EQUATION OF STATE

An elementary cell in phase space has a volume

$$\Delta x \, \Delta y \, \Delta z \, \Delta p_x \, \Delta p_y \, \Delta p_z . \tag{1}$$

According to quantum mechanics an elementary unit of phase space is h^3, where $h = 6.63 \times 10^{-27}$ erg s is Planck's constant. Therefore, the differential number of particles in a given differential volume in phase space is:

$$n\,(p)\,dp = fg\frac{4\pi p^2}{h^3}dp = \frac{g}{e^{(E-\mu)/kT} \pm 1}\frac{4\pi p^2}{h^3}dp, \tag{2}$$

where

$$f = \frac{1}{e^{(E-\mu)/kT} \pm 1},$$

is the phase space density, with a "+" sign for fermions and a "$-$" sign for bosons, and k is Boltzmann's constant, 1.38×10^{-16} erg K^{-1}. The corresponding distributions are called Fermi-Dirac and Bose-Einstein, respectively. Particles with a spin of $1/2$ are called fermions, while those with a spin $0, 1, 2...$ are called bosons. Electrons and protons are fermions, photons are bosons, while larger nuclei or atoms may be either fermions or bosons, depending on the total spin of such a composite particle. E is the particle energy, T is temperature, μ is chemical potential, and g is the particle degeneracy. The number density of particles with all momenta, contained within a unit volume, is

$$n = \int_0^\infty n\,(p)\,dp, \qquad \rho = nm \tag{3}$$

where ρ is the mass density of gas. Note, that if we know density and temperature, then we can calculate chemical potential, provided we know how particle energy depends on its momentum, i.e. the function $E(p)$.

In the above equations, g, k, T, μ, π, h are all constant, and the energy E depends on the momentum p only. As our particles may be relativistic as well as non-relativistic, we have to use a general formula for the relation between E and p. We have

$$E \equiv E_{total} = E_0 + E_k, \tag{4}$$

where the rest mass $E_0 = mc^2$, $c = 2.99792458 \times 10^{10}$ cm s^{-1} is the speed of light, and E_k is the kinetic energy of a particle. For a particle moving with arbitrary velocity there is a special relativistic relation:

$$E_{total}^2 = \left(mc^2\right)^2 + (pc)^2 . \tag{5}$$

The two simple asymptotic limits are:

$$\begin{aligned} E_k &\approx \frac{p^2}{2m}, &&\text{for } p \ll mc, &&(\text{ non-relativistic limit}), \\ E_k &\approx pc, &&\text{for } p \gg mc, &&(\text{ ultra-relativistic limit}). \end{aligned} \tag{6}$$

For the velocity, we have two simple asymptotic limits:

$$\begin{aligned} v &\approx \frac{p}{m}, &&\text{for } p \ll mc, &&(\text{ non-relativistic limit}), \\ v &\approx c, &&\text{for } p \gg mc, &&(\text{ ultra-relativistic limit}), \end{aligned} \tag{7}$$

The kinetic energy of all particles in a unit volume may be calculated as

$$U = \int_0^\infty E_k\,(p)\,n\,(p)\,dp, \qquad [\,\mathrm{erg\ cm}^{-3}]. \tag{8}$$

For a given absolute value of velocity and momentum we may calculate the dot-product of the two isotropic vectors, and average it over all angles:

$$\mathbf{vp} = vp = v_x p_x + v_y p_y + v_z p_z. \tag{9}$$

$$\langle v_x p_x \rangle = \langle v_y p_y \rangle = \langle v_z p_z \rangle = \frac{1}{3} vp. \tag{10}$$

Pressure is defined as a flux of momentum across a unit surface, integrated over all particles. For an isotropic gas we may select the unit surface to be perpendicular to the "x" axis, and we may calculate pressure as

$$P = \int_0^\infty < v_x p_x > n\,(p)\,dp = \frac{1}{3}\int_0^\infty v\,(p)\,p\ n\,(p)\,dp, \tag{11}$$

where as before the averaging is done over particles moving in all directions.

