Lecture notes on linear wave theory.
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1 Introduction.

To give an introduction to linear wave theory for surface waves lasting for a few hours is a nearly impossible challenge. There is no time for mathematical details, yet the theory is mathematical in its nature. The notes are probably going to contain more details than the lectures. Still they are rather sketchy. Consequently I shall have to rely on my listeners' ability to fill in the details that are left out.

Excellent books for further reading are for example the following:

- G.B.Whitham : *Linear and Nonlinear Waves.* John Wiley & Sons, 1974.
- J. Lighthill : *Waves in Fluids.* Cambridge University Press, 1978.
- G.D. Crapper : *Introduction to Water Waves.* Ellis Horwood Limited 1984.

2 Basic equations.

We start by assuming that our fluid is of homogeneous density ρ , and also ideal and incompressible. Consequently the continuity equation is simply

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

We shall also assume that vorticity has no major place in wave propagation. This, however, calls for a comment. From the theory of an ideal and homogeneous fluid we recall that the vorticity $\nabla \times \mathbf{v}$ is a property associated with the fluid elements. It is carried along by the fluid motion. This implies that if a particular fluid element had zero vorticity initially, it will always have zero vorticity. The main property of a wave is its ability to transport information, energy and momentum over considerable distances without transport of matter. Thus the velocity field associated with the wave is irrotational and given by a velocity potential, φ , which according to the equation (1) above satisfies the Laplace equation

$$\nabla^2 \varphi = 0 \tag{2}$$

The boundary conditions for (2) at the bottom is simply

$$\frac{\partial \varphi}{\partial z} = 0 \qquad \text{for} \qquad z = -h \tag{3}$$

where we use the oceanographic convention: the z-axis pointing vertically upwards with $z = 0$ at the equilibrium surface. The actual surface is located at

$$z = \eta(t, x, y)$$

and the kinematic surface condition states that a "fluid particle" at the surface at any given time is always at the surface:

$$\frac{d}{dt}\left(z - \eta(t, x, y)\right) = 0 \qquad \Longleftrightarrow \qquad \frac{d\eta}{dt} = \frac{\partial \varphi}{\partial z} \tag{4}$$

where we use the notation

$$\frac{dA}{dt} \equiv \frac{\partial A}{\partial t} + \nabla \varphi \cdot \nabla A$$

There is also a dynamical condition at the free surface. Above (i.e. for $z > \eta$) there is an atmospherical pressure p_a which is taken to be constant.

Below ($z < \eta$) the pressure must be calculated from the Bernoulli equation, which by our previous assumptions can be written

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + gz + \frac{p}{\rho} = \text{constant}$$

The constant must be equal to p_a/ρ if we allow for the fluid to be at rest and undisturbed at some distant region of the surface (show this!). Thus at the surface we have

$$p_a - p = \rho(\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + g\eta) \tag{5}$$

and this pressure difference would cause an infinite acceleration, if nonzero and with no forces to balance it. The surface tension give a balancing force. Using the Laplace formula we have the following condition for equilibrium between these forces

$$p - p_a = \varkappa T \tag{6}$$

Here T is the surface tension (assumed to be constant), and $\varkappa$ is the mean curvature of the surface given by

$$\varkappa = \nabla \cdot \mathbf{n} = \nabla \cdot \left(\frac{-\nabla \eta}{\sqrt{1 + (\nabla \eta)^2}} \right) \tag{7}$$

where $\mathbf{n}$ is the unit normal to the surface (pointing upwards). Thus on the free surface we have the two conditions

$$\frac{d\eta}{dt} = \frac{\partial \varphi}{\partial z} \quad , \quad \frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + g\eta = \frac{T}{\rho}\nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1 + (\nabla \eta)^2}} \right) \quad \text{at} \quad z = \eta \tag{8}$$

These equations are of course non-linear, and therefore rather difficult to handle. In the following we shall limit ourselves to the case of small pertubations from an equilibrium.

3 The linearized problem.

For waves of small steepnes (i.e. for $|\nabla \eta| << 1$) the nonlinear terms are small. We can therefore hope that a linearization procedure can give us a good approximation to the properties of these waves. As you will find out later in the course, this is not always true: even small nonlinear terms can (given sufficient time or fetch) produce large effects.

Nevertheless let us proceed to linearize the conditions (8) by neglecting quadratic and higher order terms in η and φ . We remark that

$$\varphi(t,x,y,\eta) = \varphi(t,x,y,0) + \frac{\partial \varphi}{\partial z}(t,x,y,0)\eta + \text{ third and higher order}$$

This implies that the terms containing φ is to be evaluated at $z = 0$ upon being linearized. The linearized version of (8) now becomes (fill in the details!)

$$\frac{\partial \eta}{\partial t} = \frac{\partial \varphi}{\partial z} \quad , \quad \frac{\partial \varphi}{\partial t} + g\eta = \frac{T}{\rho}\nabla^2 \eta \quad \text{at} \quad z = 0 \tag{9}$$

The linearized problem is then to solve (2) with the boundary conditions (3) and (9). We note that these can be formally derived as the lowest order equations in a development in powers of a small number ϵ which characterizes the smallness of perturbation of the surface (e.g. some characteristic value of $|\nabla \eta|$, which is a pure number). In the next section we derive a simple class of solutions, namely those corresponding to a "plane wave" of wave number k and an amplitude a. A convenient small number is then $\epsilon = ka$ called *the wave steepness.*

