Schönberg-Chandrasekhar Limit

After core hydrogen burning, the **quasi-isothermal helium core** grows by the accumulation of helium ashes from the hydrogen burning shell surrounding it. However, the core can maintain itself in hydrostatic equilbrium only if the mass of the helium core is below the so-called **"Schönberg-Chandrasekhar"** limit. If it exceeds this limit, the core starts to contract on **Kelvin-Helmholtz** timescales, inaugurating the next phase of stellar evolution. We can derive the SC critical mass by using the Virial theorem with the core boundary pressure retained and assuming an ideal gas, finding the maximum surface pressure the core can sustain (calculated in terms of core quantities), calculating the surface pressure due to the weight of the overlying envelope, and then setting these two pressures equal. A rearrangement of the resulting equation yields the ratio of the critical core mass to the total stellar mass (as long as this ratio is small) in terms of the mean molecular weights in the envelope and core. The derivation proceeds as follows:
We state the **Virial theorem**:

$$4\pi R_c^3 P_c - 2U_c = \Omega_c\,, \tag{sc.1}$$

where $U_c = \frac{3}{2}\left(\frac{M_c}{\mu_c m_p}\right) kT_c$ and the subscript c stands for "core." We will assume that $\Omega_c \sim \frac{3GM_c^2}{3R_c}$. Then, we solve for P_c, and take dP_c/dM_c, and set it to zero. The result, written for R_c, is

$$R_c = \left(\frac{2\mu_c m_p G M_c}{5kT_c}\right)\,.$$

Plugging this radius into eq. (sc.1) and solving for P_c, we have

$$P_c(crit) = \frac{375}{64\pi G^3 M_c^2}\left(\frac{kT_c}{\mu_c m_p}\right)^4\,.$$

The pressure at the surface of the core due to the envelope weight is

$$P_{c,env} = -\int_{\mathrm{M_c}}^{M} \frac{GM}{4\pi r^4} dM \sim \frac{G\left(M^2 - M_c^2\right)}{8\pi < r^4 >}\,. \tag{sc.2}$$

We assume that $< r^4 > \sim R^4/2$, where R is the outer radius of the star and M is the total mass of the star. The pressure on the core due to the envelope, $P_{c,env}$, can be written in terms of the mass density ($\rho_{c,env}$), temperature (T_c), and envelope mean molecular mass (μ_{env}), and if we assume that $\rho_{c,env} \sim \frac{3M}{4\pi R^3}$, we find (solving for R)

$$R \sim \frac{\mu_{env} m_p G M}{3kT_c} \,.$$

Plugging this into the eq. (sc.2) for the pressure, we find

$$P_c(crit) = \frac{81}{4\pi G^3 M^2} \left(\frac{kT_c}{\mu_{env} m_p} \right)^4 \,.$$

Setting the two pressures equal (note that T_c is in both expressions), we find

$$\frac{M_c}{M} \sim 0.54 \left(\frac{\mu_{env}}{\mu_c} \right)^2 \,.$$

Doing this a bit more rigorously yields a coefficient of **0.37**, not much different. If $\mu_{env} \sim 0.63$ and $\mu_c \sim 1.34$, we find that $\frac{M_c}{M} \sim 0.08$ ($\sim$8%). This is the Schönberg-Chandrasekhar ratio. Note that the subsequent KH collapse of the core leads to a **self-bound object** with central pressures much larger than the pressure in the hydrogen-burning shell. As we will state again, "once an isothermal core becomes self-gravitating, it remains self-gravitating forever, even if a new nuclear fuel ignites in the core, and the core is no longer isothermal."