SHELL BURNING STARS: Red Giants and Red Supergiants

There is a large variety of stellar models which have a distinct **core – envelope** structure. While any main sequence star, or any white dwarf, may be well approximated with a single polytropic model, the stars with the core – envelope structure may be approximated with a composite polytrope: one for the core, another for the envelope, with a very large difference in the "K" constants between the two. This is a consequence of a **very large difference in the specific entropies between the core and the envelope**. The original reason for the difference is due to a jump in chemical composition. For example, the core may have no hydrogen, and mostly helium, while the envelope may be hydrogen rich. As a result, there is a nuclear burning shell at the bottom of the envelope; hydrogen burning shell in our example. The heat generated in the shell is diffusing out with radiation, and keeps the entropy very high throughout the envelope. **The core – envelope structure is most pronounced when the core is degenerate**, and its specific entropy near zero. It is supported against its own gravity with the non-thermal pressure of degenerate electron gas, while all stellar luminosity, and all entropy for the envelope, are provided by the shell source.

A common property of stars with well developed core – envelope structure is not only a very large jump in specific entropy but also a very large difference in pressure between the center, P_c, the shell, P_{sh}, and the photosphere, P_{ph}. Of course, the two characteristics are closely related to each other. In a main sequence star, or in a white dwarf, we have $P_c/P_{ph} \gg 1$, and this makes a star a very well defined object, with a distinct surface. A star with a core – envelope structure has $P_c/P_{sh} \gg 1$, as well as $P_{sh}/P_{ph} \gg 1$, and hence the core is like a **star in its own right**, with the shell providing the outer boundary condition, just like the photosphere provides the outer boundary condition for the envelope. In the core – envelope star the boundary conditions for the envelope are provided by the photosphere on the outside, and by the core surface on the inside. In a detailed stellar model there is never a true discontinuity in any physical quantity. Nevertheless, it is useful to imagine the core and the envelope as two stars which interface each other at the shell source.

The origin of the core – envelope structure may always be traced to the development of a self-gravitating core, i.e. to the condition $P_c/P_{sh} \gg 1$, and the **"Schonberg – Chandrasekhar limit"**. Once an isothermal core becomes self-gravitating, it remains self-gravitating forever, even if a new nuclear fuel ignites in the core, and the core is no longer isothermal. The reason is that pressure required to sustain helium burning is orders of magnitude higher than pressure required to sustain hydrogen burning. Therefore, even in a composite model with a helium burning core and a hydrogen burning shell we still have $P_c/P_{sh} \gg 1$.

Stars with a core – envelope structure are called giants. They have photospheric radii which are much larger than radii of main sequence stars. They come in many varieties, the most common are red giants and blue giants. Red giants are stars which have envelopes expanded all the way to the Hayashi limit, their photospheres are cool, with T_{eff} between $2000\,\mathrm{K}$ and $4000\,\mathrm{K}$, hence their red color, and their name. Blue giants have hot photospheres, with T_{eff} between $10^4\,\mathrm{K}$ and $3\times10^4\,\mathrm{K}$, hence their blue color, and their name. There is no simple rule that might tell when a star may become a red or a blue giant. This may depend on small changes in the profile of chemical composition, e.g. on the details of variation of hydrogen abundance with mass fraction, $X(M_r)$. In fact, in many cases a star with exactly the same hydrogen profile, $X(M_r)$, may be either a red giant, or a blue giant, depending on its previous evolution. This is a direct consequence of invalidity of the so called "Vogt – Russell theorem", and a multitude of equilibrium solutions for a given profile of chemical composition. In one case up to 9 different equilibrium models were found for a fixed total mass M, and a fixed hydrogen profile $X(M_r)$ (M. Kozlowski and B. Paczynski 1973, *Acta Astronomica*, **23**, 65). Many of those were stable.

We shall concentrate on one particular red giant model, that is the simplest and also the most extreme. The **helium core will be degenerate, hydrogen burning shell will be the only energy source, and the very extended envelope will be convective**. Numerical models show that the radius of hydrogen burning shell is slightly larger than radius of a white dwarf with the mass equal to the helium core mass. It is slightly larger because the shell itself is not degenerate, and there is a thin, non degenerate layer below it. Between the shell and a very extended convective envelope there is an **intermediate radiative zone**, which has a very simple structure. There are no energy sources within this zone, and its mass is very small compared to the core or envelope mass. Therefore, we may calculate its structure assuming $M_r \approx M_c$, and $L_r = L$, where M_c is the core mass, and L is the total stellar luminosity.