3.1 Elementary solutions and the dispersion relation.

The most natural thing to do under these circumstances is probably to look for solutions where the surface is varying harmonically in time and in the horizontal spatial coordinates i.e. like a "plane" wave

$$e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

where $\mathbf{k} = (k_x, k_y)$ is a horizontal *wave vector*, and $\mathbf{x} = (x, y)$. Thus we make the ansatz

$$\eta = Ae^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + c.c. \qquad \text{and} \qquad \varphi = B(z)e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + c.c.$$

Inserting this in Laplace equation (2) B is found to satisfy the equation

$$\frac{\partial^2 B}{\partial z^2} - k^2 B = 0$$

with elementary solutions $\sinh(kz)$, $\cosh(kz)$. Here $k = |\mathbf{k}|$ is the *wavenumber* which is related to the wavelength λ by $\lambda = 2\pi/k$. A solution satisfying the lower boundary condition is then readily found to be

$$B = C\frac{\cosh\left[k(z+h)\right]}{\cosh(kh)}$$

Inserted into the linearized surface boundary conditions (9) one obtains the equations

$$i\omega A + k\tanh(kh)C = 0$$

and

$$(g + k^2\frac{T}{\rho})A - i\omega C = 0$$

The condition for existence of a plane wave solution (i.e. a nonzero solution A and C) then becomes

$$\omega^2 = (gk + \frac{T}{\rho}k^3)\tanh(kh) \tag{10}$$

This relation between the frequency ω and the wave number k is called *the dispersion relation.* In real form the plane wave solution can now be written

$$\eta = a\cos(\mathbf{k}\cdot\mathbf{x} - \Omega(\mathbf{k})t) \tag{11}$$

and

$$\varphi = \frac{\Omega(\mathbf{k})}{k}a\frac{\cosh\left[k(z+h)\right]}{\sinh(kh)}\sin(\mathbf{k}\cdot\mathbf{x} - \Omega(\mathbf{k})t) \tag{12}$$

where $\Omega(\mathbf{k})$ is a solution of the dispersion relation (10), and a is the real *amplitude* of the wave. Before looking at more general solutions of the linearized equations we consider how the dispersion relation (10) can be simplified in some special parameter domains.

3.2 Waves of different wavelengths.

We rewrite (10) as

$$\omega^2 = gk(1 + (\frac{k}{k_0})^2)\tanh(kh)$$

where

$$k_0 = \sqrt{\frac{g\rho}{T}}$$

is a characteristic wave number (corresponding to a wavelengh of 1.71cm for pure water). The different parameter regimes of the two numbers kh and k/k_0 serves to distinguish between different wavetypes:

- $kh >> 1$ *deep water waves.*
- $kh << 1$ *shallow water waves.*
- $k/k_0 << 1$ *gravity waves.*
- $k/k_0 >> 1$ *capillary waves.*

For deep water gravity waves the dispersion relation is simplified to

$$\omega^2 = gk \tag{13}$$

and for shallow water gravity waves one have

$$\omega^2 = ghk^2 \tag{14}$$

A graph of the dispersion relation for gravity waves is shown in Figure (1). The approximations (13) and (14) are also shown.

The wave pattern (crests and troughs) moves with the phase velocity $v_{ph} = \omega/k$. For waves on deep water we have

$$v_{ph}^2 = \frac{g}{k} + \frac{gk}{k_0^2}$$

from which it is found that v_{ph} has a minimum of $\sqrt{2}(Tg/\rho)^{1/4}$ ($\simeq 23cm/s$, pure water) for $k = k_0$.

3.3 Motion of the fluid particles.

The motion of a fluid particle due to the wave is found by integrating the equation of motion

$$\frac{d\mathbf{r}}{dt} = \nabla\varphi \tag{15}$$

where $\mathbf{r}=(x(t), y(t), z(t))$ gives the position of the fluid particle, and the right hand side is evaluated at that point. This is of course a nonlinear differential equation even though we shall use the linearized solution (12) for φ. We expect the motion to consist of a periodic ocillation $\mathbf{s}(t)$ and possibly a slow translatory motion of the average position, or guiding center $\mathbf{R}(t) = (X(t), Y(t), Z(t))$. We write $\mathbf{r} = \mathbf{R} + \mathbf{s}$. To lowest significant order in the wave steepness we evaluate $\nabla\varphi$ at the guiding center and neglect the variation of $\mathbf{R}$ during a wave period. The equation for $\mathbf{s}$ now becomes (taking the x-axis parallel to $\mathbf{k}$)

$$\frac{d\mathbf{s}}{dt} = \Omega(A\cos\theta, 0, B\sin\theta)$$

where $\theta = \mathbf{k}\cdot\mathbf{R} - \Omega t$ and

$$A = \frac{a\cosh\left[k(Z+h)\right]}{\sinh(kh)} \qquad \text{and} \qquad B = \frac{a\sinh\left[k(Z+h)\right]}{\sinh(kh)}$$

which is readily integrated to give

$$\mathbf{s} = (-A\sin\theta, 0, B\cos\theta)$$

The trajectory of the ocillating movement is found by eliminating θ giving

$$\left[\frac{x-X}{A}\right]^2 + \left[\frac{z-Z}{B}\right]^2 = 1$$

which is an ellipse with half axis A (horizontal) and B (vertical). It is readily seen that $B = a$ (the amplitude) for a surface particle and zero for a bottom particle. For a wave on deep water we have the simplification

$$A = B = ae^{kZ}$$

implying that the trajectories are all circles with radius a at the surface, and decreasing exponentially downwards (see Figure (2)).

