Homologous Stars: Simple Scaling Relations

Converting the equations of stellar structure from differential to difference equations, effectively doing dimensional analysis and assuming self-similar solutions, we can extract simple general relations between luminosity, mass, radius, and mean molecular weight for various assumptions concerning the EOS and opacity laws. These relations contain the essence of stellar structure.
The equation of hydrostatic equilibrium and $\rho \sim M/R^3$ yield

$$P \sim \frac{GM^2}{R^4}$$

and therefore for an ideal gas ($T \sim \frac{\mu P}{\rho}$)

$$T \sim \frac{\mu GM}{R}\,.$$

If we assume the star is radiative, the luminosity, L, scales as

$$L \sim \frac{R^2}{\kappa\rho}\left(\frac{T^4}{R}\right)$$

$$\sim R^2 \frac{R^3}{M}\left(\frac{R^3}{M}\right)^n \left(\frac{GM\mu}{R}\right)^\alpha \left(\frac{\mu GM}{R}\right)^4 /R$$

$$\sim \mu^{\alpha+4} R^{3n-\alpha} M^{\alpha+3-n}$$

where $\kappa \sim \rho^n T^{-\alpha}$.

For Kramers opacity and an ideal gas EOS, $\alpha = 3.5$, $n = 1$, resulting in

$$L \sim \mu^{7.5}\left(\frac{M^{5.5}}{R^{0.5}}\right)\,.$$

For Thomson opacity and an ideal gas EOS, $\alpha = 0$, $n = 0$, and we have

$$L \sim \mu^4 M^3\,.$$

The last two equations are classic results.

If the star is radiation-pressure dominated, $P \sim P_R \propto T^4 \sim \frac{GM^2}{R^4}$, and if Thomson opacity (constant) predominates, we have

$$L \sim R^2 \frac{R^3}{M}\frac{GM^2}{R^5} \sim M$$

another classic result, this time for very massive stars.

But what of $R(M)$? For steadily burning stars, this is obtained by setting the luminosity above equal to the thermonuclear power in the core. If the specific nuclear power, ε, goes as

$$\varepsilon \sim \rho^{u-1} T^s$$

then we can use the $\rho \sim M/R^3$ and $T \sim \mu GM/R$ relations to obtain:

$$\mu^{\alpha+4} R^{3n-\alpha} M^{\alpha+3-n} \sim M \left(\frac{M}{R^3}\right)^{u-1} \left(\frac{GM\mu}{R}\right)^s .$$

If we set $s = 20$ (CNO burning), $u = 2$, $n = 1$, and $\alpha = 3.5$, we find $R \sim M^{0.73}$, not far off. However, this model for $R(M)$ is quite crude and full numerical calculations are warranted.