$$P \approx \frac{1}{3}\int_0^\infty \frac{p^2}{m} n\,(p)\,dp, \qquad U \approx \int_0^\infty \frac{p^2}{2m} n\,(p)\,dp \approx \frac{3}{2}P, \qquad (\mathrm{non-relativistic}). \tag{12}$$

$$P \approx \frac{1}{3}\int_0^\infty pc\ n\,(p)\,dp, \qquad U \approx \int_0^\infty pc\ n\,(p)\,dp \approx 3P, \qquad (\mathrm{ultra-relativistic}). \tag{13}$$

Notice, that the relations $U \approx 3P/2$ and $U \approx 3P$ for the two limits are very general, and do not depend on the details of the distribution function $n(p)$.

Maxwell-Boltzmann distribution

This is the case when

$$mc^2 \gg E - \mu \gg kT, \tag{14}$$

i.e. the particles are non-relativistic, and the exponential term is very large, $f \ll 1$, i.e. there are very few particles per elementary cell in phase space. It is clear that the ± 1 term is negligible, and therefore the distribution will be the same for fermions and for bosons. We then have

$$n(p) = \left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT}\right] e^{-p^2/2mkT} p^2, \tag{15}$$

$$n = \left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT}\right] \int_0^\infty e^{-p^2/2mkT} p^2 dp = \tag{16}$$

$$e^{\left(\mu - mc^2\right)/kT} Z_1 =$$

$$\left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT}\right] (2mkT)^{1.5} \frac{1}{2} \int_0^\infty e^{-x} x^{0.5} dx =$$

$$\left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT}\right] (2mkT)^{1.5} \frac{\pi^{0.5}}{4} = g \left(\frac{2\pi mkT}{h^2}\right)^{3/2} e^{\left(\mu - mc^2\right)/kT},$$

$$P = \frac{1}{3} \left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT}\right] \int_0^\infty \frac{p^2}{m} e^{-p^2/2mkT} p^2 dp = \tag{17}$$

$$\left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT}\right] (2mkT)^{2.5} \frac{1}{3m} \frac{1}{2} \int_0^\infty e^{-x} x^{1.5} dx =$$

$$\left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT}\right] (2mkT)^{2.5} \frac{1}{3m} \frac{1.5\pi^{0.5}}{4} = nkT,$$

$$U = 3/2nkT, \tag{18}$$

where we substituted $x = p^2/2mkT$.

A powerful way of deriving all thermodynamic functions is through the partition function, Z_N, for N particles. For MB particles, this is related to the single-particle partition function, Z_1 (shown above), by $Z_N = Z_1^N/N!$, where the $N!$ enforces proper Boltzmann counting. The Gibbs free energy is $N\mu$ and is equal to $-kT \ln Z_N$, which gives us μ by another route. Note that the Helmholtz free energy, F, is $U - TS = G - PV$, $P = -\left(\frac{\partial F}{\partial V}\right)_T$, $S = -\left(\frac{\partial F}{\partial T}\right)_V$, and $U = kT^2 \frac{\partial \ln(Z_N)}{\partial T} = 3/2NkT$.

In practice the Maxwell distribution is used for atoms, ions and electrons with a relatively low density. The relation between the number density of particles $n[cm^{-3}]$ and physical density $\rho[\mathrm{g\ cm}^{-3}]$ depends on chemical composition. It is customary to use X for the fractional abundance of hydrogen (by mass fraction) , Y for the abundance of helium, and Z for the combined abundance of all heavier elements. Frequently, all those heavier elements are described in astrophysical literature as "metals," even though the most common among them are carbon, nitrogen and oxygen. Of course, we have $X + Y + Z = 1$. A typical chemical composition of an interstellar medium, or the main sequence stars is: $X = 0.70$, $Y = 0.28$, $Z = 0.02$, i.e. there is mostly hydrogen, and only very small fraction of heavy elements. These abundances are by mass fraction, so the fractional **number** of heavy atoms is approximately 0.002, or even less.