The equation for the slow motion of the guiding center is obtained by going to the next order in wave steepness. This seems a bit strange since we still use the expression (12) which is derived from linear wave theory. It is not difficult, however, to show that by taking into account the next order of approximation for φ one does not change the result for the guiding center. Developing the right hand side of (15) in powers of $\mathbf{s}$, and averaging over one wave period (keeping $\mathbf{R}$ constant), we obtain

$$\frac{d\mathbf{R}}{dt} = \langle \mathbf{s}\cdot\nabla\nabla\varphi\rangle = \frac{1}{2}\mathbf{k}\Omega a^2 \frac{\cosh\left[2k(Z+h)\right]}{\sinh^2(kh)}$$

This is a slow horizontal drift of the fluid particles in the wave direction called *Stokes drift.* The total mass transport $\mathbf{M}$ due to the wave is found by integrating the Stokes drift from the bottom to the surface

$$\mathbf{M} = \int_{-h}^{0} \rho \frac{d\mathbf{R}}{dt} dz = \frac{\rho\Omega a^2 \mathbf{k}}{2k\tanh(kh)} \tag{16}$$

For deep water we have the simplifications

$$\frac{d\mathbf{R}}{dt} = \frac{\mathbf{k}}{k}\Omega a^2 e^{2kz} \quad \text{and} \quad \mathbf{M} = \frac{\mathbf{k}}{2k}\rho\Omega a^2$$

4 The group velocity.

As long as we consider linear wave theory, solutions can be added to produce new solutions. Since integration is a linear operation the perturbation given by

$$\eta = \mathrm{Re} \int A(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} - \Omega(\mathbf{k})t)} d\mathbf{k}$$
$$\varphi = \mathrm{Re} \int \frac{\Omega(\mathbf{k})}{k} A(\mathbf{k}) \frac{\cosh\left[k(z+h)\right]}{\sinh(kh)} e^{i(\mathbf{k}\cdot\mathbf{x} - \Omega(\mathbf{k})t)} d\mathbf{k}$$

is still a solution provided $\Omega(\mathbf{k})$ satisfies the dispersion relation. Here $A(\mathbf{k})$ is an arbitrary function of $\mathbf{k}$. With the integration taken over the entire $\mathbf{k}$ -plane the above solution take the form of Fourier integrals, and we shall use the phrase Fourier component about the plane wave solution $e^{i(\mathbf{k}\cdot\mathbf{x} - \Omega(\mathbf{k})t)}$.

For a solution that is a sum or an integral over elementary solutions, the question of a wave velocity comes up since v_{ph} is a function of the wavenumber. Take the simple case with two waves of equal amplitude and slightly different wave vectors $\mathbf{k} - \mathbf{\Delta k}$ and $\mathbf{k} + \mathbf{\Delta k}$

$$\begin{aligned} \eta &= \mathrm{Re}\left[a(e^{i((\mathbf{k}-\mathbf{\Delta k})\cdot\mathbf{x} - \Omega(\mathbf{k}-\mathbf{\Delta k})t)} + e^{i(\mathbf{k}+\mathbf{\Delta k})\cdot\mathbf{x} - \Omega(\mathbf{k}+\mathbf{\Delta k})t})\right] \\ &\simeq 2a\cos\left[\mathbf{\Delta k}\cdot(\mathbf{x} - \frac{\partial\Omega}{\partial\mathbf{k}}t)\right]\cos(\mathbf{k}\cdot\mathbf{x} - \Omega(\mathbf{k})t) \end{aligned} \tag{17}$$

It is seen from the expression above that the combination of two waves of slightly different wave vectors behaves like a single wave with a wave vector being the average of the two and with a slowly varying amplitude given by

$$2a\cos\left[\mathbf{\Delta k}\cdot(\mathbf{x} - \frac{\partial\Omega}{\partial\mathbf{k}}t)\right]$$

It is seen from this expression that the amplitude is transported with the velocity

$$\mathbf{v}_g \equiv \frac{\partial\Omega}{\partial\mathbf{k}} = (\frac{\partial\Omega}{\partial k_x}, \frac{\partial\Omega}{\partial k_y}) \tag{18}$$

the socalled *group velocity.* The group velocity is generally different from the phase velocity. The exception is gravity waves on shallow water where there is a linear relation between ω and k (see equation (14)). For gravity waves on deep water it is readily found that $v_{ph} = 2v_g$. In Figure (3) we show how the phase and group velocities vary with the wave number for gravity waves. It is seen that they both decrease monotonically with wave number (or increase with wave length).

