.18} \approx 2\times10^{-9}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$.

(a) Estimate $n(\mathrm{H_2}^+)/n_\mathrm{H}$.

(b) Estimate $n(\mathrm{H}_3^+)/n_\mathrm{H}$.

16.5 Consider a region containing only partially-ionized hydrogen. Let ζ be the ionization rate per H atom, and let α be the recombination coefficient.

(a) Determine the steady-state ionization fraction x_{ss} in terms of $n_\mathrm{H} \equiv n(\mathrm{H}^0) + n(\mathrm{H}^+)$, ζ, and α. Express your answer in terms of the dimensionless parameter $\beta \equiv \zeta/(\alpha n_\mathrm{H})$

(b) What is the asymptotic behavior of x_{ss} for $\beta \ll 1$? Show the leading dependence on β.

(c) What is the asymptotic behavior of x_{ss} for $\beta \gg 1$? Show the leading dependence on β.

(d) Suppose that the fractional ionization at time $t = 0$ is given by $x(0) = x_{ss} + \delta$. If $|\delta| \ll x_{ss}$, determine the solution $x(t > 0)$, assuming n_H, ζ, and α to be constant. (Hint: linearize around the steady state.)

Chapter 17. Collisional Excitation

17.1 Consider the H I spin temperature in a region where the brightness temperature of the background radiation field at $\lambda \approx 21\,\mathrm{cm}$ is $T_{\rm rad}$, and the gas temperature is $T_{\rm gas}$. Suppose that $h\nu/kT_{\rm rad} \ll 1$, and $T_{\rm gas} > T_{\rm rad}$.

Consider only the two hyperfine levels of H I (i.e., ignore the effects of Lyman alpha excitation of the $2p$ levels).

Let the H nucleon density of the gas be $n_{\rm H}$, and define the "critical density" according to Eq. (17.7),

$$n_{\rm crit} \equiv (1+n_\gamma)\frac{A_{10}}{k_{10}} \quad .$$

(a) Using the approximation $h\nu/kT_{\rm rad} \ll 1$, obtain an expression for the spin temperature $T_{\rm spin}$ in terms of $T_{\rm rad}$, $T_{\rm gas}$, and the ratio $R \equiv n_{\rm H}/n_{\rm crit}$.

(b) Find the minimum value of R such that $T_{\rm spin} > (T_{\rm rad} + T_{\rm gas})/2$.
You may assume that $T_{\rm gas} \gg h\nu/k$, $T_{\rm rad} \gg h\nu/k$, and $|T_{\rm gas} - T_{\rm rad}| \gg h\nu/k$.

17.2 Consider atoms X with two levels, $j = 0$ and $j = 1$, with degeneracies g_0 and g_1, in a gas where collisions take place with some collision partners with density n_c. Let $\Delta E \equiv E_1 - E_0$, and let T be the gas temperature. Let k_{10} be the rate coefficient for collisional deexcitation, and A_{10} be the Einstein A coefficient for spontaneous decay. Suppose that there are no photons present. Let $x(t)$ be the fraction of X in the excited state.

(a) What is the fraction $x_{\rm ss}$ such that the level populations are in steady-state statistical equilibrium?

(b) Suppose that $y(t) \equiv x(t) - x_{\rm ss}$, with initial value $y(0)$. Solve for $y(t)$.

17.3 When the proton spins in H_2 are **anti**parallel, we have "para"-H_2, which can have rotational angular momentum $J = 0, 2, 4, ...$ When the proton spins are parallel, we have "ortho"-H_2, for which only odd values of J are possible. Radiative transitions between ortho-H_2 and para-H_2 are strongly forbidden; para→ortho or ortho→para transitions occur only because of collisions.

Because the nuclear spins are only weakly-coupled to collision partners such as H atoms, the rate coefficients for ortho→para or para→ortho conversion are small.

Let $\mathrm{H_2}(v,J)$ denote H_2 with vibrational and rotational quantum numbers (v,J). The rate coefficient for

$$\mathrm{H_2}(0,1) + \mathrm{H_2}(0,0) \rightarrow \mathrm{H_2}(0,0) + \mathrm{H_2}(0,0)$$

is estimated to be only $k_{10} = 1.56\times10^{-28}\,\mathrm{cm^3\,s^{-1}}$ (Huestis 2008: Plan. Sp. Sci. 56, 1733). The energy difference between $\mathrm{H_2}(0,1)$ and $\mathrm{H_2}(0,0)$ is $\Delta E/k = 170.5\,\mathrm{K}$.

(a) Use detailed balance to obtain the rate coefficient k_{01} for

$$\mathrm{H_2}(0,0) + \mathrm{H_2}(0,0) \rightarrow \mathrm{H_2}(0,1) + \mathrm{H_2}(0,0) \quad .$$

(b) In a molecular cloud with $n(\mathrm{H_2}) = 100\,\mathrm{cm^{-3}}$ and $T = 50\,\mathrm{K}$, what is the steady-state ratio of $n(J=1)/n(J=0)$ if only collisions with H_2 are acting?

(c) If the ortho-para ratio at $t = 0$ differs from the LTE value, small deviations from LTE abundances will decay exponentially on a time scale τ. Evaluate τ for $n(\mathrm{H_2}) = 10^6\,\mathrm{cm^{-3}}$ and $T = 50\,\mathrm{K}$, assuming that the only processes causing ortho-para conversion are

$$\mathrm{H_2}(0,0) + \mathrm{H_2}(0,0) \rightarrow \mathrm{H_2}(0,1) + \mathrm{H_2}(0,0)$$

$$\mathrm{H_2}(0,1) + \mathrm{H_2}(0,0) \rightarrow \mathrm{H_2}(0,0) + \mathrm{H_2}(0,0) \quad .$$

17.4 The ground term of C II has two fine structure levels: ${}^2\mathrm{P}^o_{1/2}$ and ${}^2\mathrm{P}^o_{3/2}$. Absorption line studies of an interstellar cloud give the column density of the ground state (${}^2\mathrm{P}^o_{1/2}$) $N(\mathrm{C\,II}) = 10^{16.8}\,\mathrm{cm^{-2}}$, and an upper limit on the column density of the excited state $N(\mathrm{C\,II^*}) < 10^{15.8}\,\mathrm{cm^{-2}}$. The excited fine structure level emits $158\,\mu\mathrm{m}$ photons with a spontaneous decay rate $A = 2.3\times10^{-6}\,\mathrm{s^{-1}}$. Take the electron collision strength between the ${}^2\mathrm{P}^o_{1/2}$ and ${}^2\mathrm{P}^o_{3/2}$ levels to be $\Omega = 1.5$. If the gas kinetic temperature is known to be $T = 100\,\mathrm{K}$, obtain a limit on the electron density based on the relative populations of the fine structure levels.

Chapter 18. Nebular Diagnostics

18.1 Derive the value of the constant C in the equation

$$\frac{n(\mathrm{O\,III})}{n(\mathrm{H}^+)} = C \times \frac{I([\mathrm{O\,III}]5008)}{I(\mathrm{H}\beta)} T_4^{-0.494-0.089\ln T_4}\, \mathrm{e}^{2.917/T_4} \quad .$$

in the low density limit. For what densities is your result valid?

18.2 The observed spectrum of an H II region has

$$\frac{I([\mathrm{O\,III}]4364.4\,\text{Å})}{I([\mathrm{O\,III}]5008.2\,\text{Å})} = 0.003 \quad ,$$

$$\frac{I([\mathrm{O\,II}]3729.8\,\text{Å})}{I([\mathrm{O\,II}]3727.1\,\text{Å})} = 1.2 \quad .$$

(a) If interstellar reddening is assumed to be negligible, estimate the electron temperature T and the electron density n_e. You may find it convenient to use Figs. 18.2 and 18.4.

(b) Now suppose that it is learned that there is reddening due to intervening dust with $A(4364.4\,\text{Å}) - A(5008.2\,\text{Å}) = 0.31$ mag. Re-estimate T and n_e.

Chapter 19. Radiative Trapping

19.1 By approximating the sum by an integral, evaluate the partition function for a rigid rotor,

$$Z_{\rm rot} = \sum_{J=0}^{\infty}(2J+1)e^{-B_0 J(J+1)/k_{\rm B}T_{\rm exc}} \quad ,$$

in the high temperature limit $k_{\rm B}T_{\rm exc}/B_0 \gg 1$.

19.2 Consider a uniform spherical cloud of density $n_{\rm H} = 10^3 n_3\,{\rm cm}^{-3}$ and radius $R = 10^{19}R_{19}\,{\rm cm}$, with CO abundance $n({\rm CO})/n_{\rm H} = 7\times10^{-5}$. Assume the turbulence in the cloud results in a Doppler line broadening parameter $b = b_5\,{\rm km\,s^{-1}}$.

The Einstein A coefficient for CO is given in Eq. (5.6). The rotation constant is $B_0/k_{\rm B} = 2.766\,{\rm K}$.

(a) Obtain an equation for the optical depth τ_0 (from center to edge) in the $J+1 \rightarrow J$ transition in a cloud where the CO has excitation temperature $T_{\rm exc}$. Express τ_0 in terms of n_3, R_{19}, b_5, $T_{\rm exc}$, J, and $B_0/k_{\rm B}$.

(b) Leaving n_3, R_{19}, and b_5 as variables, and assuming $T_{\rm exc} = 8\,{\rm K}$, evaluate τ_0 for the $J = 2 \rightarrow 1$ and $J = 3 \rightarrow 2$ transitions.

(c) Repeat the calculation in (b) for $T_{\rm exc} = 30\,{\rm K}$.

19.3 Recall that $X_{\rm CO} \equiv N({\rm H_2})/\int T_A dv$ gives the relation between $N({\rm H_2})$ and the "antenna temperature" T_A integrated over radial velocity v of the CO 1–0 line in a resolved source.

(a) Suppose that we observe CO 1–0 line emission from an unresolved galaxy at distance D, with an integrated flux in the 1-0 line $W_{\rm CO} \equiv \int F_\nu dv$, where F_ν is the flux density, v is radial velocity, and the integral extends over the full range of radial velocities in the galaxy.

Derive an expression giving the mass $M({\rm H_2})$ of $\rm H_2$ in terms of $W_{\rm CO}$, $X_{\rm CO}$, λ, and D (and fundamental constants).

(b) NGC 7331, at a distance $D = 14.7\,{\rm Mpc}$, has $W_{\rm CO} = 4090\,{\rm Jy\,km\,s^{-1}}$. Calculate $M({\rm H_2})$. Assume that $X_{\rm CO} = 4\times10^{20}\,{\rm cm^{-2}}({\rm K\,km\,s^{-1}})^{-1}$.

19.4 If L is the line luminosity of a spherical cloud of radius R, and M is its mass, calculate the ratio of the mean line intensity $\langle I\rangle$ (averaged over the solid angle subtended by the cloud) to the mean surface density of the cloud $\Sigma = M/\pi R^2$. Note: this is easy – just an exercise to make you think about factors of 4π.

