SIMPLE RADIATION TRANSFER FOR SPHERICAL STARS

Under LTE (Local Thermodynamic Equilibrium) condition radiation has a Planck (black body) distribution. Radiation energy density is given as

$$U_{r,\nu}d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{e^{h\nu/kT}-1}, \qquad \text{(LTE)}, \tag{tr.1}$$

and the intensity of radiation (measured in ergs per unit area per second per unit solid angle, i.e. per steradian) is

$$I_\nu = B_\nu(T) = \frac{c}{4\pi}U_{r,\nu}, \qquad \text{(LTE)}. \tag{tr.2}$$

The integrals of $U_{r,\nu}$ and $B_\nu(T)$ over all frequencies are given as

$$U_r = \int_0^\infty U_{r,\nu}d\nu = aT^4, \qquad \text{(LTE)}, \tag{tr.3a}$$

$$B(T) = \int_0^\infty B_\nu(T)\,d\nu = \frac{ac}{4\pi}T^4 = \frac{\sigma}{\pi}T^4 = \frac{c}{4\pi}U_r = \frac{3c}{4\pi}P_r, \qquad \text{(LTE)}, \tag{tr.3b}$$

where $P_r = aT^4/3$ is the radiation pressure.

Inside a star conditions are very close to LTE, but there must be some anisotropy of the radiation field if there is a net flow of radiation from the deep interior towards the surface. We shall consider intensity of radiation as a function of radiation frequency, position inside a star, and a direction in which the photons are moving. We shall consider a spherical star only, so the dependence on the position is just a dependence on the radius r, i.e. the distance from the center. The angular dependence is reduced to the dependence on the angle between the light ray and the outward radial direction, which we shall call the polar angle θ. The intensity becomes $I_\nu(r,\theta)$.

The specific intensity is the fundamental quantity in classical radiative transfer. The radiation energy density is its zeroth angular moment, the flux is its first angular moment, and the radiation pressure is its second angular moment.

Let us consider a change in the intensity of radiation in the direction θ at the radial distance r when we move along the beam by a small distance $dl = dr/\cos\theta$. The intensity will be reduced by the amount proportional to the opacity per unit volume, $\kappa_\nu \rho$ multiplied by dl, where ρ is density of matter and κ_ν is monochromatic total opacity in units $\mathrm{cm}^2\,\mathrm{g}^{-1}$. κ_ν is comprised of an absorptive (κ_ν^a) and a scattering (κ_ν^s) part. The former is connected to the emissivity by Kirchhoff's law and the intensity will increase by the amount proportional to the emissivity of gas. Under nearly LTE condition the emissivity per unit volume is given as a product $\kappa_\nu^a \rho B_\nu(T)$. The equation of monochromatic radiation transfer is written as

$$\frac{\partial I_\nu(\theta,r)}{\partial l} = \cos\theta \frac{\partial I_\nu(\theta,r)}{\partial r} = -\kappa_\nu \rho I_\nu(\theta,r) + \kappa_\nu^a \rho B_\nu(T) + \frac{\kappa^s}{4\pi} \int I'_\nu(\theta,r)\Phi(\Omega',\Omega)d\Omega' \,, \qquad \text{(tr.4)}$$

where the last term represents the redistribution of radiation by scattering back into the beam with angular indicatrix (coupling) $\Phi(\Omega',\Omega)$. Note that the dependence of the various opacities on photon frequency, as well as on the density, temperature, and chemical composition of the gas, is implied.

Monochromatic radiation energy density may be calculated as

$$U_{r,\nu} = \frac{1}{c} \int\limits_{4\pi} I_\nu\left(\theta,r\right) d\Omega, \qquad \text{(tr.5)}$$

where the integration is extended over the whole 4π solid angle. This is the zeroth angular moment of the radiation field. Because of azimuthal symmetry this integral may be written as

$$U_{r,\nu} = \frac{2\pi}{c} \int\limits_0^\pi I_\nu\left(\theta,r\right) \sin\theta d\theta. \qquad \text{(tr.6)}$$

The total radiation energy density is given as

$$U_r = \int\limits_0^\infty U_{r,\nu} d\nu. \qquad \text{(tr.7)}$$

Monochromatic flux of radiation in the direction r is the first angular moment of the specific intensity and may be calculated as

$$F_\nu = \int\limits_{4\pi} I_\nu\left(\theta, r\right)\cos\theta d\omega = 2\pi \int\limits_0^\pi I_\nu\left(\theta, r\right)\cos\theta\sin\theta d\theta, \tag{tr.8}$$

and the total flux of radiation, measured in $\mathrm{erg\ cm}^{-2}\,\mathrm{s}^{-1}$, is given as

$$F = \int\limits_0^\infty F_\nu d\nu. \tag{tr.9}$$

The radiation pressure is the second angular moment of I_ν, integrated over frequency.