4.1 Energy and momentum.

It is intuitively evident that the physical quantities energy and momentum associated with the wave is transported in the same way as the amplitude i.e. with the group velocity. We shall now derive the expressions for the average energy and momentum in an elementary wave. For simplicity we consentrate on gravity waves. The average (available) potential energy E_p is given by

$$E_p = \left\langle \int_0^{\eta} g\rho z dz \right\rangle = \frac{g\rho}{2} \left\langle \eta^2 \right\rangle$$

The average kinetic energy E_k (energy per unit surface area) is given by

$$E_k = \left\langle \frac{\rho}{2} \int_{-h}^{\eta} (\nabla\varphi)^2 dz \right\rangle$$

where $\langle\rangle$ denotes averaging over a wave period.Since we shall limit ourselves to linear wave theory, only the quadratic part of E_k is relevant. The upper boundary of the integral can therefore be taken to be 0, as the difference is of third order in the perturbation. Inserting the expressions (11) and (12) into the relations above we get

$$E_k = \frac{\rho(a\Omega)^2}{4k\tanh(kh)} \tag{19}$$

and

$$E_p = \frac{g\rho a^2}{4}$$

Using the dispersion relation it is seen from these expressions that $E_k = E_p$, which is a general result in linear wave theory. Since the average kinetic- and potential energies are equal for a propagating wave, the total wave energy E can be written as

$$E = g\rho \left\langle \eta^2 \right\rangle = \frac{1}{2} g\rho a^2 \tag{20}$$

The *energy flux*, $\mathbf{F}$, associated with the wave ((11) and (12)) is

$$\mathbf{F} = \left\langle \int_{-h}^{0} p\mathbf{u} dz \right\rangle = \rho g a^2 \frac{\Omega \mathbf{k}}{4k^2} \left[1 + \frac{2kh}{\sinh(2kh)} \right] = E\mathbf{v}_g$$

The velocity of energy transport is therefore the group velocity.

The average momentum associated with an elementary wave is

$$\mathbf{P} = \left\langle \int_{-h}^{\eta} \rho \nabla\varphi dz \right\rangle \simeq \rho \left\langle \eta \nabla\varphi \right\rangle_{z=0} = \mathbf{k} \frac{\rho\Omega a^2}{2k\tanh(kh)} = \frac{\mathbf{k}}{\Omega} E \tag{21}$$

By comparing this with the expression (16) it is seen that the average momentum is equal to the total mass transport.

These expressions for the average energy and momentum remain valid as an approximation even if the amplitude of the wavetrain is slowly varying. If L is a characteristic length for a significant variation of the amplitude (corresponding to $1/|\Delta\mathbf{k}|$ in the example above) then (20) and (21) are correct to the order $(kL)^{-2}$. By a slow variation it is understood that the amplitude has a very small relative variation during a wave period.

Since both E and $\mathbf{P}$ are quadratic in the amplitude, the transport velocity for these quantities is also the group velocity as already anticipated.

5 The initial value problem.

The linear superposition of elementary waves can be used to solve initial value problems. In the following we consider such a problem for a gravity wave on deep water travelling along a channel (one dimensional propagation). Let the fluid be at rest initially (i.e. $\varphi = 0$ at $t = 0$) with a perturbed surface. We take the initial perturbation for η to be an impuls function. If that problem can be solved (as it was by Cauchy and Poisson in 1816) the solution for a more general initial value can be found by convolution. The initial conditions are now

$$\eta(0,x) = \delta(x) \qquad , \qquad \frac{\partial \eta}{\partial t}(0,x) = 0$$

Consider now a general solution of the linearized equations

$$\eta(t,x) = \int_{-\infty}^{\infty} R_1(k) e^{i(kx - \Omega(k)t)} dk + \int_{-\infty}^{\infty} R_2(k) e^{i(kx + \Omega(k)t)}$$

written as Fourier integrals. The first integral represents a wave moving to the right, and the second a wave moving to the left (corresponding to the two solutions $\pm\Omega(\mathbf{k})$ of the dispersion relation). From the initial conditions we obtain

$$\frac{\partial \eta}{\partial t}(0,x) = \int_{-\infty}^{\infty} -i\Omega(k)(R_1(k) - R_2(k)) e^{ikx} dk = 0$$

and

$$\eta(0,x) = \int_{-\infty}^{\infty} (R_1(k) + R_2(k)) e^{ikx} dk = \delta(x)$$

implying that

$$R_1 = R_2 = \frac{1}{4\pi}$$

Thus the solution becomes

$$\eta(t,x) = \frac{1}{\pi}\int_0^\infty \cos(kx)\cos(\Omega(k)t)dk \tag{22}$$

which is an exact, nice and compact solution. The content of it is , however, far from easy to see. Next we show how some important information can be extracted from it.

5.1 Asymptotic solution of the initial value problem.

In the above initial value problem we consentrate on the waves moving to the right. The challenge is then to find an asymptotic approximation (far from the initial impuls) to the Fourier integral representing these waves

$$\eta(t,x) = \frac{1}{4\pi}\int_{-\infty}^\infty e^{i(kx-\Omega(k)t)}dk = \frac{1}{4\pi}\int_{-\infty}^\infty e^{itw(k)}dk \tag{23}$$

where

$$w = k\frac{x}{t} - \Omega(k)t$$

The leading term in an asymptotic development of this integral for large t is known to come from a small area around the points of *stationary phase* i.e. the solution of the equation

$$\frac{dw}{dk} = 0 \qquad \Leftrightarrow \qquad \frac{d\Omega}{dk} = \frac{x}{t} \tag{24}$$