Chapter 20. Optical Pumping

20.1 Consider UV-pumping of the rotationally-excited states of para-H_2.
$\mathrm{H}_2(v=0, J=2)$ can be pumped by UV in the 912–1108 Å wavelength range as follows:

1. Ground-electronic state $\mathrm{H}_2(v=0, J=0)$ absorbs a photon (via a permitted electric-dipole transition) that excites it to a $J=1$ state of one of the many vibrational levels of either the $\mathrm{B}^1\Sigma_u^+$ or $\mathrm{C}^1\Pi_u$ states (see Fig. 5.1).
2. This is followed by spontaneous emission of a UV photon in a transition back to the ground electronic state. A fraction $f_{\rm diss} \approx 0.15$ of these transitions are to the vibrational continuum of the ground electronic state, leading to immediate dissociation $\mathrm{H}_2 \rightarrow \mathrm{H} + \mathrm{H}$.
3. A fraction $(1 - f_{\rm diss})$ of the UV decays are to one of the bound vibration-rotation levels (either $J=0$ or $J=2$) of the ground electronic state. A large fraction (close to 100%) of these UV decays are to excited vibrational states $v \geq 1$ (either $J=0$ or $J=2$), and are then followed by a "vibrational cascade" that returns the H_2 to the $v=0$ level, typically after emission of several infrared photons.
4. Suppose that a fraction $\phi_{\rm para}$ of the vibrational cascades of para-H_2 end up in one of the rotationally-excited levels $J = 2, 4, 6, ...$ of the ground vibrational state $v=0$. At low densities, the $J = 4, 6, ...$ rotationally-excited levels will decay by rotational quadrupole transitions ($J \rightarrow J-2$) down the rotational ladder, eventually populating the $v=0, J=2$ level.

Now consider a plane-parallel cloud, and suppose that each face of this cloud is illuminated by a radiation field, isotropic over 2π steradians, with $\lambda u_\lambda = 2{\times}10^{-14}\chi\,\mathrm{erg\,cm^{-3}}$, where χ is a dimensionless intensity scale factor. Suppose that a fraction $f_{\rm H_2}$ of the incident UV photons in the 1110–912 Å range are absorbed by H_2 (rather than by dust), and suppose that a fraction $h_{\rm para}$ of the H_2 absorptions are due to $\mathrm{H}_2(v=0, J=0)$.

Suppose that an observer views the cloud with the line-of-sight making an angle θ with respect to the cloud normal.

If collisions can be neglected, and UV pumping is the only mechanism for populating the $J \geq 2$ levels of H_2, obtain a formula for the surface brightness of the cloud in the H_2 0–0S(0) line at 28.22 μm (your result should depend on χ, $f_{\rm H_2}$, $f_{\rm diss}$, $h_{\rm para}$, $\phi_{\rm para}$, and the inclination angle θ).

20.2 Consider an ion X in an H II region around a star radiating $h\nu > 13.6\,\mathrm{eV}$ photons at a rate Q_0. Let $L_\star$ and $T_\star$ be the stellar luminosity and effective temperature.

Suppose that the ion X in level ℓ has an absorption line with wavelength $\lambda_{\ell u}$ and oscillator strength $f_{\ell u}$ to upper level u with $(E_u - E_\ell) < 13.6\,\mathrm{eV}$.

(a) If intervening absorption can be neglected, and the $h\nu < 13.6\,\mathrm{eV}$ radiation from the star can be approximated by a blackbody, obtain a formula for the photoabsorption probability/time $\zeta_{\ell u}$ for an ion X at a distance r from the star.

(b) Now suppose that the H II region has uniform density $n_{\rm H}$ and recombination rate coefficient α_B. Obtain an expression for the pumping rate at the "half-mass" radius $r = 2^{-1/3}R_{S0}$, where R_{S0} is the Strömgren radius.

(c) The N II ion has a permitted absorption line out of the ground state $^3\mathrm{P}_0$ to the $^3\mathrm{D}_1^\mathrm{o}$ level (see Fig. 6.1) with $f_{\ell u}$ given in Table 9.4.
Consider an H II region around an O9V star with Q_0, $L_\star$, and effective temperature as given in Table 15.1. Suppose the H II region has $\alpha_B = 3\times10^{-13}\,\mathrm{cm^3\,s^{-1}}$ (corresponding to electron temperature $\sim$8150 K). Evaluate ζ_{0u} at the half-mass radius as a function of $n_{\rm H}$. Evaluate ζ_{0u} for $n_{\rm H} = 1\,\mathrm{cm^{-3}}$.

(d) The $^3\mathrm{D}_1^\mathrm{o}$ state has three allowed decay channels: $^3\mathrm{D}_1^\mathrm{o} \rightarrow {}^3\mathrm{P}_0$, with $A_{u0} = 2.10\times10^8\,\mathrm{s^{-1}}$, $^3\mathrm{D}_1^\mathrm{o} \rightarrow {}^3\mathrm{P}_1$, with $A_{u1} = 1.54\times10^8\,\mathrm{s^{-1}}$, and $^3\mathrm{D}_1^\mathrm{o} \rightarrow {}^3\mathrm{P}_2$, with $A_{u2} = 9.96\times10^6\,\mathrm{s^{-1}}$.
Ignoring intervening absorption, what is the UV pumping rate β_{01} at the "half-mass radius" for populating the $^3\mathrm{P}_1$ fine structure level by photoexcitation out of $^3\mathrm{P}_0$? Include radiative transitions that pass through the $^3\mathrm{P}_2$ state.

(e) Compare this UV pumping rate with the rate for collisional excitation of N II $^3P_1$ by thermal electrons. Collision strengths are available in Table F.2.

Chapter 21. Interstellar Dust: Observed Properties

21.1 Suppose that dust produced extinction $A(\lambda)$ directly proportional to the frequency of the light. What would be the value of $R_V \equiv A_V/E(B-V)$? Assume $\lambda_V = 0.55\,\mu\mathrm{m}$ and $\lambda_B = 0.44\,\mu\mathrm{m}$.

21.2 If the extinction were to vary as a power law, $A \propto \nu^\beta$, what power-law index β would give $R_V = 3.1$? Assume $\lambda_V = 0.55\,\mu\mathrm{m}$ and $\lambda_B = 0.44\,\mu\mathrm{m}$.

21.3 The interstellar extinction at $\lambda = 0.55\,\mu\mathrm{m}$ is observed to be proportional to the hydrogen column density N_{H}, with $A_{0.55\,\mu\mathrm{m}}/N_{\mathrm{H}} \approx 3.52 \times 10^{-22}\mathrm{mag\,cm}^2$. Suppose that this extinction is produced by spherical dust grains with a single radius a. Let the usual dimensionless efficiency factor $Q_{\mathrm{ext}}(\lambda) \equiv C_{\mathrm{ext}}(\lambda)/\pi a^2$, where $C_{\mathrm{ext}}(\lambda)$ is the cross section for extinction at wavelength λ.

(a) If the grains have internal density ρ, calculate $M_{\mathrm{dust}}/M_{\mathrm{H}}$, the ratio of total dust mass to total hydrogen mass, in terms of unknown $Q_{\mathrm{ext}}(0.55\,\mu\mathrm{m})$, ρ, and a.

(b) Calculate the numerical value of $M_{\mathrm{dust}}/M_{\mathrm{H}}$ if $a = 0.1\,\mu\mathrm{m}$, $Q_{\mathrm{ext}} = 1.5$, and $\rho = 3\,\mathrm{g\,cm}^{-3}$.

Chapter 22. Scattering and Absorption by Small Particles

22.1 Suppose that for $\lambda = 10\,\mu\mathrm{m}$, amorphous silicate material has dielectric function $\epsilon = \epsilon_1 + i\epsilon_2$ given in the following table:

$\lambda(\,\mu\mathrm{m})$	ϵ_1	ϵ_2
9.50	0.731	1.987
9.60	0.774	2.131
9.70	0.831	2.260
9.80	0.891	2.373
9.90	0.946	2.476
10.0	0.996	2.575
10.1	1.040	2.678
10.2	1.085	2.792
10.3	1.141	2.920
10.4	1.224	3.056
10.5	1.333	3.184

For a spherical grain of amorphous silicate with radius $a = 0.1\,\mu\mathrm{m}$:

(a) Calculate the absorption efficiency factor Q_{abs} and the absorption cross section per volume C_{abs}/V at $\lambda = 10\,\mu\mathrm{m}$.

(b) Calculate the scattering efficiency factor Q_{sca} at $\lambda = 10\,\mu\mathrm{m}$.

22.2 For particles in the electric dipole limit, Eq. (22.12) gives the absorption cross section in terms of the complex polarizability α, and Eq. (22.14) provides an expression for the polarizability α_{jj} for the electric field parallel to principal axis j.

(a) Consider an oblate ellipsoid with shape factors L_j. Show that in the electric dipole limit, the absorption cross section for radiation polarized with the electric field parallel to axis j is

$$C_{\mathrm{abs},j} = \frac{2\pi V}{\lambda} \frac{\epsilon_2}{\left[1 + (\epsilon_1 - 1)L_j\right]^2 + (\epsilon_2 L_j)^2} \quad .$$

(b) Show that the difference in extinction cross section for radiation polarized parallel to axes a and b is

$$C_{\mathrm{abs},a} - C_{\mathrm{abs},b} = \frac{2\pi V \epsilon_2}{\lambda}(L_b - L_a) \times \frac{(\epsilon_1 - 1)^2(L_b + L_a) + 2(\epsilon_1 - 1) + \epsilon_2^2(L_b + L_a)}{\left\{\left[1 + (\epsilon_1 - 1)L_a\right]^2 + \epsilon_2^2 L_a^2\right\}\left\{\left[1 + (\epsilon_1 - 1)L_b\right]^2 + \epsilon_2^2 L_b^2\right\}} \quad .$$

22.3 Consider a oblate spheroidal grain with axial ratios $a : b : b :: 0.1\,\mu\mathrm{m} : 0.15\,\mu\mathrm{m} : 0.15\,\mu\mathrm{m}$. In the electric dipole limit ($\lambda \gg a$), the absorption cross section for radiation polarized with the electric field parallel to axis j is

$$C_{\mathrm{abs},j} = \frac{2\pi V}{\lambda} \frac{\epsilon_2}{\left[1 + (\epsilon_1 - 1)L_j\right]^2 + (\epsilon_2 L_j)^2} \quad .$$

(a) Using Eq. (22.15–22.18), evaluate the shape factors L_a and L_b for applied electric fields parallel to the short axis a or a long axis b.

(b) If the complex dielectric function of amorphous silicate material at $\lambda = 10\,\mu\mathrm{m}$ is $\epsilon = 0.996 + 2.575i$, where $i \equiv \sqrt{-1}$, calculate $C_{\mathrm{abs},a}/V$, $C_{\mathrm{abs},b}/V$, and $\left[C_{\mathrm{abs},b} - C_{\mathrm{abs},a}\right]/V$ for radiation with $\lambda = 10.0\,\mu\mathrm{m}$.

22.4 In the electric dipole limit, the absorption cross section for an ellipsoidal grain is given by

$$C_{\mathrm{abs},j} = \frac{2\pi V}{\lambda} \frac{\epsilon_2}{\left[1 + (\epsilon_1 - 1)L_j\right]^2 + (\epsilon_2 L_j)^2} \quad .$$

Consider a spheroidal grain with axial ratios $a:b:b::1:1.5:1.5$ composed of "astrosilicate" material. For $\lambda \gtrsim 100\,\mu\mathrm{m}$, suppose that the dielectric function of this material can be approximated by

$$\epsilon \approx 11.6 + 3.0\left(\frac{100\,\mu\mathrm{m}}{\lambda}\right)i \quad .$$

If such grains are perfectly aligned with short axis $\hat{\mathbf{a}} \parallel \mathbf{B}_0$, calculate the degree of polarization of optically-thin $450\,\mu\mathrm{m}$ thermal emission along a sightline $\perp \mathbf{B}_0$.