The equations of hydrostatic and radiative equilibria may be written as

$$\frac{dP}{dr} = -\frac{GM_c}{r^2}\rho, \tag{rg1.1}$$

$$\frac{d\left(\frac{a}{3}T^4\right)}{dr} = -\frac{\kappa L}{4\pi c r^2}\rho. \tag{rg1.2}$$

Assuming that opacity is given by electron scattering, i.e. $\kappa = 0.2(1+X) = \text{const}$, and dividing equations (rg1.2) and (rg1.1) side by side we obtain

$$\frac{d\left(\frac{a}{3}T^4\right)}{dP} = \frac{\kappa L}{4\pi c G M_c} = \text{const}. \tag{rg1.3}$$

The last equation may be integrated to obtain

$$\frac{a}{3}T^4 = \frac{\kappa L}{4\pi c G M_c}P + C_1, \tag{rg1.4}$$

where C_1 is the integration constant. In many numerical models the intermediate radiative zone may extend from $T = 10^8\,\text{K}$ at the hydrogen burning shell out to $T = 10^5\,\text{K}$ at the bottom of the convective envelope, i.e. over many orders of magnitude variation of T^4. The constant C_1 can be calculated from the matching conditions between the radiative zone and the outer convective envelope, and it is very important near the radiative – convective boundary. However, deep inside the radiative zone the other two terms in the equation (rg1.4) become much larger than the constant C_1, and C_1 may be neglected. Therefore, as long as we are well below the radiative – convective boundary, we have the approximate relation

$$1 - \beta = \frac{\frac{a}{3T^4}}{P} = \frac{\kappa L}{4\pi c G M_c} = \frac{L}{L_{Edd,c}} = \text{const.}, \tag{rg1.5}$$

where

$$L_{Edd,c} \equiv \frac{4\pi c G M_c}{\kappa} = \frac{64780 L_\odot}{1+X}\frac{M_c}{M_\odot}, \tag{rg1.6}$$

is the Eddington luminosity for the core.

From the equation of state

$$P = \frac{k}{\mu H}\rho T + \frac{a}{3}T^4, \tag{rg1.7}$$

we may calculate density to be

$$\rho = \frac{a\mu H}{3k}\frac{\beta}{1-\beta}T^3. \tag{rg1.8}$$

This may be combined with the equation of radiative equilibrium (rg1.2) to obtain

$$\frac{d\left(\frac{a}{3}T^4\right)}{dr} = \frac{4a}{3}T^3\frac{dT}{dr} = -\frac{\kappa L}{4\pi c r^2}\frac{a\mu H}{3k}\frac{\beta}{1-\beta}T^3. \tag{rg1.9}$$

The last equation may be written as

$$\frac{dT}{dr} = -\frac{\mu H}{4k}\frac{\kappa L}{4\pi c r^2}\frac{\beta}{1-\beta}, \tag{rg1.10}$$

and taking into account equation (rg1.5) this becomes

$$\frac{dT}{dr} = -\frac{\mu H\beta}{4k}\frac{GM_c}{r^2}, \tag{rg1.11}$$

which may be integrated to obtain

$$T = \frac{\mu H\beta}{4k}\frac{GM_c}{r} + C_2, \tag{rg1.12}$$

where C_2 is the integration constant. Just as before, the numerical value of the constant C_2 may be obtained from the interface between the radiative zone and the outer convective envelope. This constant is very important near the interface, but at small radii, near the hydrogen burning shell source the other two terms in the equation (rg1.12) are much larger than C_2, and C_2 may be neglected.

The equations (rg1.8) and (rg1.12) may be written as

$$T \approx 5.78 \times 10^6 \beta\mu \frac{M_c}{M_\odot}\frac{R_\odot}{r} \quad [\mathrm{K}], \tag{rg1.13}$$

$$\rho \approx 0.0059\frac{\beta^4}{1-\beta}\mu^4\left(\frac{M_c}{M_\odot}\right)^3\left(\frac{R_\odot}{r}\right)^3 \quad [\mathrm{g\ cm}^{-3}], \tag{rg1.14}$$

where $\mu^{-1} = 2X + 0.75Y + 0.5Z$.