The physical interpretation of this relation is that the main contribution at time t and location x comes from the Fourier component whose group velocity is exactly right for travelling the distance x in the timespan t. Let $K(\frac{x}{t})$ be the relevant solution of (24) with respect to k. The leading term of the asymptotic expansion of the integral (23) is (using the socalled stationary phase method and observing that $w'' > 0$ for a gravity wave)

$$\eta(t,x) \simeq \frac{1}{\sqrt{2\pi t w''(K)}}\cos\left[tw(K) + \frac{\pi}{4}\right] = A\cos\theta \tag{25}$$

where A is a slowly varying amplitude and θ is the wave phase. For deep water waves (i.e. $\Omega = \sqrt{g\,|k|}$) an explicite solution $K(\frac{x}{t})$ of equation (24) can be found as

$$K = g(\frac{t}{2x})^2$$

giving (show this)

$$\theta = -\frac{gt^2}{4x} \qquad \text{and} \qquad A = \frac{t}{2}\sqrt{\frac{g}{\pi x^3}}$$

The solution (25) represent a slowly varying wavetrain that locally looks like a plane wave with a local wave number K and frequency $\Omega(K)$ where

$$\frac{\partial\theta}{\partial x} = K \qquad \text{and} \qquad \frac{\partial\theta}{\partial t} = -\Omega(K)$$

Show that K and Ω thus defined satisfies the dispersion relation. It is also straight forward to show directly that the relative variations in the quantities K, Ω and A over a wave period is small as long as $\Omega t >> 1$.

Out of all this emerges the following picture: Initially a compact region is disturbed. The Fourier spectrum of the initial disturbance is rather broadband. The waves corresponding to each Fourier component starts moving. Its part of the energy is transported with the corresponding group velocity. At first all these waves add up to some rather involved pattern. After a while, since they move with different velocities, an ordering takes place and increases with time: The longest waves in front and the shortest in the rear. In fact if after a long time one observes the train going by, the local frequency is increasing linearly with time since

$$\Omega = -\frac{\partial\theta}{\partial t} = \frac{gt}{2x} \tag{26}$$

Although we developed these results under rather special conditions (one dimensional propagation and an impulsive initial condition) they can readily be generalized. For two dimensional propagation a corresponding version of the stationary phase method can be used and the condition for stationary phase is then

$$\frac{\partial\Omega}{\partial\mathbf{k}} = \frac{\mathbf{x}}{t} \qquad \text{with solutions} \qquad \mathbf{K} = g\left(\frac{t}{2r}\right)^2\frac{\mathbf{x}}{r} \qquad \text{for deep water waves}$$

where $r = |\mathbf{x}|$. The asymptotic wave phase θ now becomes

$$\theta = -\frac{gt^2}{4r} \qquad \text{with} \qquad \mathbf{K} = \nabla\theta \qquad \text{and} \qquad \Omega = -\frac{\partial\theta}{\partial t}$$

An example of the effects discussed above is the common experience when throwing a stone into a pond. The circular symmetric wavetrain resulting from the splash have the long waves in front and the short waves in the rear. As another example consider someone who is recording the frequency of the swell arriving at a beach from a storm distant both in time and space (see Figure (4)). If the storm was of short duration compared with the transit time of the swell, and if all the incoming swell came from that source one would expect the frequency to be a linear function of time. Fitting a straight line to the measured points one can determine the distance to- and the time of occurence of the storm. Indeed such measurements have been conducted (Snodgrass et al., 1966, see Figure (5)).

5.2 The wave front.

The stationary phase method is an effective tool for extracting asymptotic information from the solution of an initial value problem. In some cases we have to modify the tools. Take for example the case of one dimensional propagation of gravity waves in a channel discussed above. The question: what does the front of the propagating disturbance look like? cannot be answered by using the standard stationary phase method. The mathematical reason for this is the fact that the stationary points $\pm K(\frac{x}{t})$ merges at the origine as $\frac{x}{t}$ tends to the maximum wavespeed $\sqrt{gh}$. This happens when $k \rightarrow 0$. Thus the secrets of the front seems to be buried in the neighbourhood of $k = 0$ in the integral solution (23). Developing w around this point to third order in k we have

$$w \simeq k(\frac{x}{t} - \sqrt{gh}) + \frac{\sqrt{gh}h^2}{6}k^3$$

Inserting the expansion into the integral (23) we have

$$\eta \simeq \frac{1}{4\pi}\int_{-\infty}^{\infty} \exp\left[ik(x - \sqrt{gh}t) + i\frac{t\sqrt{gh}h^2}{6}k^3\right] dk$$

Comparing this to the integral representation of the Airy function

$$Ai(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[i(\tau p + \frac{1}{3}p^3)\right] dp$$

we obtain the following approximation valid around the wavefront

$$\eta \simeq \frac{1}{2Ct^{1/3}} Ai(\frac{x - \sqrt{gh}t}{Ct^{1/3}}) \tag{27}$$

where

$$C = \left(\frac{h^2\sqrt{gh}}{2}\right)^{1/3}$$

This is exibited in Figur (6) at some different times.