Chapter 23. Composition of Interstellar Dust

23.1 Suppose that in the optical and near-UV the extinction efficiency can be approximated

$$Q_{\rm ext}(a,\lambda) \approx 2(\pi a/2\lambda)^{\beta} \quad \text{for } \pi a/\lambda < 2$$
$$\approx 2 \quad \text{for } \pi a/\lambda > 2 \quad .$$

This is imprecise, but you can see from Fig. 22.3 that for $\beta \approx 1.5$ it roughly approximates the essential behavior if $|m-1| \approx 0.5$: a rapid increase in $Q_{\rm ext}$ with increasing a/λ until it reaches $\sim$2. Note that this is *not* a good approximation for $a/\lambda \lesssim 0.05$, but in the present problem we consider only the extinction at B and V, which is dominated by the larger particles.

Suppose that the dust density is proportional to $n_{\rm H}$, with a simple power-law size distribution

$$\frac{1}{n_{\rm H}}\frac{dn}{da} = \frac{A_0}{a_0}\left(\frac{a}{a_0}\right)^{-p} \qquad 0 < a \le a_{\rm max} \quad ,$$

where $a_0 = 0.1\,\mu{\rm m}$ is a fiducial length, A_0 is dimensionless, and $p < 4$.
The V and B bands have wavelengths $\lambda_V = 0.55\,\mu{\rm m}$ and $\lambda_B = 0.44\,\mu{\rm m}$.

Let $\sigma_{\rm ext}(\lambda)$ be the extinction cross section per H at wavelength λ.

(a) Assume that $a_{\rm max} < 0.28\,\mu{\rm m}$ (i.e., $\pi a_{\rm max}/\lambda_V < \pi a_{\rm max}/\lambda_B < 2$). Obtain an expression for

$$\frac{\sigma_{\rm ext}(\lambda)}{A_0\pi a_0^2}$$

that would be valid for $\lambda = \lambda_V$ or λ_B. Evaluate this ratio for $\beta = 1.5$, $p = 3.5$, $a_{\rm max} = 0.25\,\mu{\rm m}$, and $\lambda = \lambda_V$.

(b) For $a_{\rm max} < 0.28\,\mu{\rm m}$, using your result from (a), obtain an expression for the ratio

$$\frac{\sigma_{\rm ext}(\lambda_B)}{\sigma_{\rm ext}(\lambda_V)} \quad ,$$

and evaluate this for $\beta = 1.5$.

(c) Assuming $a_{\rm max} < 0.28\,\mu{\rm m}$, obtain an expression for $R_V \equiv A_V/(A_B - A_V)$, and evaluate this for $\beta = 1.5$.

(d) Now suppose that $a_{\rm max} > 2\lambda/\pi$. Obtain an expression for

$$\frac{\sigma_{\rm ext}(\lambda)}{A_0\pi a_0^2} \quad .$$

(e) If $a_{\rm max} = 0.35\,\mu{\rm m}$, $p = 3.5$, and $\beta = 2$,
(i) Evaluate $\sigma_{\rm ext}(\lambda_V)/A_0\pi a_0^2$,
(ii) Evaluate $\sigma_{\rm ext}(\lambda_B)/A_0\pi a_0^2$,
(iii) Evaluate R_V.

Chapter 24. Temperatures of Interstellar Grains

24.1 Consider particles with number density n_c, mass m_c, and kinetic temperaure T_c colliding with a neutral grain. The collision rate is

$$\frac{dN}{dt} = \pi a^2 n_c \left(\frac{8k_{\rm B}T_c}{\pi m_c}\right)^{1/2} \quad .$$

Let E be the kinetic energy of an impacting particle. What is $\langle E^n \rangle$, where the average is over the impacting particles, for general n?

Evaluate the result for $n = 1$.

24.2 The Debye model for the heat capacity of a solid has the thermal energy given by

$$E(T) = \frac{3N_v}{(k_{\rm B}\Theta)^3} \int_0^{k_{\rm B}\Theta} \frac{x}{\exp(x/k_{\rm B}T) - 1} x^2 dx \quad .$$

where N_v is the number of vibrational degrees of freedom of the solid, and Θ is the "Debye temperature" ($k_{\rm B}\Theta = \hbar\omega_{\rm max}$, where $\omega_{\rm max}$ is the frequency of the highest frequency vibrational mode of the solid). In the low-temperature limit $T \ll \Theta$, the thermal energy becomes

$$\begin{aligned} E(T) &\approx \frac{3N_v (k_{\rm B}T)^4}{(k_{\rm B}\Theta)^3} \int_0^{\infty} \frac{y^3 dy}{{\rm e}^y - 1} \\ &= \frac{N_v \pi^4}{5} \frac{k_{\rm B}T^4}{\Theta^3} \quad . \end{aligned}$$

(a) Suppose a grain contains N_a atoms, with 3 translational, 3 rotational, and $N_v = 3N_a - 6$ vibrational degrees of freedom. Suppose that the vibrational modes are approximated by the Debye model. Consider a grain with $N_a = 10^3$ atoms and $\Theta = 300\,{\rm K}$. If the grain is initially at $E_v = 0$, what is the temperature after absorbing a photon with energy $h\nu = 10\,{\rm eV}$?

(b) Obtain an expression for the heat capacity $C(T)$ of a grain with N_a atoms in the low-temperature limit $T \ll \Theta$.

(c) At high temperatures, $C(T) \to N_v k_{\rm B}$. By equating the low-temperature form you obtained in (b) with this high temperature limit, determine the value of T/Θ above which the low-temperature form of the Debye heat content ($E \propto T^4$) is no longer a good approximation. Evaluate this for a solid with $\Theta = 400\,{\rm K}$

24.3 Suppose that interstellar dust grains have $Q_{\rm abs} \propto \lambda^{-2}$ for $\lambda > 1\mu{\rm m}$. When exposed to the local interstellar radiation field (LISRF), these grains are heated to $T \approx 18\,{\rm K}$ and radiate with λI_λ peaking at $\lambda = 140\mu{\rm m}$.

In a region where the starlight has the same spectrum as the LISRF but is stronger by a numerical factor U:

(a) What will be the grain temperature?

(b) If $U = 10^3$, what will be the wavelength where λI_λ peaks?

24.4 Suppose that interstellar dust grains have $Q_{\rm abs} \propto \lambda^{-\beta}$ for $\lambda > 1\mu{\rm m}$. Suppose that when exposed to the local interstellar radiation field (LISRF), these grains are heated to $T \approx 18\,{\rm K}$ and radiate with λI_λ peaking at $\lambda_{\rm peak} = 140\mu{\rm m}$.

In a region where the starlight has the same spectrum as the LISRF but is stronger by a numerical factor U:

(a) What will be the grain temperature? (give your result in terms of U and β).

(b) If $U = 10^2$ and $\beta = 1.5$, what will be the wavelength where λI_λ peaks?

Chapter 25. Grain Physics: Charging and Sputtering

25.1 Consider a grain with radius $a = 0.1\,\mu\mathrm{m}$, located in an H I cloud with $n_e = 0.01\,\mathrm{cm}^{-3}$, $T = 100\,\mathrm{K}$, and a starlight background given by the MMP83 estimate for the solar neighborhood. Assume that the "sticking efficiency" for colliding electrons $s_e = 1$.

(a) Estimate the probability per unit time t_{e0}^{-1} for electron capture by a neutral grain.

(b) For the MMP83 radiation field [see Table 12.1 and Eq. (12.7)] the number density of 10-13.6 eV photons is $n_{\mathrm{FUV}} \approx 8\times10^{-4}\,\mathrm{cm}^{-3}$ (see Problem 12.2). If the $a = 0.1\,\mu\mathrm{m}$ grain has an absorption efficiency factor $Q_{\mathrm{abs}} \approx 1$, and the mean photoelectric yield for $h\nu > 10\,\mathrm{eV}$ is $Y_{\mathrm{pe}} = 0.1$, estimate the photoelectron emission rate t_{pe}^{-1}.

(c) As the grain becomes positively charged, Coulomb focusing will increase the rate of electron collisions. If the rate of photoelectron emission t_{pe}^{-1} does not change when the grain becomes positively charged, to what potential U will the grain charge?
How many unit charges does this correspond to?

25.2 Consider a grain of radius a. Suppose that the balance between photoelectron emission and electron capture results in charging to an average potential $U > 0$. The grain is located in gas with electron density n_e and gas temperature T, and the electron "sticking coefficient" is a constant s_e.

(a) What is the time-averaged rate of electron capture $\dot{N}_e$ by the grain? Give your result for general $U > 0$, $n_e s_e$, T, and a, and then evaluate this result for $n_e = 0.01\,\mathrm{cm}^{-3}$, $T = 10^2\,\mathrm{K}$, $s_e = 1$, $a = 0.1\,\mu\mathrm{m}$, and $U = 0.3\,\mathrm{V}$.

(b) What is the mean charge on the grain, in units of the charge quantum e?

(c) If the photoelectric emission rate is approximately independent of small variations of U, then the grain charge Z will fluctuate around $\langle Z\rangle$, with a standard deviation $\sim \sqrt{\langle Z\rangle}$, and a charge correlation time $\tau_Q \approx \langle Z\rangle \dot{N}_e^{-1}$, where $\dot{N}_e$ is the time-averaged electron capture rate.
Calculate the dimensionless number $\omega\tau_Q$, where ω is the grain gyrofrequency in the local magnetic field. Assume the grain material to have a density $\rho = 3\,\mathrm{g\,cm}^{-3}$. Evaluate $\omega\tau_Q$ for $a = 0.1\,\mu\mathrm{m}$, $n_e s_e = 0.01\,\mathrm{cm}^{-3}$, $T = 10^2\,\mathrm{K}$, and $B_0 = 5\,\mu\mathrm{G}$.

25.3 Sputtering acts to erode grains at a rate $da/dt = -n_{\mathrm{H}}\beta$ independent of a. Suppose that the grain size distribution at $t = 0$ is a power-law

$$\frac{1}{n_{\mathrm{H}}}\frac{dn}{da} = \frac{A_0}{a_{\max}}\left(\frac{a}{a_{\max}}\right)^{-p} \qquad 0 \le a \le a_{\max} \quad .$$

(a) Let V_0 be the initial volume of grain material per H nucleon. Express V_0 in terms of A_0, $a_{\max}$, and p.

(b) Obtain an algebraic expression for $V(t)/V_0$ in terms of $y \equiv \Delta a/a_{\max} = n_{\mathrm{H}}\beta t/a_{\max}$ and p.

25.4 For the previous problem, now assume $p = 3.5$, and $a_{\max} = 0.3\,\mu\mathrm{m}$.

(a) Plot $V(t)/V_0$ as a function of $\Delta a/a_{\max}$.

(b) Graphically estimate $\Delta a/a_{\max}$ such that $V/V_0 = 1/2$.

(c) If $a_{\max} = 0.30\,\mu\mathrm{m}$, $\beta = 10^{-2}\,\mathrm{cm}^3\,\text{Å}\,\mathrm{yr}^{-1}$, and $n_{\mathrm{H}} = 10^{-2}\,\mathrm{cm}^{-3}$, what time Δt is required to sputter away 50% of the mass in grains?

25.5 Suppose that at $t = 0$ the dust has a size distribution

$$\frac{1}{n_{\mathrm{H}}}\frac{dn}{da} = \frac{A_0}{a_0}\left(\frac{a}{a_0}\right)^{-p} \qquad \text{for } a \le a_{\max} \quad .$$

Suppose that sputtering has continued for some time t, at a sputtering rate $da/dt = -n_{\mathrm{H}}\beta$. Let $Q_{\mathrm{ext}}(a,\lambda)$ be the extinction efficiency factor at wavelength λ for a grain of radius a

Let $\sigma_{\rm ext}(\lambda)$ be the dust extinction cross section per H. Write down an integral expression for $\sigma_{\rm ext}(\lambda)$ at some fixed time $t < a_{\rm max}/|da/dt|$.