In most stellar applications it is safe to assume that all atoms are fully ionized. The only exceptions are generally stellar atmospheres and sub-atmospheric layers. A convenient unit of mass is $H = 1.67 \times 10^{-24}\,\mathrm{g}$, i.e. the mass of a single hydrogen atom (also written m_p). H is approximately equal to the reciprocal of Avogadro's number, N_A, which is exactly equal to the reciprocal of the AMU. The mass of a helium atom is $4H$, and the mass of an element with a nuclear charge Z^* is approximately $2Z^*$. An average charge of all heavier elements is approximately $< Z^* > \approx 8$. In one gram of matter there are XN_A hydrogen nuclei, $YN_A/4$ helium nuclei, and approximately $ZN_A/(2 < Z^* >)$ heavier nuclei. Therefore, the number of nuclei per gram of matter, i.e. the number density of ions is

$$n_i = \rho N_A \left[X + \frac{Y}{4} + \frac{Z}{2 < Z^* >} \right] \approx \rho N_A \left(X + \frac{Y}{4} + \frac{Z}{16} \right). \tag{19}$$

Fully ionized hydrogen provides 1 electron per nucleus, i.e. 1 electron per nucleon. Helium provides 2 electrons per nucleus, i.e. 1 electron per 2 nuclei. A heavier element with a charge Z^* provides Z^* electrons per nucleon, which in a typical case has $A \approx 2Z^*$ nucleons, i.e. we get 1 electron per 2 nucleons, just like for helium. The number density of electrons may be calculated as

$$n_e = \rho N_A \left(X + \frac{Y}{2} + \frac{Z}{2} \right) = \rho N_A \frac{1+X}{2}. \tag{20}$$

The number density of all particles, if completely ionized, is

$$n = n_i + n_e \approx \rho N_A \left(2X + 3/4Y + 0.5Z \right). \tag{21}$$

It is customary to define mean molecular weight, μ, mean molecular weight per ion, μ_i, and mean molecular weight per electron, μ_e, as

$$\mu \equiv \frac{\rho N_A}{n} \approx \frac{1}{2X + 3/4Y + 1/2Z}, \tag{22a}$$

$$\mu_i \equiv \frac{\rho N_A}{n_i} = \frac{1}{X + Y/4 + Z/16}, \tag{22b}$$

$$\mu_e \equiv \frac{\rho N_A}{n_e} = \frac{1}{Y_e} = \frac{2}{1+X}. \tag{22c}$$

With the new definitions of mean molecular weights (not to be confused with a chemical potential, which is also written as μ) we may write the equation of state for an ideal gas as

$$P_e = \frac{kN_A}{\mu_e} \rho T, \tag{23a}$$

$$P_i = \frac{kN_A}{\mu_i} \rho T, \tag{23b}$$

$$P_g = P_e + P_i = \frac{kN_A}{\mu} \rho T = \frac{R}{\mu} \rho T, \tag{23c}$$

where P_e, P_i, and P_g are the electron pressure, the ion pressure, and the gas pressure, respectively. Note that $\frac{k}{\mu H} = \frac{kN_A}{\mu}$ and that $kN_A = R$, the gas constant per mole ($= 8.314 \times 10^7$ ergs g^{-1} K^{-1}). We also have the corresponding kinetic energy densities given as

$$U_e = 3/2 P_e, \qquad U_i = 3/2 P_i, \qquad U = 3/2 P. \tag{24}$$

Note that the coefficient for an ideal gas is in general $\frac{d}{X}$, where d is the number of spatial dimensions (3) and X is either 2 (NR) or 1 (Rel.). The latter comes from power in the E_p versus p relation.