6 Current and refraction.

It is only possible to give a sketchy introduction to this theme. Let us start with the case of a uniform (horizontal) current $\mathbf{U}$. It can be considered as a Gallilei transformation to a coordinate system moving with the velocity -$\mathbf{U}$ with respect to the fluid. Going back to the linearized equations it is readily seen that the only difference is that $\frac{\partial}{\partial t}$ is changed to $\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla$. In the dispersion relation this implies the transformation $\omega \rightarrow \omega - \mathbf{U} \cdot \mathbf{k}$. Thus the dispersion relation (10) becomes

$$(\omega - \mathbf{U} \cdot \mathbf{k})^2 = (gk + \frac{T}{\rho} k^3) \tanh(kh) \tag{28}$$

The solutions of this equation we write

$$\pm \Omega(\mathbf{k}) + \mathbf{U} \cdot \mathbf{k}$$

where $\Omega(\mathbf{k})$ is again a solution of the original dispersion relation (10). The extra term, representing a frequency shift is just the Doppler shift. The group velocity for the wave $\omega = \Omega(\mathbf{k}) + \mathbf{U} \cdot \mathbf{k}$ is seen to be

$$\frac{\partial \omega}{\partial \mathbf{k}} = \frac{\partial \Omega}{\partial \mathbf{k}} + \mathbf{U}$$

Let us next investigate the changes in the mean energy and momentum from the expressions (20) and (21). We first remark that the potential energy is unaltered. The kinetic energy is now

$$\begin{aligned} E_k &= \left\langle \frac{\rho}{2} \int_{-h}^{\eta} (\nabla \varphi + \mathbf{U})^2 dz \right\rangle - \frac{\rho}{2} \mathbf{U}^2 h \\ &\simeq \frac{\rho a^2 (\omega - \mathbf{U} \cdot \mathbf{k})^2}{4k \tanh(kh)} + \langle \rho \eta \mathbf{U} \cdot \nabla \varphi \rangle \\ &= \frac{\rho a^2 (\omega^2 - (\mathbf{U} \cdot \mathbf{k})^2)}{4k \tanh(kh)} \end{aligned}$$

which is not equal to the potential energy. Thus we have learned that the equipartition of potential and kinetic energy only works when these quantities are referred to a coordinate system at rest with respect to the fluid (the "rest frame").

The mean total energy E_u in the "lab frame" (i.e. in the coordinate system where the fluid is flowing with the uniform velocity $\mathbf{U}$) is found to be

$$E_u = \frac{\rho a^2 \omega (\omega - \mathbf{U} \cdot \mathbf{k})}{2k \tanh(kh)} = \frac{\Omega + \mathbf{U} \cdot \mathbf{k}}{\Omega} E$$

where E is the mean energy density in the rest frame (see equation (20)). This shows that the total mean energy is not invariant under a Gallilei transformation. It is also seen that the quantity $N = E/\Omega$ called *action density* is invariant since

$$N = \frac{E}{\Omega} = \frac{E_u}{\Omega + \mathbf{U} \cdot \mathbf{k}} \tag{29}$$

The averaged momentum density is seen from (21) to be invariant

$$\mathbf{P} = \mathbf{k}\frac{E}{\Omega} = \mathbf{k}N$$

6.1 Refraction

In reality a current is hardly uniform in space and time, also depth h varies with the horizontal dimensions. However, in many cases the time and length scales associated with these quantities are much larger than those of the wave. This can be expressed as

$$k >> \left|\frac{1}{h}\nabla h\right| \qquad , \qquad k >> \left|\frac{1}{U}\nabla U\right| \qquad \text{and} \qquad \omega >> \left|\frac{1}{U}\frac{\partial U}{\partial t}\right|$$

It is then rather natural to assume that locally (in time and space) the wave properties are the same as that of a corresponding plane wave under uniform conditions. Formally this can be shown to be true by an asymptotic expansion in a small parameter made from the ratio of wave and current scales (or depth scales). The lowest order result from such an exercise is the socalled geometric optics approximation or ray theory. Since the medium through which the wave is propagating is slowly varying in space (and possibly in time) the amplitude, wave number and (possibly) the frequency are varying too. Starting with a "locally plane" wave represented by

$$\eta = \mathrm{Re}(a(\varepsilon\mathbf{x},\varepsilon t)e^{i\theta})$$

where θ is the wave phase and a is a slowly varying amplitude (the slow variation is made explicite by a small parameter ε). The local frequency and wave vector are then (as earlier) defined by

$$\omega(\varepsilon\mathbf{x},\varepsilon t) = -\frac{\partial\theta}{\partial t} \qquad \text{and} \qquad \mathbf{k}(\varepsilon\mathbf{x},\varepsilon t) = \nabla\theta \tag{30}$$

and are also assumed to be slowly varying. Making a similar anzats for the potential φ and inserting this into the linearized equations one obtain to the zero order in ε just the dispersion relation (10). To the first order in ε one obtains an equation governing the variation of the amplitude which can be written (after a lot of work!)

$$\frac{\partial N}{\partial t} + \nabla \cdot (\frac{\partial \omega}{\partial \mathbf{k}} N) = 0 \tag{31}$$

where $N = E/\Omega$ is the action density and E is given in terms of the amplitude by the expression (20). This has the form of a conservation equation and tell us that action is conserved and the action density is transported with the group velocity, like some "wave fluid" density. Now the stream lines of this wave fluid are the *rays,* which can also be thought of as trajectories of wave packets. They are defined through the equation

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \omega}{\partial \mathbf{k}} \tag{32}$$