25.6 Consider hot plasma with density $n_{\rm H}$ in an elliptical galaxy. Suppose that planetary nebulae and other stellar outflows are injecting dust into the plasma with a rate per unit grain radius

$$\left(\frac{d\dot{N}_{\rm dust}}{da}\right)_{\rm inj} = \frac{A_0}{a_{\rm max}}\left(\frac{a}{a_{\rm max}}\right)^{-p} \quad .$$

(a) Obtain an expression for the total rate $(dM_{\rm dust}/dt)_{\rm inj}$ at which dust mass is being injected into the plasma, in terms of A_0, $a_{\rm max}$, p, and the density ρ of the grain material.

(b) Upon injection into the plasma, the grains are subject to sputtering at a rate $da/dt = -\beta n_{\rm H}$, where β is a constant. Find the steady state solution for $dN_{\rm dust}/da$, where $N_{\rm dust}(a)$ is the number of dust grains present with radii $\leq a$.

(c) Obtain an expression for the steady-state dust mass, $M_{\rm dust}$.

(d) Obtain an expression for the characteristic mass survival time $\tau_{\rm survival} \equiv M_{\rm dust}/(dM_{\rm dust}/dt)_{\rm inj}$ in terms of $a_{\rm max}$, p, and da/dt.

(e) Consider the "passive" elliptical galaxy NGC 4564 containing hot plasma $kT \approx 0.5\,{\rm keV}$ ($T \approx 6\times10^6\,{\rm K}$) and a core density $n_{\rm H} \approx 0.01\,{\rm cm}^{-3}$ (Soria et al. 2006, ApJ 640, 126). From Figure 25.4, the sputtering rate for refractory grains would be $da/dt = -\beta n_{\rm H}$, with $\beta \approx 10^{-6}\,\mu{\rm m\,cm^3\,yr^{-1}}$.

Suppose that the injected dust has $p = 3.5$ and $a_{\rm max} = 0.3\,\mu{\rm m}$. Estimate the mass survival time $\tau_{\rm survival}$. If the dust injection rate from evolved stars in the central kpc is $1.3\times10^{-4}\,M_\odot\,{\rm yr}^{-1}$ (Clemens et al. 2010: A&A 518, L50), estimate the estimated steady-state dust mass $M_{\rm dust}$.

Compare to the observed upper limit $M_{\rm dust} < 8700\,M_\odot$ from Clemens et al. (2010).

25.7 Suppose that interstellar gas contains dust grains consisting of two populations: "large" grains of radius $a_1 = 1\times10^{-5}\,{\rm cm}$ and number density $n_1 = 2\times10^{-12}n_{\rm H}$, and "small" grains of radius $a_2 = 5\times10^{-7}\,{\rm cm}$ and number density $n_2 = 1\times10^{-9}n_{\rm H}$.

(a) Suppose that every grain is charged to a potential $U \approx +2\,{\rm V}$. If the gas as a whole is electrically neutral, compute $(n_e - n_I)/n_{\rm H}$, where n_e is the free electron density, and n_I is the density of free ions (where we do *not* consider the charged grains to be "ions").

(b) Discuss whether your answer to (a) is seriously affected by charge quantization.

25.8 Suppose that a spherical dust grain of radius $a \approx 0.1\,\mu{\rm m}$ is charged to a potential +1.0 Volt.

(a) If the net charge on the grain is Q, calculate the number of excess charges $Z = Q/e$, where e is the unit charge.

(b) If the grain is moving perpendicular to the local magnetic field B, it will move in a circular orbit. If the local field is $B = 5\,\mu{\rm G}$, and the grain has solid density $\rho = 3\,{\rm g\,cm^{-3}}$, calculate the period of this orbital motion (assume only the Lorentz force acts on the grain).

(c) If the grain is moving at a speed $v_\perp = 1\,{\rm km\,s^{-1}}$ perpendicular to the magnetic field, what is the "gyroradius" of the orbit?

25.9 Consider hot plasma with density $n_{\rm H}$ in an elliptical galaxy. Suppose that planetary nebulae and other stellar outflows are injecting dust grains with a single initial size $a = a_{\rm max}$ into the plasma with a rate $(dN_{\rm dust}/dt)_{\rm inj}$, where $N_{\rm dust}$ = number of dust grains.

Upon injection into the plasma, the grains are subject to sputtering at a rate $da/dt = -\beta n_{\rm H}$, where β is a constant.

(a) Find the steady state solution for $dN_{\rm dust}/da$, where $N_{\rm dust}(a)$ is the number of dust grains present with radii $\leq a$. Express your result in terms of β, $n_{\rm H}$, and the injection rate $(dN_{\rm dust}/dt)_{\rm inj}$.

(b) If the rate of injection of dust mass is

$$\left(\frac{dM_{\rm dust}}{dt}\right)_{\rm inj} = \frac{4\pi\rho}{3} a_{\rm max}^3 \left(\frac{dN_{\rm dust}}{dt}\right)_{\rm inj}$$

where ρ is the internal density of the dust, obtain an expression for the steady-state dust mass, $M_{\rm dust}$. Express your result in terms of $(dM_{\rm dust}/dt)_{\rm inj}$, β, $n_{\rm H}$, and $a_{\rm max}$.

(c) Obtain an expression for the characteristic "mass survival time"

$$\tau_{\rm mass\ survival} \equiv \frac{M_{\rm dust}}{(dM_{\rm dust}/dt)_{\rm inj}} \quad .$$

(d) Consider the "passive" elliptical galaxy NGC 4564 containing hot plasma $kT \approx 0.5\,{\rm keV}$ ($T \approx 6\times10^6\,{\rm K}$) and a core density $n_{\rm H} \approx 0.01\,{\rm cm}^{-3}$ (Soria et al. 2006, ApJ 640, 126). From Figure 25.4, the sputtering rate for refractory grains would be $da/dt = -\beta n_{\rm H}$, with $\beta \approx 10^{-6}\,\mu{\rm m}\,{\rm cm}^3\,{\rm yr}^{-1}$.

Suppose that the injected dust has $a = 0.3\,\mu{\rm m}$. If the dust injection rate from evolved stars in the central ${\rm kpc}$ is $1.3\times10^{-4}\,M_\odot\,{\rm yr}^{-1}$ (Clemens et al. 2010: A&A 518, L50), estimate the steady-state dust mass $M_{\rm dust}$ in the central ${\rm kpc}$.

Compare to the observed upper limit $M_{\rm dust} < 8700\,M_\odot$ from Clemens et al. (2010).

(e) Estimate the dust mass survival time $\tau_{\rm mass\ survival}$ in the core of NGC 4564.

Chapter 26. Grain Dynamics

26.1 Suppose that a silicate dust grain has a radius $a = 0.1\,\mu\mathrm{m}$. Suppose that the dust grain has $Q_{abs} = 0.11$, $Q_{sca} = 0.69$, and that the scattered light has $\langle\cos\theta\rangle = 0.31$, where θ is the angle between the direction of incidence and the direction of propagation; these values have been estimated for "astronomical silicate" grains at $\lambda = 5500\,\text{Å}$ (Draine 1985: *Ap.J.Suppl.*, **57**, 587). Take $\rho = 3\,\mathrm{g\,cm^{-3}}$ for the grain density.

(a) Ignoring the wavelength dependence of these quantities, what is the value of L/M (the ratio of luminosity to mass) for a star such that the radiation pressure force on such a grain close (but not *too* close) to the star exactly balances the gravitational force due to the star? Give L/M in solar units ($L_\odot/M_\odot$).

(b) Now suppose that such silicate grains are mixed with gas, with the dust mass equal to 0.7% of the gas mass, and that the grains are "well-coupled" to the gas through collisions or magnetic fields. What must be the ratio of L/M for the star (in solar units) such that radiation pressure on the grains will exert a repulsion equal in magnitude to the gravitational attraction on the gas-dust mixture?

26.2 In this problem you will get some feeling for how anisotropic the radiation field in interstellar space is likely to be.

(a) Estimate the anisotropy of the radiation field in an interstellar cloud by pretending that it consists of an isotropic component with energy density $0.4\,\mathrm{eV\,cm^{-3}}$ plus radiation from an imaginary source of luminosity $L \approx 10^{10}\,L_\odot$ located at the galactic center (at a distance 8 kpc). What is the energy density ($\mathrm{eV\,cm^{-3}}$) of the radiation associated with the anisotropic component?

(b) Obviously one should worry about the contribution of the single apparently brightest star to the anisotropy of the local radiation field. Suppose that the brightest star in the sky is an A1V star with a luminosity $L = 50\,L_\odot$ at a distance $d = 2.7\,\mathrm{pc}$ [e.g., Sirius in our sky!]. Calculate the ratio of the energy density contributed by this star to the energy density contributed by the "galactic center" pseudosource.

26.3 Consider a dust grain with the properties of the $a = 0.1\,\mu\mathrm{m}$ "astronomical silicate" grain of problem 26.1. Suppose this grain to be located in a diffuse cloud of density $n_\mathrm{H} = 20\,\mathrm{cm^{-3}}$ and temperature $T = 100\,\mathrm{K}$, with $n(\mathrm{He})/n(\mathrm{H}) = 0.1$. Assume the starlight background to have an energy density of 0.5 eV $\mathrm{cm^{-3}}$, with 80% of the energy in an isotropic component, and 20% in a unidirectional component [cf. the "Galactic Center" contribution from problem 26.2(a)].

(a) Neglecting any forces other than gas drag and radiation pressure, what will be the "terminal" drift velocity of the grain relative to the gas if the grain is uncharged? Approximate the gas drag by the formula appropriate for subsonic motion (see Eq. 26.1-26.3):

$$F_\mathrm{drag} \approx C \cdot (\pi a^2) \cdot (n k_\mathrm{B} T) \frac{v}{\sqrt{k_\mathrm{B} T/\mu}} \quad ,$$

where $C = 16/3\sqrt{2\pi} \approx 2.13$, n is the gas particle density, and μ is the mass per gas particle.

(b) Approximately how long does it take the grain to reach terminal speed? Assume the grain density to be $\rho = 3\,\mathrm{g\,cm^{-3}}$.

(c) Moving at the terminal speed, how long would it take the grain to drift a distance of 1 pc?

26.4 Suppose the magnetic field strength in an interstellar cloud is $B = 3\,\mu\mathrm{G}$.

(a) Estimate the gyroradius for a grain with radius $a = 0.1\,\mu\mathrm{m}$, density $\rho = 3\,\mathrm{g\,cm^{-3}}$, charged to a potential $U = 2\,\mathrm{V}$, moving with a velocity of $1\,\mathrm{km\,s^{-1}}$ perpendicular to the magnetic field.

(b) What is the gyroperiod for this grain?

(c) Estimate the gyroradius for a 100 MeV proton moving perpendicular to the field.

26.5 β Pictoris is an A5 ZAMS star with substantial amounts of solid matter in a circumstellar disk. An A5 ZAMS star has luminosity $L \approx 20\,L_\odot$ and mass $M \approx 2\,M_\odot$. Assume that there is no gas present in the disk – we want to consider the motion of solid particles under the influence of radiation and gravity.

(a) Estimate $\tau_{\rm PR}$ for an $a = 10\,\mu{\rm m}$ grain (with $Q_{abs} \approx 1$ and $\rho \approx 3\,{\rm g\,cm^{-3}}$) in an orbit with radius $r = 3\times10^{13}\,{\rm cm}$. (Neglect scattering).

(b) Briefly discuss the dynamics of an $a = 0.1\,\mu{\rm m}$ silicate grain in the neighborhood of this star. Assume the optical properties given in problem 26.1.