Planck distribution

This is the distribution of photons under LTE (local thermodynamic equilibrium) conditions. It is customary to characterize photons with the frequency of their oscillations ν. Their wavelength is given as $\lambda = c/\nu$, energy as $E = h\nu$, and momentum as $p = h\nu/c$. Photons always move with the speed of light, have zero rest mass, and are their own antiparticles. Therefore, under LTE conditions their chemical potential is zero. Their spin is 1, so they are bosons. A photon has two quantum states with the same energy; they correspond to two different polarization states. Therefore $g = 2$ for photons, and we have

$$n\left(p\right) dp = \frac{8\pi}{h^3} \frac{p^2}{e^{pc/kT} - 1} dp, \tag{25a}$$

or, in a more customary form

$$n_\nu d\nu = \frac{8\pi}{c^3} \frac{\nu^2}{e^{h\nu/kT} - 1} d\nu\,. \tag{25b}$$

The total energy associated with radiation is

$$U_r = \int_0^\infty n_\nu h\nu\; d\nu = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu = \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}, \tag{26a}$$

where we substituted $x = h\nu/kT$. The last integral has a value of $\frac{\pi^4}{15} \approx 6.088$, and we may write

$$U_r = aT^4, \qquad \text{where} \quad a = \frac{8\pi^5}{15} \frac{k^4}{h^3 c^3} \approx 7.565 \times 10^{-15}\, \text{erg cm}^{-3}\, \text{K}^{-4}. \tag{26b}$$

Radiation pressure is given as

$$P_r = \frac{1}{3} U_r = \frac{a}{3} T^4, \tag{27}$$

because photons are ultra-relativistic.

It is customary to describe radiation with its intensity $B_\nu(T)$ $(= I_\mu(T))$ and

$$B\left(T\right) = \int_0^\infty B_\nu \left(T\right) d\nu. \tag{28}$$

In LTE radiation intensity is the same in all directions, over the whole solid angle $\Omega = 4\pi$. The radiation energy density may be calculated as

$$U_r = \frac{1}{c} \int B\left(T\right) d\Omega = \frac{4\pi}{c} B\left(T\right), \tag{29}$$

where the integration covers all directions; in our case radiation is isotropic, i.e. $B(T)$ is the same in all directions. We may combine equations (st.26b) and (st.29) to obtain

$$B\left(T\right) = \frac{c}{4\pi} U_r = \frac{ac}{4\pi} T^4, \tag{30}$$

and also

$$B_\nu\left(T\right) d\nu = \frac{c}{4\pi} U_\nu d\nu = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu. \tag{31}$$

$B_\nu(T)$ is called the **Planck function**. It gives the intensity of radiation of a black body at a given temperature T.

Let us calculate the flux of radiation emitted from a surface of a black body which has a temperature T. The intensity of radiation integrated over all frequencies is given as $B(T)$ and it is the same in any direction pointing out from the surface. Let us introduce angle θ, between the normal to the surface and the direction of a light ray, and an azimuthal angle ϕ. An element of a solid angle is given as $d\Omega = \sin\theta d\phi d\theta$. The flux of black body radiation comming in the direction normal to the surface may be calculated integrating over all angles the component of $B(T)$ orthogonal to the surface, i.e. $B(T)\cos\theta$. We have:

$$F_{BB} = \int_0^{\pi/2} \left[\int_0^{2\pi} B\left(T\right) \cos\theta \sin\theta d\phi \right] d\theta = 2\pi B\left(T\right) \int_0^{\pi/2} \cos\theta \sin\theta d\theta = \pi B\left(T\right). \tag{32}$$

Combining the last equation with (st.30) we obtain the flux of radiation from a unit surface of a black body

$$F_{_{BB}} = \sigma T^4, \qquad \sigma = \frac{ac}{4} = 5.67 \times 10^{-5} \, \mathrm{erg \ cm^{-2} \, s^{-1} \, K^{-4}}, \tag{33}$$

where σ is the Stefan-Boltzmann constant.