Now we may check how reasonable was our assumption that mass of the radiative zone described with equations (rg1.13) and (rg1.14) is small. Let us calculate mass between two radii, r_1 and r_2 using density distribution given with equation (rg1.14) ; we obtain:

$$M_{1,2} = \int_{r_1}^{r_2} 4\pi r^2 \rho \, dr = 0.0126 M_\odot \frac{\beta^4 \mu^4}{1-\beta} \left(\frac{M_c}{M_\odot} \right)^3 \ln \frac{r_2}{r_1}. \tag{rg1.15}$$

Adopting $X = 0.7$, $Y = 0.28$, and $Z = 0.02$ we have $\mu^{-1} = 1.62$, and the equation (rg1.15) becomes

$$\frac{M_{1,2}}{M_\odot} = 0.00184 \frac{\beta^4}{1-\beta} \left(\frac{M_c}{M_\odot} \right)^3 \ln \frac{r_2}{r_1}. \tag{rg1.16}$$

We find that mass of the radiative zone varies logarithmically with the ratio of its radii, and it is very small indeed if β is not too close to 1, i.e. provided the luminosity is not too small (cf. equation rg1.5)

Hydrogen burning shell in our red giant model is **very thin**, in mass as well as in radius. This is a consequence of **high temperature sensitivity of hydrogen burning rate**. At temperature above 1.5×10^7 K hydrogen burning is dominated by CNO cycle, and we have

$$\epsilon \approx \epsilon_0 \rho T^\nu, \qquad \nu \approx 15. \tag{rg1.17}$$

As a crude approximation we shall take the density and temperature distribution within hydrogen burning shell as given with equations (rg1.13) and (rg1.14), even though within the shell L_r and X vary. Within this approximation we have

$$L = \int_{r_{sh}}^{\infty} 4\pi \rho \epsilon r^2 \, dr \approx 4\pi \epsilon_0 \int_{r_{sh}}^{\infty} r^2 \rho^2 T^\nu dr \sim \tag{rg1.18}$$

$$\sim \frac{\beta^{\nu+8}}{(1-\beta)^2} M_c^{\nu+6} \int_{r_{sh}}^{\infty} r^{-\nu-4} dr \sim \frac{\beta^{\nu+8}}{(1-\beta)^2} M_c^{\nu+6} r_{sh}^{-\nu-3},$$

where r_{sh} is the shell radius. It is the **very rapid variation of energy generation rate** with radius, $dL_r/dr \sim r^{-\nu-4}$, that makes the shell so thin. At a radius $2 \times r_{sh}$ the hydrogen burning rate is almost 10^6 times smaller than at r_{sh}.

Radiative equilibrium in the radiative zone above the shell source gives $L \sim (1-\beta)M_c$ (cf. equations rg1.5 and rg1.6). Combining this relation with equation (rg1.18) we obtain

$$(1-\beta)\, M_c \sim \frac{\beta^{\nu+8}}{(1-\beta)^2} M_c^{\nu+6} r_{sh}^{-\nu-3}. \qquad \text{(rg1.19)}$$

If the shell radius is only somewhat larger than a **radius of degenerate dwarf** (polytrope of index 1.5) with a mass M_c, then $r_{sh} \sim M_c^{-1/3}$, and equation (rg1.19) may be written as

$$\frac{(1-\beta)^3}{\beta^{\nu+8}} \sim M_c^{\frac{4}{3}\nu+6}, \qquad \nu \approx 15. \qquad \text{(rg1.20)}$$

We may look at equation (rg1.20) as a core mass – luminosity relation. According to this equation an increase of the core mass M_c leads to a decrease of β, and according to equations (rg1.5) and (rg1.6) to an increase of stellar luminosity , $L \sim (1-\beta)M_c$. Full scale numerical computations confirm this conclusion. A luminosity of a red giant with a degenerate core increases with the core mass, and it is independent on the envelope mass. The core mass – luminosity relation may be approximated with

$$\frac{L}{L_\odot} \approx 200 \left(\frac{M_c}{0.3M_\odot}\right)^{7.6}, \qquad 0.15 < \frac{M_c}{M_\odot} < 0.45, \qquad (RGB)\,, \qquad \text{(rg1.21)}$$

for **red giants with a degenerate helium cores and a hydrogen burning shell** source (S. Refsdal and A. Weigert, 1971, *Astronomy and Astrophysics*, **6**, 426), and with

$$\frac{L}{L_\odot} \approx 59000 \left(\frac{M_c}{M_\odot} - 0.522\right), \qquad 0.6 < \frac{M_c}{M_\odot} < 1.4, \qquad (AGB)\,, \qquad \text{(rg1.22)}$$

for **red supergiants with a degenerate carbon – oxygen core and hydrogen and helium burning shell sources** (B. Paczynski, 1970, *Acta Astronomica*, **20**, 47).