26.6 Consider a dust grain with internal density $\rho \approx 2\,{\rm g\,cm^{-3}}$ (appropriate for carbonaceous material). Suppose the grain to be spherical with radius $a = 10^{-7}a_{-7}\,{\rm cm}$.

(a) If the gas kinetic temperature is $T = 10^2T_2\,{\rm K}$, what is the r.m.s. translational velocity of the dust grain due to thermal excitation alone?

(b) If the grain rotation is in thermal equilibrium with the gas, what will be the r.m.s. rotation rate?

(c) If the grain is neutral, and is located in an H I region with density $n_{\rm H} = 10^2n_2\,{\rm cm^{-3}}$, what is the timescale τ_M for the grain to collide with its own mass of gas? (If the only process acting to change the linear and angular momentum of the grain is direct collisions with neutral atoms, the translational and rotational motion of the grain will "thermalize" on this timescale.)

26.7 The relative velocity of the Sun and the local interstellar medium is estimated to be $26\,{\rm km\,s^{-1}}$ (Möbius et al. 2004: A&A 426, 897): from the standpoint of the Sun there is an "interstellar wind" with a speed $v_{\rm ISW} = 26\,{\rm km\,s^{-1}}$. The local density of the interstellar medium can be inferred from observations of backscattered solar Lyman α and Helium resonance line radiation; if the local helium is primarily neutral, then the inferred density is $n_{\rm H} \approx 0.22\,{\rm cm^{-3}}$ (Lallement et al. 2004; A&A 426, 875). Suppose that the local gas contains dust grains with a mass equal to 0.01 of the hydrogen mass. Suppose that these grains are in a size distribution with $dn/da \propto a^{-3.5}$ for $.005 < a < 0.25\,\mu{\rm m}$ (this is the "MRN" size distribution).

(a) For this size distribution, what fraction $f_M(a > 0.1\,\mu{\rm m})$ of the grain mass is in particles with $a > 0.1\,\mu{\rm m}$?

(b) At the radius of Jupiter, estimate the mass flux $({\rm g\ cm^{-2}\,s^{-1}})$ due to $a > 0.1\,\mu{\rm m}$ interstellar grains if they are not deflected after passing through the "heliopause" where the interstellar medium and the interplanetary medium are both shocked. The location of the heliopause is uncertain; it is estimated to be at $\sim 100\,{\rm AU}$.

(c) Now suppose the grains have internal densities of $\rho = 2\,{\rm g\,cm^{-3}}$, and suppose that sunlight charges them to a potential $U = 5\,{\rm V}$. Let the solar wind be in the radial direction with a speed $v_{\odot{\rm W}} = 450\,{\rm km\,s^{-1}}$. Assume that in the frame of reference where the solar wind is locally at rest, the local electric field vanishes. Further assume, for simplicity, that the interplanetary magnetic field is perpendicular to the direction of the interstellar wind, and has a strength $B = 2\,\mu{\rm G}$ (at $\sim$100 AU). With the above assumptions, calculate the gyroradius of an $a = 0.1\,\mu{\rm m}$ interstellar grain once it has entered the region containing the solar wind. How does the gyroradius depend on the grain radius a?

Chapter 27. Heating and Cooling of H II Regions

27.1 Consider an H II region consisting only of hydrogen. Suppose that the source of ionizing photons is a blackbody with temperature $T = 32000\,\mathrm{K}$. Assume the nebula is in thermal and ionization equilibrium.

(a) Near the center of the nebula, at what temperature will heating by photoionization balance cooling?

(b) Estimate the mass-weighted *average* temperature of the gas in the nebula.

27.2 The central regions of the Orion Nebula have $n_{\mathrm{H}} \approx 4{\times}10^3\,\mathrm{cm}^{-3}$. Suppose that MHD waves are being dissipated in the Orion Nebula. If the energy density in the waves Δu_{wave} is less than 10% of the gas pressure, what value of the damping length L_{damp} is required for the wave heating to equal 10% of the photoelectric heating rate Γ_{pe}? You may assume that $v_{\mathrm{wave}} \approx 10\,\mathrm{km\,s}^{-1}$.

27.3 Suppose that the cosmic ray flux within the Orion Nebula corresponds to a cosmic ray ionization rate $\zeta_{\mathrm{CR}} < 10^{-15}\,\mathrm{s}^{-1}$ for an H atom, with the ionization dominated by $\sim 1\,\mathrm{GeV}$ protons. Compare the heating rate due to plasma drag on the cosmic rays with the photolectric heating rate Γ_{pe}. Assume $n_{\mathrm{H}} \approx 4000\,\mathrm{cm}^{-3}$ and $T \approx 8000\,\mathrm{K}$ for the gas, and assume the H is fully ionized.

Chapter 28. The Orion H II Region

28.1 The free-free emission from the Orion Nebula has been measured with radio telescopes. At $\nu = 1.4\,\mathrm{GHz}$ the integrated flux density from M42 is $F_\nu = 495\,\mathrm{Jy}$. Assuming a distance $D = 414\,\mathrm{pc}$, estimate the hydrogen photoionization rate $\dot{N}_L$ required to keep this gas ionized, if the gas temperature is $T = 9000\,\mathrm{K}$. Assume helium to be singly-ionized, with $n_{\mathrm{He}}/n_{\mathrm{H}} = 0.10$.

28.2 The peak emission measure in M42 is $EM = 5 \times 10^6\,\mathrm{cm}^{-6}\,\mathrm{pc}$. If M42 is approximated as a uniform density sphere of diameter $0.5\,\mathrm{pc}$, calculate the total rate of H recombinations occuring within this sphere. Assume a gas temperature $T = 10^4\,\mathrm{K}$, and assume He is singly ionized with He/H=0.1

Chapter 29. H I Clouds: Observations

29.1 Suppose the H I gas to be in a plane-parallel slab geometry, with full thickness $6 \times 10^{20}\,\mathrm{cm}^{-2}$, and take the velocity distribution be Gaussian with a one-dimensional velocity dispersion $\sigma_V = 10\,\mathrm{km\,s}^{-1}$. Neglect the effects of Galactic rotation.

(a) If the spin temperature is $T_{\rm spin} = 100\,\mathrm{K}$, for what galactic latitudes is the line-center optical depth $\tau < 0.5$, as seen from a point in the mid-plane?

(b) If the full-thickness of the H I disk is 300 pc, out to what radius (in the plane) can it be observed with line-center optical depth $\tau < 0.5$?

(c) What is the maximum $N(\mathrm{H\,I})$ that can be observed with $\tau < 0.5$ at all radial velocities?

29.2 Let $dN(\mathrm{H\,I})/du \times \Delta u$ be the column density of H I in the radial velocity interval Δu. Show that the optical depth in the 21-cm line can be written

$$\tau = \frac{3}{32\pi} A_{u\ell} \frac{hc\lambda^2}{k_{\rm B} T_{\rm spin}} \frac{dN(\mathrm{H\,I})}{du}$$
$$= 0.552 \left(\frac{100\,\mathrm{K}}{T_{\rm spin}}\right) \frac{dN(\mathrm{H\,I})/du}{10^{20}\,\mathrm{cm}^{-2}/(\,\mathrm{km\,s}^{-1})} \quad .$$

29.3 Suppose we observe a background radio continuum point source through a layer of "foreground" H I with $dN(\mathrm{H\,I})/du = 3 \times 10^{20}\,\mathrm{cm}^{-2}/(20\,\mathrm{km\,s}^{-1})$, where u is the radial velocity. If the measured flux density of the background continuum source changes by less than 1% on-line to off-line, what can be said about the spin temperature of the H I? Assume the beamsize is very small. You may use the result from problem 29.2:

$$\tau = 0.552 \left(\frac{100\,\mathrm{K}}{T_{\rm spin}}\right) \frac{dN(\mathrm{H\,I})/du}{10^{20}\,\mathrm{cm}^{-2}/(\,\mathrm{km\,s}^{-1})} \quad .$$

Chapter 30. H I Clouds: Heating and Cooling

30.1 The local X-ray background (see Figure 12.1) can be approximated by

$$\nu u_\nu \approx 1\times10^{-18}\left(\frac{h\nu}{400\,\mathrm{eV}}\right)^{\beta}\,\mathrm{erg\,cm^{-3}}$$

for $400 \lesssim h\nu \lesssim 1\,\mathrm{keV}$.

(a) Using the photoionization cross section from eq. (13.3), obtain an expression for the rate for photoionization of H by the 0.4–1 keV X-ray background, showing explicitly the dependence on β. To keep the algebra simple, define $u_0 \equiv 10^{-18}\,\mathrm{erg\,cm^{-3}}$ and $\sigma_0 \equiv 6.3\times10^{-18}\,\mathrm{cm}^2$, and leave your result in terms of u_0, σ_0, and c.

(b) Evaluate the rate for $\beta = 2$. Is the photoionization rate dominated by the low-energy X-rays or the high-energy X-rays?

(c) What is the mean energy of the absorbed photons for the above X-ray spectrum?

(d) The photoelectrons resulting from X-ray ionization of H and He have sufficient energy to produce secondary ionizations. If the fractional ionization $x_e \approx 4 \times 10^{-4}$, use Eq. (13.6) to estimate the number of secondary ionizations per photoelectron.

Chapter 31. Molecular Hydrogen

31.1 The radiative attachment reaction

$$\mathrm{H} + e^- \rightarrow \mathrm{H}^- + h\nu$$

has a rate coefficient $k_{\mathrm{ra}} = 1.9\times10^{-16}T_2^{0.67}\,\mathrm{cm^3\,s^{-1}}$. The associative detachment reaction

$$\mathrm{H}^- + \mathrm{H} \rightarrow \mathrm{H}_2 + e^-$$

is a fast ion-molecule reaction with rate coefficient $k_{\mathrm{ad}} = 1.3\times10^{-9}\,\mathrm{cm^3\,s^{-1}}$, but H^- also undergoes photodetachment

$$\mathrm{H}^- + h\nu \rightarrow \mathrm{H} + e^-$$

with a rate $\zeta_{\mathrm{pd}} = 2.4\times10^{-7}\,\mathrm{s^{-1}}$ in the interstellar radiation field (rates are from Le Teuff et al. 2000: A&A Suppl., 146, 157).

Consider an H I cloud with density $n_{\mathrm{H}} = 30\,\mathrm{cm^{-3}}$ and electron density $n_e = 0.02\,\mathrm{cm^{-3}}$. The temperature is $T = 10^2 T_2\,\mathrm{K}$ (show the dependence of your results on T_2).

(a) What is the steady-state ratio $n(\mathrm{H}^-)/n_{\mathrm{H}}$?

(b) What fraction of the H^- ions undergo the reaction $\mathrm{H}^- + \mathrm{H} \rightarrow \mathrm{H}_2 + e^-$?

(c) Evaluate the quantity

$$R_{\mathrm{H}^-} \equiv \frac{k_{\mathrm{ad}}n(\mathrm{H}^-)n(\mathrm{H})}{n_{\mathrm{H}}n(\mathrm{H})}$$

Compare this to the empirical "rate coefficient" for formation of H_2 by dust grain catalysis.

31.2 Consider a region containing a mixture of H and H_2. Let the rate per volume of formation of H_2 from H via grain surface recombination be $Rn_{\mathrm{H}}n(\mathrm{H})$ (i.e., $[dn(\mathrm{H}_2)/dt]_{\mathrm{gr.form.}} = Rn_{\mathrm{H}}n(\mathrm{H})$ is the contribution to $dn(\mathrm{H}_2)/dt$ from formation on grains). Let β be the rate for photodissociation of $\mathrm{H}_2 \rightarrow 2\mathrm{H}$.

(a) What is the steady-state solution y_s for $y \equiv 2n(\mathrm{H}_2)/n_{\mathrm{H}}$?