If stellar luminosity is L, and its radius is R, then its surface radiates at a rate of $F = L/4\pi R^2$. The effective temperature is defined with a relation:

$$\sigma T_{eff}^4 \equiv \frac{L}{4\pi R^2}. \tag{34}$$

If we have a mixture of gas and radiation then total pressure and total kinetic energy density are given as

$$P = P_g + P_r = \frac{kN_A}{\mu}\rho T + \frac{a}{3}T^4, \tag{35a}$$

$$U = U_g + U_r = 3/2\frac{kN_A}{\mu}\rho T + aT^4, \tag{35b}$$

It is useful to define the ratio of gas pressure to total pressure

$$\beta \equiv P_g/P, \qquad 1-\beta = \frac{P_r}{P}, \qquad\qquad 0 < \beta < 1. \tag{36}$$

At a low temperature gas pressure dominates, while at a high temperature radiation pressure dominates. The two contributions are equal, i.e. we have $P_g = P_r$ when

$$\frac{kN_A}{\mu}\rho T = \frac{a}{3}T^4. \tag{37}$$

For a standard chemical composition, i.e. for $X = 0.70$, $Y = 0.28$, $Z = 0.02$, the mean molecular weight is $\mu^{-1} = 1.62$, and the gas pressure constant is $kN_A/\mu = 5.1 \times 10^7 \, \mathrm{erg \ g^{-1} \, K^{-1}}$. The density where gas pressure equals radiation pressure is

$$\rho \approx 5 \times 10^{-23} \ T^3 \left[\mathrm{g \ cm^{-3} \, K^{-3}}\right] = \left(\frac{T}{3 \times 10^7 \, \mathrm{K}}\right)^3 \left[\mathrm{g \ cm^{-3}}\right]. \tag{38}$$

Fermi-Dirac distribution

Now, we shall consider a distribution of electrons, i.e. spin 1/2 particles, when density is high. An electron may be in two spin states: $+1/2$ and $-1/2$, so the number of different quantum states per unit cell in a phase space is $g = 2$, and the average number of electrons per unit cell and spin is

$$f = \frac{1}{e^{(E-\mu)/kT} + 1}, \tag{39a}$$

The number density of electrons is

$$n_e = \int_0^\infty n_e(p)\, dp = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{e^{(E-\mu)/kT} + 1}, \tag{39b}$$

with electron energy E a function of its momentum, p. If $kT \ll \mu$, the gas is considered **degenerate** and for $E \ll \mu$ the occupation fraction (f) is near one. We define the Fermi energy, E_F, using $\mu = mc^2 + E_F$. For the completely degenerate situation, we have

$$n_e = \frac{8\pi}{h^3} \int_0^{p_F} p^2 dp = \frac{8\pi}{3} \left(\frac{p_F}{h}\right)^3, \tag{40}$$

where p_F is the Fermi momentum. This formula gives $p_F = \left(3\pi^2 \hbar^3 n_e\right)^{1/3}$, where $\hbar$ is $h/2\pi$. It is correct for all v/c at zero kT. The general relation between E_F and p_F may be written as

$$\left(\frac{E_F}{mc^2} + 1\right)^2 = \left(\frac{p_F}{mc}\right)^2 + 1, \tag{41}$$

which yields $E_F = p_F^2/2m_e$ or cp_F in the non-relativistic or relativistic limit. m_e is the mass of the electron, 0.911×10^{-28} g. A convenient gauge of degeneracy is the quantity

$$\frac{p_F^2}{2mkT},$$

which is approximately equal to the square of the ratio of the mean thermal DeBroglie wavelength and the mean interparticle spacing ($1/n_e^{1/3}$). Therefore, when the DeBroglie wavelengths of the electrons overlap, the gas becomes degenerate.

We may express pressure and kinetic energy density in terms of Fermi momentum as

$$P_e = \frac{8\pi}{3mh^3} \int_0^{p_F} \frac{p^4\, dp}{\left[1 + \left(\frac{p}{mc}\right)^2\right]^{1/2}}, \tag{42a}$$

$$U_e = \frac{8\pi mc^2}{h^3} \int_0^{p_F} \left[\left(1 + \left(\frac{p}{mc}\right)^2\right)^{1/2} - 1\right] p^2 dp, \tag{42b}$$

In the **degenerate, non-relativistic** limit:

$$P_e = 2/5 E_F n_e = 0.991 \times 10^{13} \left(\rho/\mu_e\right)^{5/3}, \tag{43a}$$

$$U_e = 3/5 E_F n_e = \frac{3}{2} P_e. \tag{43b}$$

The relation between P_e and U_e is just that expected for any non-relativistic gas.