The core mass – luminosity relation (rg1.21) describes evolution of low and intermediate mass stars ($M < 2M_\odot$) along the red giant branch up to helium ignition in the core. In these stars helium ignition is known as a **helium flash** — a rather violent, almost dynamical event. The violence of helium ignition is due to degeneracy of the core. When the core mass increases its central density increases too. The temperature within hydrogen burning shell and within the core also increase. At some point the rate of energy release due to helium burning becomes important for the heat balance near the center, and leads to an additional increase of the central temperature. This rise in temperature does not produce any increase in gas pressure, because pressure of degenerate gas does not depend on temperature. Therefore, there is no expansion of the core. Instead, rising temperature increases the rate of helium burning, and this in turn rises temperature even more. We have a thermal run-away: a self-accelerating rise in temperature and helium burning. This is somewhat similar to the instability of semi-degenerate stars on the "high density branch" of main sequence. In that case the unstable models were separated from stable models by a turning point corresponding to a minimum in total stellar mass. In the helium flash case the unstable and stable models are separated by a turning point corresponding to a maximum of a degenerate helium core mass. Helium flash corresponds to the tip of red giant branch in the H—R diagram.

Helium flash leads to such a large rise in temperature, that degeneracy of electron gas is removed. At this point farther release of heat increases pressure and makes the core expand. The expansion lowers pressure, density and temperature, and we reach stable conditions for helium burning. In the process of reaching this stable condition approximately 5% of all helium is converted into carbon — that much energy is required to remove electron degeneracy. This may be estimated as follows. Kinetic energy of degenerate electrons is equal to minus one half of gravitational energy of the core. Helium flash develops when $M_c \approx 0.45M_\odot$, and the core radius $r_c \approx 0.014R_\odot$. Gravitational binding energy per unit mass in the core, which is like $n = 1.5$ polytrope, is about $0.5GM_c/r_c \approx 5 \times 10^{16}\,\mathrm{erg\ g}^{-1}$. Therefore, energy density of electron gas is about $2.5 \times 10^{16}\,\mathrm{erg\ g}^{-1}$. Burning 1 gram of helium into carbon releases $6 \times 10^{17}\,\mathrm{erg\ g}^{-1}$. Hence, about 1/20 of all helium has to be burnt to provide enough heat to generate thermal pressure comparable to degenerate pressure.

During stable core helium burning stage of stellar evolution there are two sources of energy, because hydrogen is burning in a shell source. Therefore, mass of the helium core increases. **Horizontal branch stars in globular clusters**, as well as **"red clump"** stars in old open clusters are in this stage of evolution.

When helium is exhausted in the center the mass surrounded by the hydrogen shell is approximately $0.6M_\odot$. About 50% of that is in the inner carbon – oxygen core, made of helium burning products, while the other 50% is in the outer helium core. At this time we have two nuclear shell sources: hydrogen and helium. Initially, helium burning proceeds more rapidly, and most of the outer helium core is burnt into carbon and oxygen. The helium burning shell catches up with the hydrogen burning shell, and very little mass is left between them. The core of approximately $0.6M_\odot$ becomes isothermal and degenerate, as there are no nuclear energy sources within it. The star becomes a red supergiant. In globular clusters these are called **"asymptotic giant branch" (AGB) stars**. Their structure is similar to that of red giants: they have degenerate core, extended convective envelope, and intermediate radiative zone. Structure of the radiative zone is well described with equations (rg1.13) and (rg1.14). There is one major difference: red giants have only one shell source burning hydrogen. Asymptotic branch giants have two shell sources: hydrogen and helium. The two core mass – luminosity relations given with equations (rg1.21) and (rg1.22) correspond to these two types of stars.