(b) If $y(t=0) = y_s + \delta y$, show that $y(t>0) = y_s + \delta y\ \mathrm{e}^{-t/\tau}$ (assuming n_{H}, R, and β to remain constant). Obtain an expression for the "relaxation time" τ in terms of n_{H}, R, and β.

(c) Estimate the timescale τ for $n_{\mathrm{H}} = 20\,\mathrm{cm^{-3}}$, $R = 3\times10^{-17}\,\mathrm{cm^3\,s^{-1}}$, and β such that $y_s = 0.5$.

31.3 In the early universe, near redshift $z \approx 100$, the H nucleon density

$$n_{\mathrm{H}} \approx 0.20\,\mathrm{cm^{-3}} \quad ,$$

and the fractional ionization of hydrogen has dropped to

$$\frac{n_e}{n_{\mathrm{H}}} \approx 3\times10^{-4} \quad .$$

The CMB temperature is $T \approx 275\,\mathrm{K}$; the gas temperature is close to the CMB temperature.

The radiative attachment reaction

$$\mathrm{H} + e^- \rightarrow \mathrm{H}^- + h\nu$$

has a rate coefficient

$$k_{\mathrm{ra}} = 3.7\times10^{-16}\left(\frac{T}{275\,\mathrm{K}}\right)^{0.67}\,\mathrm{cm^3\,s^{-1}} \quad .$$

The associative detachment reaction

$$\mathrm{H}^- + \mathrm{H} \rightarrow \mathrm{H}_2 + e^-$$

is a fast ion-neutral reation with a rate coefficient

$$k_{\rm ad} = 1.3 \times 10^{-9}\,{\rm cm}^3\,{\rm s}^{-1} \ \ ,$$

and $\rm H^-$ is also destroyed by

$$\rm H^- + H^+ \to H + H \ \ ,$$

with a rate coefficient

$$k_{\rm n} \approx 7.8 \times 10^{-8} \left(\frac{T}{275\,{\rm K}}\right)^{-1/2} {\rm cm}^3\,{\rm s}^{-1} \ \ .$$

(a) If the Universe were not continuing to expand and recombine, what would be the steady-state density $n({\rm H}^-)$ for $n_{\rm H} = 0.20\,{\rm cm}^{-3}$, $T = 275\,{\rm K}$, and $n_e/n_{\rm H} = 3\times10^{-4}$?

(b) Assuming this steady-state abundance of $\rm H^-$, calculate the rate per volume of $\rm H_2$ formation ($\rm H_2\,cm^{-3}\,s^{-1}$) at this time.

Chapter 32. Molecular Clouds: Observations

32.1 The mass distribution of GMCs in the Galaxy is given by [eq. (32.1) in the textbook]:

$$\frac{dN_{\rm GMC}}{d\ln M_{\rm GMC}} \approx N_u \left(\frac{M_{\rm GMC}}{M_u}\right)^{-\alpha} \quad 10^3\,M_\odot \lesssim M_{\rm GMC} < M_u$$

with $M_u \approx 6\times 10^6\,M_\odot$, $N_u \approx 63$, and $\alpha \approx 0.6$ (Williams & McKee 1997, Astrophys. J. 476, 166).

(a) Calculate the total mass in GMCs in the Galaxy.

(b) Calculate the number of GMCs in the Galaxy with $M > 10^6\,M_\odot$.

Chapter 33. Molecular Clouds: Chemistry and Ionization

33.1 Consider a diffuse molecular cloud with $n_{\rm H} = 10^2\,{\rm cm}^{-3}$. The hydrogen is predominantly molecular, with $n({\rm H}_2) = 50\,{\rm cm}^{-3}$. Assume that 30% of the total C (250 ppm) abundance is in ${\rm C}^+$: $n({\rm C}^+) \approx 7.5\times10^{-5} n_{\rm H} = 7.5\times10^{-3}\,{\rm cm}^{-3}$. Assume that $n_e \approx 10^{-4} n_{\rm H} = 0.01\,{\rm cm}^{-3}$. Assume that $n({\rm O})/n_{\rm H} \approx 4\times10^{-4} = 0.04\,{\rm cm}^{-3}$. Treat $T_2 \equiv T/10^2\,{\rm K}$ as a free parameter.

Consider the reactions in the reaction network (33.6-33.13).

(a) Calculate the steady-state abundance of ${\rm CH}_2^+$.
(b) Calculate the steady-state abundance of CH.
(c) Calculate the steady-state abundance of CO, leaving $f_{\rm shield}({\rm CO})$ as a free parameter. What fraction of all of the carbon is in CO?

33.2 Consider a diffuse molecular cloud with $n_{\rm H} = 10^2\,{\rm cm}^{-3}$. The hydrogen is predominantly molecular, with $n({\rm H}_2) = 50\,{\rm cm}^{-3}$. The oxygen is primarily atomic, with $n({\rm O}) \approx 4\times10^{-4} n_{\rm H}$. Assume that cosmic ray ionization maintains an abundance $n({\rm H}_3^+) \approx 5\times10^{-8} n_{\rm H}$, and cosmic ray ionization plus starlight photoionization of metals maintains $n_e \approx 10^{-4} n_{\rm H} = 0.01\,{\rm cm}^{-3}$. Let the gas temperature be $T = 10^2 T_2\,{\rm K}$.

Consider the reactions in the reaction network (33.14-33.19).

(a) What is the steady-state density $n({\rm OH}^+)$?
(b) What is the steady-state density $n({\rm H_2O}^+)$?
(c) What is the steady-state OH abundance relative to hydrogen, $n({\rm OH})/n_{\rm H}$?
(d) There is more than one reaction that can produce OH. Which is most important for the given conditions?

33.3 Consider a hypothetical molecule ${\rm XH}^+$. Suppose that the principal channel for its formation in a diffuse cloud is the radiative association reaction

$$
{\rm X}^+ + {\rm H} \rightarrow {\rm XH}^+ + h\nu
$$

with a rate coefficient $k_{ra} = 5\times10^{-17}\,{\rm cm}^3\,{\rm s}^{-1}$. Suppose that the two principal reactions for destroying ${\rm XH}^+$ are dissociative recombination

$$
{\rm XH}^+ + e^- \rightarrow {\rm X} + {\rm H}
$$

with a rate coefficient $k_{dr} = 2\times10^{-7}\,{\rm cm}^3\,{\rm s}^{-1}$ and photodissociation

$$
{\rm XH}^+ + h\nu \rightarrow {\rm X}^+ + {\rm H}
$$

with a rate $\beta = 5\times10^{-10}\,{\rm s}^{-1}$ due to the ambient starlight background.

(a) If only these processes act, compute the steady-state density n_s of ${\rm XH}^+$ in a diffuse cloud with $n({\rm H}) = 20\,{\rm cm}^{-3}$, $n({\rm X}^+) = 5\times10^{-3}\,{\rm cm}^{-3}$, and $n_e = 0.01\,{\rm cm}^{-3}$.
(b) Suppose that at time $t = 0$ we have $n({\rm XH}^+) = n_s + \Delta_0$. Assume that $n({\rm H})$, $n({\rm X}^+)$, and n_e can all be approximated as constant. It is easy to show that for $t > 0$, $n({\rm XH}^+) = n_s + \Delta_0 e^{-t/\tau}$. Calculate the value of τ.

33.4 Consider a hypothetical molecule ${\rm XH}^+$. The principal channel for its formation in a diffuse cloud is

$$
{\rm X}^+ + {\rm H}_2 \rightarrow {\rm XH}^+ + {\rm H}
$$

with a rate coefficient $k_{\rm f} = 1\times10^{-12}\,{\rm cm}^3\,{\rm s}^{-1}$. Suppose that the two principal reactions for destroying ${\rm XH}^+$ are dissociative recombination

$$
{\rm XH}^+ + e^- \rightarrow {\rm X} + {\rm H}
$$

with a rate coefficient $k_{\rm dr} = 2\times10^{-7}\,{\rm cm}^3\,{\rm s}^{-1}$ and photodissociation

$$
{\rm XH}^+ + h\nu \rightarrow {\rm X}^+ + {\rm H}
$$

with a rate $\beta = 5\times10^{-10}\,{\rm s}^{-1}$ due to the ambient starlight background.

(a) If only these processes act, compute the steady-state density n_s of ${\rm XH}^+$ in a diffuse cloud with $n({\rm H}) = 10\,{\rm cm}^{-3}$, $n({\rm H}_2) = 5\,{\rm cm}^{-3}$, $n({\rm X}^+) = 5\times10^{-3}\,{\rm cm}^{-3}$, and $n_e = 0.01\,{\rm cm}^{-3}$.
(b) What fraction $f_{\rm dr}$ of the ${\rm XH}^+$ destructions are due to dissociative recombination?

Chapter 34. Physical Processes in Hot Gas

34.1 Consider a spherical cloud of radius R_c immersed in hot gas with temperature and density (far from the cloud) T_h and n_h. In the regime where classical evaporation applies, and the evaporative mass loss rate is $\dot{M} = 16\pi\mu R_c \kappa_h / 25 k_{\rm B}$, estimate the velocity $v(r)$ of the evaporative flow. Express your answer for $v(r)$ in terms of n_h, T_h, κ_h, R_c, and (r/R_c).

34.2 Consider a slab of gas, with surfaces at $x = \pm L/2$. Suppose that the gas at the two surfaces of the slab has density $n_{\rm H} = n_0$ and temperature $T = T_0 > 10^5\,{\rm K}$, and is collisionally ionized. Assume that no magnetic field is present.

(a) If the slab is thin, thermal conduction will keep the temperature within the slab close to the value at the slab surface. Suppose that the gas within the slab loses heat by radiative cooling, with radiative power per unit volume $\Lambda(T)$. Suppose that the temperature profile within the slab is

$$T \approx T_0 \left[1 + \beta \left(\frac{2x}{L} - 1 \right) \left(\frac{2x}{L} + 1 \right) \right]$$

where $\beta > 0$ is a constant. This has the property that $T = T_0$ at $x = \pm L/2$, and $T = T_0 - \beta T_0$ at $x = 0$. If the thermal conductivity κ and cooling function Λ are both taken to be constants, $\kappa \approx \kappa_0 \equiv \kappa(T_0)$ and $\Lambda \approx \Lambda_0 \equiv \Lambda(n_0, T_0)$, find β as a function of L, T_0, κ_0 and Λ_0.

(b) Classical thermal conduction is given by eq. (34.5):

$$\kappa \approx 0.87 \frac{k^{7/2} T^{5/2}}{m_e^{1/2} e^4 \ln \Lambda_c}$$

where $\ln \Lambda_c \approx 25$ is the Coulomb logarithm. Suppose that the cooling function

$$\Lambda = 1.3 \times 10^{-22} n_0^2 T_6^{-0.7}\,{\rm erg\,cm^3\,s^{-1}}$$

where $n_0 \equiv n_{\rm H}/\,{\rm cm}^{-3}$, and $T_6 \equiv T/10^6\,{\rm K}$. Using the result for (a) (i.e., treating κ and Λ as constant), evaluate the length scale L_F such that $\beta = 1$. L_F is known as the "Field length".
Give your answer in terms of n_0 and T_6.

(c) If $\beta \ll 1$ the assumption of constant κ and Λ are reasonable. Qualitatively, what do you expect to happen if the slab thickness L were to be such that β is of order unity?

34.3 Suppose that hot interstellar gas contains dust grains of radius $a = 1 \times 10^{-5}\,{\rm cm}$ and number density $n_{\rm gr} = 2 \times 10^{-12} n_{\rm H}$. Suppose that the grains are uncharged, and that every ion or electron that collides with the grain surface transfers a fraction α of its original kinetic energy to the grain, which then cools radiatively.