In the **ultra-relativistic degenerate** limit:

$$P_e = 1/4 E_{_F} n_e = 1.231 \times 10^{15} \, (\rho/\mu_e)^{4/3} \, , \tag{44a}$$

$$U_e = 3/4 E_{_F} n_e = 3 P_e. \tag{44b}$$

Again, the relation between P_e and U_e is that expected for any ultra-relativistic gas.

The transition between the non-relativistic and the relativistic regimes is near $\rho \approx 10^6$ g cm^{-3}. In the transition region, one may use the following approximate formula, which is accurate to better than about 2%:

$$P_{e,nr} = K_1 \rho^{5/3}, \qquad K_1 = 0.991 \times 10^{13} \mu_e^{-5/3}, \tag{45a}$$

$$P_{e,r} = K_2 \rho^{4/3}, \qquad K_2 = 1.231 \times 10^{15} \mu_e^{-4/3}, \tag{45b}$$

$$P_{e,d} = \left[P_{e,nr}^{-2} + P_{e,r}^{-2} \right]^{-1/2} , \tag{45c}$$

where $P_{e,d}$ stands for pressure of degenerate electron gas. The interpolation formula selects the smaller of the two limiting cases.

In the non-degenerate and non-relativistic limit, i.e. for $mc^2 \gg E - \mu \gg kT$ the integrals can be evaluated as in the case of Maxwell distribution function, and we obtain

$$P_{e,nd} = \frac{k N_A}{\mu_e} \rho T, \qquad U_{e,nd} = \frac{3}{2} P_{e,nd}, \tag{46}$$

where the subscript "e,nd" stands for **non-degenerate electron gas**.

In the non-relativistic, partly-degenerate region, i.e. when $E - \mu \approx kT$, there is no exact analytical formula, but there is an analytical interpolation formula that is accurate to about 2%. The pressure of a partly-degenerate electron gas P_e, may be calculated as

$$P_e = \left[P_{e,nd}^2 + P_{e,d}^2 \right]^{1/2} . \tag{47}$$

The last formula selects the larger of the two limiting case formulae.

Let us now find the region in the density - temperature plane where there is a transition from non-degenerate to degenerate electron gas. This may be estimated setting $P_{e,nd} = P_{e,d} \approx P_{e,nr}$, which gives

$$\frac{k N_A}{\mu_e} \rho T = K_1 \rho^{5/3} \qquad \Longleftrightarrow \qquad T \approx 10^5 (\rho / 1 \text{ g cm}^{-3})^{2/3} K. \tag{48}$$

Electron-positron pairs

At very high temperatureis, when $kT > mc^2$, electrons become relativistic. It becomes energetically possible to form electron - positron pairs, i.e. to create new particles. Now we have to write down the distribution functions for both types of electrons. As one is antiparticle of another their chemical potential differ by two rest mass energies, i.e. we have

$$\mu^- = \mu_0 + mc^2, \qquad \mu^+ = -\mu_0 - mc^2. \tag{49}$$

Note that the sum of the chemical potentials is zero, the same as a thermal photon's.
The average number of particles per unit cell is

$$n_{av}^- = \frac{2}{e^{(E-\mu^-)/kT}+1}, \qquad n_{av}^+ = \frac{2}{e^{(E-\mu^+)/kT}+1}. \tag{50}$$

The number density per unit volume is given as

$$n_e^- = \int_0^\infty n_e^-(p)\,dp = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2 dp}{e^{(E-\mu^-)/kT}+1}, \tag{51a}$$

$$n_e^+ = \int_0^\infty n_e^+(p)\,dp = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2 dp}{e^{(E-\mu^+)/kT}+1}, \tag{51b}$$

where superscripts "-" and "+" refer to electrons and positrons, respectively.