To integrate (32) we need a similar equation for $\mathbf{k}$. It follows from the relations (30) that

$$\frac{\partial \mathbf{k}}{\partial t} = \nabla \frac{\partial \theta}{\partial t} = -\nabla \omega$$

Through the dispersion relation ω is a function of $\mathbf{k}$. It may also be an explicite function of time and space through the physical parameters entering the dispersion relation, i.e. the depth h and the current velocity $\mathbf{U}$. The right hand side of the equation above can now be developed as

$$-\frac{\partial^2 \theta}{\partial x_i \partial t} = \sum_j \frac{\partial \omega}{\partial k_j} \frac{\partial k_j}{\partial x_i} + \frac{\partial \omega}{\partial x_i} = \sum_j \frac{\partial \omega}{\partial k_j} \frac{\partial k_i}{\partial x_j} + \frac{\partial \omega}{\partial x_i}$$

where the last equality follows from the fact that $\mathbf{k}$ is a gradient vector. We now get the equation

$$\frac{d\mathbf{k}}{dt} = -\frac{\partial \omega}{\partial \mathbf{x}} \tag{33}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \omega}{\partial \mathbf{k}} \cdot \nabla$. The two equations (32) and (33) can be considered to be dynamical relations for the motion of wave groups. In fact they have the canonical form of the Hamiltonian equations with ω corresponding to the Hamiltonian and $\mathbf{k}$ to the momentum variable. This correspondence is not very surprising considering the wave-particle correspondence first suggested by Louis de Broglie. The conservation equation for action (31) is the corresponding evolution equation for the amplitude. Together these three equations constitute the "geometrical optics" relations for surface gravity waves.

In the following we give some examples.

6.2 Examples.

6.2.1 Swell approaching beach.

The frequency change of swell (that we considered earlier) is a slow process. When considering the transition of swell from deep water to shallow water we may consider the frequency of the incoming waves to be constant. We shall also neglect any background current. The frequency during the transition is also constant because we have

$$\frac{d\omega}{dt} = \frac{\partial\omega}{\partial t} + \frac{d\mathbf{k}}{dt}\cdot\frac{\partial\omega}{\partial\mathbf{x}} + \frac{d\mathbf{x}}{dt}\cdot\frac{\partial\omega}{\partial\mathbf{k}} = \frac{\partial\omega}{\partial t} = 0$$

which follows from the equations (32) and (33). Consequently it is the wave vector that must change in order to satisfy the dispersion relation with a changing depth. Consider a straight beach parallel to the y-axis, with a depth that depends on the x-coordinate only. From (33) we then have

$$\frac{dk_y}{dt} = 0$$

thus k_y is a constant. Since we now have a steady state problem equation (31) can be integrated to give

$$\frac{\partial\omega}{\partial k_x}E = \frac{\omega k_x}{2k^2}(1 + \frac{2kh}{\sinh(2kh)})E = \text{ constant}$$

where the constant is equal to the component of the energy flux in the x-direction. Using the subscript 0 to denote the wave properties of the incoming wave at deep water we obtain from the dispersion relation and the equations above

$$\frac{E}{E_0} = \frac{k_x k_0^2}{k_{x0}k^2}(1 + \frac{2kh}{\sinh(2kh)})$$

and

$$\frac{k}{k_0} = \coth(kh) = \frac{\cos(\psi_0)}{\cos(\psi)}$$

where ψ is the angle between $\mathbf{k}$ and the x-axis. Explicite solutions can be found when the wave has arrived at shallow water i.e. $kh << 1$. In this approximation we have

$$\frac{E}{E_0} = \left(\frac{a}{a_0}\right)^2 = \frac{\cos(\psi_0)}{2}(k_0 h)^{-1/2}$$
$$\frac{k}{k_0} = \frac{\cos(\psi_0)}{\cos(\psi)} = (k_0 h)^{-1/2}$$
$$\frac{ak}{a_0 k_0} = \sqrt{\frac{\cos(\psi_0)}{2}}(k_0 h)^{-3/4}$$

It is seen that as $k_0 h$ decreases:

- the energy (and thus the amplitude) increases,
- the wavelength decreases (and thus the wavenumber increases),
- the wave direction turns towards normal incidence (i.e. $\psi \to \pi/2$),
- the most dramatic increase is experienced by the *wave steepnes* ak.

6.2.2 Waves against current.

Consider waves on deep water moving into a region where there is a current. We shall assume a simple geometry with the waves moving parallel to the x-axis in the positive direction and the current $-U(x)$ going in the opposite direction (the incoming waves come from $x = -\infty$ and we assume $U(-\infty) = 0$. As in the previous example we assume that the incoming waves have a constant frequency and denote the wave properties of the incoming waves (far from the current region) by subscript 0. The dispersion relation is now

$$\omega = \sqrt{gk} - kU(x) = \sqrt{gk_0} = \text{ constant}$$

which is a quadratic equation in $\sqrt{k}$ with the solutions

$$\sqrt{\frac{k}{k_0}} = \frac{1}{p}(1 \pm \sqrt{1-2p}) \qquad \text{where} \qquad p = \frac{U(x)}{v_{g0}} = 2U(x)\sqrt{\frac{k_0}{g}}$$

There is a lot to be learnt from these solutions. First the question is why there are two of them. The incoming wave must satisfy the condition that $k \to k_0$ when $x \to -\infty$ (i.e. when $p \to 0$). The solution with the minus sign therefore corresponds to the incoming wave. It is also seen that solutions only exist when $p < 1/2$, i.e. when $U(x) < v_{g0}/2$. At the level $x = x_c$ where $U(x_c) = v_{g0}/2$ the wave is reflected (or blocked). The reflected wave corresponds to the solution with the plus sign, and at the level of reflection the wavenumber is seen to be the same for the two branches and equal to $4k_0$. Thus the wavelength of the incoming wave decreases to $\lambda_0/4$. After reflection it decreases much faster. This can be seen from Figure (7).