Estimate $\Lambda =$ the rate per volume at which the gas loses thermal energy due to this process, for density $n_{\rm H} = n_0\,{\rm cm}^{-3}$ and temperature $T = 1 \times 10^7 T_7\,{\rm K}$. Assume the H and He to be fully ionized, and He/H=0.1. Give your answer in terms of α, n_0 and T_7.

Chapter 35. Fluid Dynamics

35.1 Show that the term $(c/4\pi\sigma)\nabla \times \partial\mathbf{D}/\partial t$ that has been omitted in Eq. (35.46) is smaller than $(c^2/4\pi\sigma)\nabla^2\mathbf{B}$ by a factor $\sim (v/c)^2$, where v is a characteristic velocity in the flow.

35.2 The discussion leading to the expression Eq. (35.49) for $\tau_{\rm decay}$ assumed a fully-ionized gas. In partially-ionized gas, electrons can be scattered by neutrals as well as by ions. Define a dimensionless quantity x_c by

$$x_c \equiv \frac{x_e}{(1-x_e)} \times \frac{\text{scattering by neutrals}}{\text{scattering by ions}} \quad ,$$

where x_c is a constant. The conductivity can then be written

$$\sigma \approx \frac{\sigma(x_e=1)}{1+(1-x_e)(x_c/x_e)} \quad .$$

Thus, if $x_c \ll 1$, when $x_e \approx x_c$ the neutrals and ions are equally important for limiting the electrical conductivity.

(a) Obtain an estimate for x_c as a function of temperature. Electron-neutral scattering is discussed in §2.5. Using the rate coefficient (2.41) for electron scattering by H_2, and the electron-ion scattering rate from Eq. (2.23), estimate the value of x_c as a function of T. Ignore scattering by He, and take the "Coulomb logarithm" to have the value $\ln\Lambda \approx 25$.

(b) For $T = 100\,{\rm K}$, estimate the fractional ionization x_e below which scattering of electrons by neutrals is more important than scattering of electrons by ions.

35.3 The "cooling time" $\tau_{\rm cool} \equiv |d\ln T/dt|^{-1}$. Suppose the power radiated per unit volume Λ can be approximated by

$$\Lambda \approx A n_{\rm H} n_e \left[T_6^{-0.7} + 0.021 T_6^{1/2}\right]$$

for gas of cosmic abundances, where $A = 1.1\times10^{-22}\,{\rm erg\,cm^3\,s^{-1}}$, and $T_6 \equiv T/10^6\,{\rm K}$. Assume the gas to have $n_{\rm He} = 0.1 n_{\rm H}$, with both H and He fully ionized.

Compute the cooling time (at constant pressure) due to radiative cooling

(a) in a supernova remnant at $T = 10^7\,{\rm K}$, $n_{\rm H} = 10^{-2}\,{\rm cm^{-3}}$.

(b) for intergalactic gas within a dense galaxy cluster (the "intracluster medium") with $T = 10^8\,{\rm K}$, $n_{\rm H} = 10^{-3}\,{\rm cm^{-3}}$.

35.4 Show that the surface integral (36.16) is equivalent to the volume integral (36.15).

Chapter 36. Shock Waves

36.1 Consider a strong shock wave propagating into a medium that was initially at rest. Assume the gas to be monatomic ($\gamma = 5/3$). Consider the material just behind the shock front. The gas has an energy density $u_{\rm thermal}$ from random thermal motions, and an energy density $u_{\rm flow}$ from the bulk motion of the shocked gas. If cooling is negligible, calculate the ratio $u_{\rm flow}/u_{\rm thermal}$ in the frame of reference where the shock front is stationary.

36.2 Consider a 2-fluid shock with preshock neutral density n_{n0}, preshock magnetic field B_0, and preshock electron density n_{e0}. The extent L of the magnetic precursor is given by eq. (36.43).

(a) Obtain an estimate for N_i = number of times that a given ion will undergo scattering by a neutral in the precursor.

(b) Obtain an estimate for N_n = number of times that a given neutral will undergo scattering by an ion in the precursor.

(c) Obtain an estimate for Δp, the total momentum loss per neutral.

(d) Estimate the change Δv_n in the flow speed of the neutrals *before* arrival at the viscous subshock.

36.3 Suppose that a shock wave propagates at velocity v_s through a fluid with preshock number density n_0, preshock temperature $T_0 = 0$, and preshock magnetic field $B_0 = 0$. Take the fluid to be a monatomic ideal gas of molecular weight μ.

(a) What is the density n_s just behind the shock?

(b) What is the temperature T_s just behind the shock?

(c) What is the ratio of the thermal pressure $n_s kT_s$ to the preshock "ram pressure" $n_0 \mu v_s^2$?

(d) Suppose that the postshock gas is subject to radiative cooling with a loss rate per unit volume $\Lambda = An^2T^\alpha$, where A and α are constants. Assume that the shock is steady and plane-parallel, neglect heat conduction, and make the simplifying assumption that the postshock cooling occurs at constant pressure, i.e., $nT = n_sT_s$.

For what values of α does a fluid element cool to $T = 0$ in a finite time $t_{\rm cool}$ after being shocked? Obtain a formula for $t_{\rm cool}$ as a function of n_s, T_s, A, and α. Would this hold true for bremsstrahlung cooling, in particular?

(e) With the same assumptions as in (c), for what values of α does the fluid element cool to $T = 0$ within a finite distance $x_{\rm cool}$ of the shock front?

Hint: Remember that the distance x traveled from the shock and the time t elapsed since passing through the shock are related by $dx = v\,dt$, where v is related to the shock speed v_s through mass conservation, $nv = n_0v_s$. Thus $dx = (n_0/n)v_s\,dt$.

36.4 Consider spherically-symmetric accretion of matter from "infinity" onto a white dwarf of mass $M = 1\,M_\odot$ and radius $R = 5.5 \times 10^8$ cm. Assume that the accretion flow is cold, but fully-ionized. Suppose the accretion rate to be $\dot{M} = 10^{-9}\,M_\odot\,{\rm yr}^{-1}$, with He/H=0.1 . The "accretion shock" is assumed to be just above the stellar surface.

(a) What is the temperature T_s and H nucleon density $n_{\rm H}$ just after passing through the accretion shock? Express kT in keV.

(b) What is the luminosity of the star due to accretion alone?

(c) What is the effective temperature $T_{\rm eff}$ of the star if accretion energy dominates the luminosity?

36.5 Consider a strong shock with velocity v_s propagating into a monatomic ($\gamma = 5/3$) gas. The preshock gas contains dust grains that are at rest relative to the gas. Immediately after passage of the shock front, the grains still have their original velocity. What is the velocity of the grains relative to the shocked gas?

36.6 A spherically-symmetric galaxy cluster has a mass $M = 5 \times 10^{14}\, M_\odot$ interior to radius $R = 2\,\mathrm{Mpc}$. Cold gas from the intergalactic medium is falling freely (from "infinity") toward the cluster until it hits the intracluster medium and forms a shock front at $R = 2\,\mathrm{Mpc}$.

(a) Assume that the standing shock is at rest relative to the center of the cluster. Assume that the cluster mass exterior to $2\,\mathrm{Mpc}$ can be neglected. If the infalling H-He mixture is fully ionized, what is the temperature of the infalling gas after it is shocked?

(b) Now instead assume that the *shocked* gas is at rest relative to the center of the cluster, with the shock front at $R = 2\,\mathrm{Mpc}$ moving *outward*. The infalling gas is as before. What would be the shock speed, and the post-shock temperature?

Chapter 37. Ionization/Dissociation Fronts

37.1 Eq. (37.15) gives the velocity V_i of the ionization front propagating outward from a source of ionizing photons that turned on at $t = 0$ in a uniform, initially neutral, medium.

Consider early times when $t/\tau \ll 1$. For $Q_0 = 10^{48}\,\mathrm{s}^{-1}$ and $n_0 = 10^3\,\mathrm{cm}^{-3}$, evaluate V_i for $t/\tau = 10^{-4}$. Discuss the physical significance of the result, and comment on the validity of the analysis leading to this result.

37.2 At $z = 10$ the age of the Universe is $\sim$482 Myr, and the average H density is $n_{\mathrm{H}} = 2.52 \times 10^{-4}\,\mathrm{cm}^{-3}$. Consider a QSO at $z = 10$ that suddenly "turns on" with a spectrum

$$\nu L_\nu = 10^{44} L_{44} \left(\frac{h\nu}{I_{\mathrm{H}}}\right)^{1-\alpha} \quad \mathrm{erg\,s^{-1}} \ .$$

Suppose that the region around the QSO is initially neutral with uniform density $n_{\mathrm{H}} = 10^{-4} n_{-4}\,\mathrm{cm}^{-3}$.

(a) Let Q_0 be the rate of emission of photons with $h\nu > I_{\mathrm{H}}$. Relate Q_0 to L_{44} and α.

(b) For the above spectrum, what is the mean energy of the photons with $h\nu > 13.6\,\mathrm{eV}$? Evaluate this for $\alpha = 1.2$.

(c) Let the temperature of the photoionized gas be $T = 10^4 T_4\,\mathrm{K}$. Supposing that photoelectric absorption is the only process producing ionization of H, obtain an expression for the Strömgren radius R_{S0} in terms of n_{-4}, L_{44}, α, and T_4. Evaluate this for $n_{-4} = 3$, $L_{44} = 1$, $\alpha = 1.2$, and $T_4 = 2$.

(d) One validity criterion for the Strömgren sphere approximation is that $\tau_{\mathrm{S0}} = n_{\mathrm{H}} \sigma_{\mathrm{pi}} R_{\mathrm{S0}} \gg 1$. Is this fulfilled in the present problem? (Assume $L_{44} = 1$, $n_{-4} = 3$, and $\alpha = 1.2$).

(e) Suppose that the Hubble expansion can be ignored, so that the density can be approximated as remaining constant. The ionization front radius should then be given by Eq. (37.14). Assume $L_{44} = 1$, $n_{-4} = 3$, $\alpha = 1.2$, and $T_4 = 2$. Estimate the radius and velocity of the ionization front at $t = 10^6\,\mathrm{yr}$, $t = 10^7\,\mathrm{yr}$, and $t = 10^8\,\mathrm{yr}$.

(f) If the photoionized gas has $T_4 \approx 2$, what is the R-critical velocity u_R? Approximate the He as being fully-ionized, no magnetic fields, take the neutral gas to be cold, and primordial $n_{\mathrm{He}}/n_{\mathrm{H}} = 0.082$.

(g) Ignoring the Hubble expansion (i.e., assuming the density to remain constant), estimate the time when the ionization front would make the transition from R-type to D-type.

Compare to the age of the Universe at that time. Comment on whether or not it is reasonable to neglect the Hubble expansion.

Chapter 38. Stellar Winds

38.1 Suppose that a star has spent $10^6\,\mathrm{yr}$ as a red supergiant with a mass loss rate $\dot{M}_{rg} = 10^{-6}\,M_\odot\,\mathrm{yr}^{-1}$ and a wind velocity $v_{rg} = 10\,\mathrm{km\,s}^{-1}$. At time $t = 0$ the star suddenly begins producing a fast wind with $\dot{M}_{fw} = 10^{-7}\,M_\odot\,\mathrm{yr}^{-1}$ and $v_{fw} = 10^3\,\mathrm{km\,s}^{-1}$. Assume that radiative cooling and heat conduction are negligible. The resulting structure will contain four zones:

1. unshocked fast wind;
2. shocked fast wind;
3. shocked slow wind;
4. unshocked slow wind.