There is a constraint on the total number of particles, as the difference between the number density of electrons and positrons must be equal to the number of electrons that came from ionizing all the ions, i.e.

$$n_e = n_e^- - n_e^+ = \frac{\rho N_A}{\mu_e}, \qquad \mu_e = \frac{2}{1+X}, \tag{52}$$

where X is hydrogen abundance by mass fraction.

Given the distribution functions we may calculate electron pressure end electron energy density. In the latter we shall include not only the kinetic energy of all electrons and positrons, but also the annihilation energy of all electron - positron pairs, but not the rest mass energy of the "original" electrons. We may write

$$P_e = P_{e^-} + P_{e^+} = \frac{1}{3}\int_0^\infty v(p)\,p\,n_e^-(p)\,dp + \frac{1}{3}\int_0^\infty v(p)\,p\,n_e^+(p)\,dp, \tag{53a}$$

$$U_e = U_{e^-} + U_{e^+} + 2n_e^+ mc^2 = \tag{53b}$$

$$\int_0^\infty E_k(p)\,n_e^-(p)\,dp + \int_0^\infty E_k(p)\,n_e^+(p)\,dp + 2n_e^+ mc^2.$$

As usual, the integrals are too complicated to evaluate analytically, but there are simple limiting cases.

First, we shall consider a case of a relatively low temperature and low density: $mc^2 \gg E - \mu \gg kT$, just like the Maxwell distribution, but we shall look for a small correction in order to estimate the number of pairs. We have, under this approximation:

$$e^{(E-\mu)/kT} \gg 1, \qquad e^{\left(E-\mu+2mc^2\right)/kT} \gg 1, \qquad E \approx mc^2 + \frac{p^2}{2m}, \tag{54}$$

and therefore the number density of electrons and positrons may be calculated as

$$n_e^- = e^{(\mu_0/kT)} \frac{8\pi}{h^3} \int_0^\infty p^2 dp e^{-E_k/kT}, \tag{55a}$$

$$n_e^+ = e^{-(\mu_0/kT + 2m_e c^2/kT)} \frac{8\pi}{h^3} \int_0^\infty p^2 dp e^{-E_k/kT}. \tag{55b}$$

Taking the product of n_e^- and n_e^+, $e^{-\mu_0/kT}$ cancels and we obtain

$$n_e^- n_e^+ = 4 \left(\frac{2\pi m_e kT}{h^2} \right)^3 e^{-\frac{2m_e c^2}{kT}}.$$

Assuming that $n_e^- \approx n_e$, we have a formula for n_e^+. This is the associated Saha equation.

It is obvious how to calculate small corrections to electron pressure and electron energy density. Notice, that the corrections to energy density are relatively larger than the corrections to pressure because the rest mass energy of a small number of pairs is much larger than their thermal energy.

Another simple case is when $kT \gg mc^2$, and therefore most electrons and positrons are ultra-relativistic, their velocities are approximately equal c, and their energies $E \approx pc \gg mc^2 \approx \mu$. In this limit we have

$$n_e^- \approx n_e^+ \approx \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{e^{pc/kT} + 1} \gg n_e, \tag{56}$$

i.e. there are many more pairs than original electrons. The energy density due to all these pairs may be calculated as

$$U_e \approx \frac{16\pi}{h^3} \int_0^\infty \frac{pc\; p^2 dp}{e^{pc/kT} + 1} = 16\pi \frac{(kT)^4}{(hc)^3} \int_0^\infty \frac{x^3 dx}{e^x + 1} = \tag{57}$$

$$\frac{14\pi^5}{15} \frac{(kT)^4}{(hc)^3} = \frac{7}{4} U_r,$$

where we substituted $x = pc/kT$, the last integral has a value $7\pi^4/120$, and U_r is the radiation energy density. As the pairs are ultra-relativistic in this limit, their thermodynamic properties are very much like those of radiation. In particular, in this limit $P_e = U_e/3 = 7P_r/4$.