The conclutions above are only slightly modified when the waves and the current are not colinear. In Figure (7) the waves originate from a point source outside the current area. Note that waves coming in at different angles with respect to the the current direction are reflected at almost the same level.

It is left to the reader to show that the group velocity changes sign at the level of reflection. Note that the incoming- and reflected waves both have phase velocities opposite to the current velocity. On the other hand: in a steady state the incoming wave energy must be carried away by the reflected wave. Thus we arrive at the conclution that the reflected wave must have its phase- and group velocities pointing in opposite directions. Such a "backward" wave looks quite strange. Imagine that we are looking at the surface near the reflecion level. Because our visual interpretation associate the wave speed with the motion of the wave crests i.e. with the phase velocity, it appears that two waves of different wavelength is approaching the reflection level and then suddenly disappears without trace.

The amplitude and steepness of the waves are also increasing as one approaches the reflection level. In a steady state this evolution is found using the conservation equation (21) which now implies that the energy flux is constant i.e.

$$v_g E = (\frac{1}{2}\sqrt{\frac{g}{k}} - U(x))E = v_{g0}E_0 = \frac{1}{2}\sqrt{\frac{g}{k_0}}E_0$$

which gives

$$\frac{E}{E_0} = (\sqrt{\frac{k}{k_0}} - p)^{-1}$$

This shows that the average energy density (and therfore the amplitude) increases as we approaches the level of reflection. The geometrical optics approximation is not valid at such an "internal reflection" (or more generally at a caustic). Still it is possible to find uniformly valid solutions of the linearized equations in such regions. Even if the steepness of the incoming wave (far from the reflection level) is small this may no longer be so as the wave approaches reflection. Consequently the linear approximation may become invalid.

In Figure (8) and (9) it is shown how some wave parameters are evolving according to the relations above, when the wave meet with a counter-current whose variation is taken to be Gaussian (see bottom of the figures). In Figure (8) the current maximum is higher than the critical one (i.e. $v_{g0}/2$), such that reflection occurs at a level $x = x_c$. In Figure (9) the current maximum is subcritical so that the wave is able to pass through the current region.

6.2.3 Ship waves.

It is wellknown that a ship moving at a constant velocity generate a V-shaped wave pattern that is stationary in the reference frame moving with the ship. It is somewhat less wellknown that the half-angle of the V is 19.5° and that the waves at the edge of the wave pattern (caustic) is moving at an angle of 35.3° to that of the ship, regardless of the ship form and speed. This was explained by Lord Kelvin more than a hundred years ago.

The explanation starts from two observations

- In the reference frame of the ship the wave pattern is stationary i.e. the frequency is zero.
- The ship is the wave source, therefore (still in the ship-frame) the wave energy at a point of the pattern must have travelled along a straight line from the ship.

In the ship-frame the water has a uniform velocity $-\mathbf{U}$ and the condition of zero frequency (assuming deep water) becomes

$$\omega = \sqrt{gk} - \mathbf{k} \cdot \mathbf{U} = \sqrt{gk} - Uk\cos\psi = 0 \tag{34}$$

The group velocity is then (with the x-axis parallel to the ship track, see Figure (10))

$$\mathbf{v}_g = \frac{1}{2}\sqrt{\frac{g}{k}}\frac{\mathbf{k}}{k} - \mathbf{U} = (-\frac{1}{2}\sqrt{\frac{g}{k}}\cos\psi + U, \frac{1}{2}\sqrt{\frac{g}{k}}\sin\psi)$$

Simple trigonometry, and using equation (34) give us

$$\tan\theta = \frac{v_g \sin\psi}{U - v_g\cos\psi} = \frac{\sin\psi\cos\psi}{1+\sin^2\psi} = \frac{\tan\psi}{1+2\tan^2\psi} \tag{35}$$

where the angles θ and ψ are explained in Figure (10a). Solving for $\tan\psi$ we obtain the two real solutions

$$\tan\psi = \frac{1}{4\tan\theta}(1 \pm \sqrt{1 - 8\tan^2\theta})$$

corresponding to the two wave-systems in the "Kelvin" wake (see Figure (10b)). These different waves meet in cusps along the perifery of the wave pattern. As seen from the formula above there are no solutions for $|\tan\theta| > 2\sqrt{2}$ i.e. $|\theta| > 19.3°$. Further, the angle that the cuspwaves at the perifery make with the ship track, is given by $\tan^{-1}(\frac{1}{\sqrt{2}}) \simeq 35.3°$. Figure (11) shows a picture of a ship wake pattern, where only the transversal wave system is in evidence. Which parts of the pattern are amplified and which are suppressed, depend on the form and speed of the ship.