So long as the shock has not reached the outer boundary of the slow wind, the radius of the (outer) shock wave propagating into the unshocked slow wind material will vary as some power of t: $R_{sw} \propto t^\alpha$. You can use simple dimensionless analysis to obtain the value of α.

Proceed by assuming that the radius $R_{sw}(t)$ of the shock wave propagating into the slow wind material varies as some power of time: $R_{sw}(t) \propto t^\alpha$. If $M_{sw}(t)$ is the mass of shocked slow wind material (i.e., slow wind material that has been overtaken by the shock front), this will also vary as some power of time; similarly, the kinetic energy of the shocked slow wind material will increase as a power of time. Since we have assumed that there are no radiative losses, the total energy (kinetic energy of the ordered motion plus thermal kinetic energy) $E_{sw}(t)$ of the shocked slow wind material must be some (constant) fraction of the energy input from the fast wind up to time t; use this to determine the value of α. Let $E(t) = (1/2)\dot{M}_{fw}v_{fw}^2 t$ be the total energy input from the fast wind up to time t. If you now assume that $E_{sw}(t)$ is some (as yet unknown, but constant) fraction β of $E(t)$ [i.e., $E_{sw}(t) = \beta E(t)$], you can now obtain an estimate of $R_{sw}(t)$.

(a) Use simple "dimensional analysis" to determine the value of the power-law index α.

(b) Estimate the radius R_{sw} of the region of shocked slow wind at $t = 10^4\,\mathrm{yr}$.

(c) Estimate the temperature of the shocked slow wind material. (Assume the gas to be fully-ionized with He/H=0.1).

(d) Estimate the temperature of the shocked fast wind material.

38.2 The local ISM is estimated to have a density $n_{\mathrm{H}} \approx 0.22\,\mathrm{cm}^{-3}$, and flowing at $V_{\mathrm{ISM}} \approx 26\,\mathrm{km\,s}^{-1}$ relative to the Sun. The local ISM partially ionized, with an isothermal sound speed $c_{\mathrm{ISM}} \approx 7\,\mathrm{km\,s}^{-1}$, and it is magnetized, with an Alfvén speed $v_A \approx 10\,\mathrm{km\,s}^{-1}$. The solar wind varies over the solar cycle, but characteristic values of the wind speed and mass loss rate are $\dot{M} \approx 2.5 \times 10^{-14}\,M_\odot\,\mathrm{yr}^{-1}$, and $V_w \approx 700\,\mathrm{km\,s}^{-1}$. The solar wind is hypersonic – thermal and magnetic pressures are neglible compared to the ram pressure $\rho_w V_w^2$. Estimate the distance to the stagnation point between the termination shock and the bowshock. Express this in AU.

Chapter 39. Effects of Supernovae on the ISM

39.1 Obtain an estimate of the dimensionless factor A in eq. (39.8) by assuming that 50% of the total energy will be in ordered kinetic energy, and that the ordered kinetic energy is $\approx (1/2)Mv_s^2$, where M is the swept-up mass. Compare the resulting estimate for A with the exact solution.

39.2 Above we considered the case of uniform ambient density ρ and constant total energy E. Suppose that we instead assume that the ambient density decreases as

$$\rho = \rho_0 \left(r/r_0\right)^\delta \qquad (\delta > -3) \quad ,$$

and energy is increasing with time as a power law:

$$E = E_0 \left(t/t_0\right)^\epsilon \qquad (\epsilon \geq 0) \quad .$$

The radius of the blastwave will vary as $R_s = \mathrm{const}\, t^\gamma$.

(a) Find γ in terms of δ and ϵ.
Hint: To proceed, suppose once again that

$$R_s = AE^\alpha \rho^\beta t^\eta \quad ,$$

where A is a dimensionless constant of order unity, $\rho \equiv \rho(R_s)$. We have seen above from dimensional analysis that $\alpha = 1/5$, $\beta = -1/5$, and $\eta = 2/5$. Taking into account the variation of E and ρ with t and R_s, you can find the exponent γ.

(b) If $R_s \propto t^\gamma$, how does the shock temperature T_s vary with time?

(c) Suppose that the density profile in the ambient medium is $\rho \propto r^{-2}$, as would apply to a constant-velocity steady stellar wind present before the explosion. Suppose that there is a sudden explosion (e.g., a nova explosion) depositing an energy $E_0 = constant$. What will be γ for this case?

Chapter 40. Cosmic Rays and Gamma Rays

40.1 Observations of 1.809 MeV γ rays resulting from the decay of ^{26}Al indicate that the ISM of the Milky Way contains $\sim$$2.7 \pm 0.7\, M_\odot$ of ^{26}Al. The total mass of H in the ISM today is $4.9 \times 10^9\, M_\odot$ (see Table 1.2). What is the ratio of $^{26}\mathrm{Al}/^{27}\mathrm{Al}$ in the ISM today?

40.2 The 511 keV positronium annihilation line from the central regions of the Galaxy has an observed photon flux from a "disk" component $F_{511} = 7.3^{+2.6}_{-1.9} \times 10^{-4}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ (Weidenspointer et al. 2008: New Astr. Rev. 52, 454).

(a) Estimate the the total positronium formation rate $\dot{N}_{\mathrm{Ps}}$, and the positronium annihilation luminosity, assuming that all of the interstellar material is at the 8.5 kpc distance of the Galactic Center.

(b) Compare the total positronium formation rate $\dot{N}_{\mathrm{Ps}}$ with the rate of creation of positrons from decay of ^{26}Al, $\dot{N}(^{26}\mathrm{Al}) \approx 4 \times 10^{42}\,\mathrm{s}^{-1}$.

(c) If the Ps forms by radiative recombination, the radiative recombination process will be analogous to that for hydrogen. What will be the wavelength of the analogs to $\mathrm{H}\alpha$ and $\mathrm{Ly}\alpha$?

(d) The positronium recombinations will be "case A". Suppose that a fraction $f(3-2) \approx 0.2$ of the case A recombinations produces a 3$\rightarrow$2 photon. Estimate the Galactic luminosity in this line.

Chapter 41. Gravitational Collapse and Star Formation: Theory

41.1 Consider a plane-parallel slab of gas. At $t = 0$, suppose that the slab is of uniform density ρ, with half-thickness H. Let z be a coordinate perpendicular to the slab, with the center of the slab at $z = 0$. Suppose that the gas is at zero temperature, but supported (against its own self-gravity) entirely by magnetic pressure.

(a) The gravitational potential Φ satisfies Poisson's equation $\nabla^2\Phi = 4\pi G\rho$. What is the gravitational acceleration $g = -\nabla\Phi$ as a function of z?

(b) If the magnetic field strength at the slab surface is B_0, what must be the magnetic field $B(z)$ within the slab (i.e., $-H < z < H$) in order to provide the necessary support against gravity for the overall fluid (neutrals + ions)?

(c) Assume that at $t = 0$ the ionization fraction is uniform throughout the slab: $n_i = x_i\rho/m_n$, where $x_i \ll 1$ is the ionization fraction and m_n is the molecular mass of the neutrals (which are assumed to provide essentially all of the mass density ρ). If $\langle\sigma v\rangle_{\rm mt}$ is the "momentum transfer rate coefficient" for ion-neutral scattering (i.e., the force per volume exerted on the neutrals by the ions is $n_i n_n \langle\sigma v\rangle_{\rm mt}[m_n m_i/(m_n + m_i)](\vec{v}_i - \vec{v}_n)$), obtain an expression for the ambipolar diffusion drift velocity v_{in} as a function of z.

(d) Obtain an expression for the ambipolar diffusion timescale z/v_{in}. Evaluate this timescale for $m_n = 2m_{\rm H}$, $x_i = 10^{-6}$, $m_i = 9m_n$, and $\langle\sigma v\rangle_{\rm mt} = 1.9 \times 10^{-9}\,{\rm cm^3\,s^{-1}}$.

41.2 Observations of H II regions in metal-poor galaxies indicate that the primordial He abundance is $n_{\rm He}/n_{\rm H} \approx 0.082$. The WMAP 7 yr data analysis (Komatsu et al 2010, arXiv:1001.4538) finds $H_0 = 70.2\,{\rm km\,s^{-1}}$ and $\Omega_{\rm baryon} = 0.0458$, corresponding to

$$n_{\rm H} = 1.91 \times 10^{-7}(1+z)^3\,{\rm cm^{-3}} = 0.197\left(\frac{1+z}{101}\right)^3 \quad .$$

After recombination and decoupling of matter and radiation, adiabatic cooling of the baryons and residual electrons results in a gas temperature

$$T(z) \approx 180\left(\frac{1+z}{101}\right)^2 {\rm K} \qquad \text{for } z \lesssim 150 \quad .$$

Suppose that in some small region we can ignore the expansion of the universe, and the dynamics of the dark matter can be ignored (this is not actually true, but let's make these assumptions for the sake of discussion). Evaluate the Bonnor-Ebert mass $M_{\rm BE}$ as a function of redshift z for $z \lesssim 150$, assuming the validity of Eq. (41.43).

41.3 The Taurus Molecular Cloud has regions with H nucleon density $n_{\rm H} = 1 \times 10^3\,{\rm cm^{-3}}$, temperature $T = 12\,{\rm K}$. The hydrogen is almost entirely molecular. Assume the gas remains isothermal. If the magnetic field can be neglected, calculate the maximum mass of a self-gravitating non-rotating density peak in such gas.

Chapter 42. Star Formation: Observations

42.1 Star formation with a specified IMF implies steady production of massive stars which, although short-lived, emit large numbers of ionizing photons. Using the stellar models in the Starburst99 code (Leitherer et al. 1999, ApJS, 123, 3), the time-averaged emission of $h\nu > 13.6\,\mathrm{eV}$ photons from steady star formation is found to be

$$Q_0 = 1.37\times 10^{53}\left(\frac{\mathrm{SFR}}{M_\odot\,\mathrm{yr}^{-1}}\right)\mathrm{s}^{-1} \quad .$$

(e.g., Murphy et al. 2011, ApJ, 737, 67).

(a) Suppose that we observe a galaxy at a distance D, and measure an integrated Hα energy flux $F(\mathrm{H}\alpha)$. If dust is not important, and the H II regions in the galaxy have an electron temperature $10^4 T_4\,\mathrm{K}$, show that star formation rate SFR can be obtained from the observed $F(\mathrm{H}\alpha)$:

$$\frac{\mathrm{SFR}}{M_\odot\,\mathrm{yr}^{-1}} = \frac{4\pi D^2 F(\mathrm{H}\alpha)}{1.91\times 10^{41}\,\mathrm{erg\,s}^{-1}} \times T_4^{0.126+0.010\ln T_4} \quad .$$

State any important assumptions.

(b) Suppose that the thermal radio free-free emission from a galaxy at distance D, is observed to have a flux density F_ν at frequency $\nu = \nu_9\,\mathrm{GHz}$. Show that the star formation rate can be deduced from the observed F_ν using

$$\frac{\mathrm{SFR}}{M_\odot\,\mathrm{yr}^{-1}} = 5.53\times 10^{-27}\nu_9^{0.118} T_4^{-0.493} \times \frac{D^2 F_\nu}{\mathrm{erg\,s^{-1}\,Hz^{-1}}} \quad .$$

State any important assumptions.

42.2 We observe a flux $F(\mathrm{H}\alpha) = 4\times 10^{-11}\,\mathrm{erg\,cm^{-2}\,s^{-1}}$ of Hα photons from a galaxy at distance $D = 10\,\mathrm{Mpc}$. Estimate $Q_0 =$ the rate of emission of H-ionizing photons by the stars in the galaxy. State any assumptions.