In the Eddington/diffusive approximation, the second moment is one-third of the zeroth moment. It can then be shown that the flux is driven by the spatial gradient of the Planck function:

$$F_\nu = -\frac{4\pi}{3\kappa_\nu\rho}\frac{\partial B_\nu\left(T\right)}{\partial r} = -\frac{4\pi}{3\kappa_\nu\rho}\frac{\partial B_\nu\left(T\right)}{\partial T}\frac{dT}{dr}.$$

The total radiative energy flux is an integral of F_ν over all frequencies, i.e

$$F = -\frac{4\pi}{3\rho}\frac{dT}{dr}\int\limits_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu\left(T\right)}{\partial T}d\nu = -\frac{4\pi}{3\kappa_R\rho}\frac{dB\left(T\right)}{dr} = -\frac{c}{\kappa_R\rho}\frac{dP_r}{dr}, \tag{tr.10}$$

where the **Rosseland mean opacity**, κ_R, is defined as

$$\frac{1}{\kappa_R} \equiv \frac{\int\limits_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu(T)}{\partial T}d\nu}{\int\limits_0^\infty \frac{\partial B_\nu(T)}{\partial T}d\nu} = \frac{\int\limits_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu(T)}{\partial T}d\nu}{\frac{ac}{\pi}T^3}. \tag{tr.11}$$

Of course, the Rosseland mean opacity is a function of density, temperature and chemical composition, $\kappa_R(\rho, T, X)$. We shall at times write the Rosseland opacity as κ.

We now have

$$\frac{L_r}{4\pi r^2} = F = -\frac{c}{3\kappa\rho}\frac{dU_r}{dr} = -\left(\frac{4acT^3}{3\kappa\rho}\right)\frac{dT}{dr} = -\lambda\frac{dT}{dr}, \tag{tr.12}$$

where $L_r = 4\pi r^2 F$ is stellar luminosity at a radius r, i.e. the total amount of radiation energy flowing across a spherical surface with a radius r, and U_r is the radiation energy density. The last equation looks just like the equation for heat diffusion, with the coefficient of thermal conductivity λ related to the coefficient of opacity with a relation

$$\lambda = \frac{4acT^3}{3\kappa\rho}. \tag{tr.13}$$

As heat may be transferred not only by photons, but also by electrons, it may be safer to write the last equation as

$$\lambda_{rad} = \frac{4acT^3}{3\kappa_{rad}\rho}, \tag{tr.14}$$

where λ_{rad} and κ_{rad} are explicitly related to radiation. We may write a similar relation for electrons:

$$\lambda_{el} = \frac{4acT^3}{3\kappa_{el}\rho}, \tag{tr.15}$$

where λ_{el} and κ_{el} are the coefficients of thermal conductivity and "opacity" for the electrons.

In general, we may have some heat transferred by photons, and some by electrons. As the two means of heat transport are additive, the combined coefficient of thermal conductivity may by calculated as

$$\lambda = \lambda_{rad} + \lambda_{el}, \tag{tr.16}$$

or equivalently, we may write a formula for the combined coefficient of opacity as

$$\frac{1}{\kappa} = \frac{1}{\kappa_{rad}} + \frac{1}{\kappa_{el}}. \tag{tr.17}$$

Near the stellar surface the luminosity and radius can be taken as L and R, and the radiation energy flux is $F = L/4\pi R^2$. Let us define optical depth τ as

$$d\tau \equiv -\kappa\rho dr, \qquad \tau = 0 \quad \text{at} \quad r = R. \tag{tr.18}$$

Now, we may write the equation as

$$\frac{dT^4}{d\tau} = \frac{3F}{ac} \approx \text{const}, \tag{tr.19}$$

and therefore

$$T^4 = T_0^4 + \frac{3F}{ac}\tau, \tag{tr.20}$$

where T_0 is the temperature at the stellar surface.

Consider now a surface radiating as a black body with a temperature T. At a point just above the surface the radiation comes from one hemisphere only, and we may use equations (tr.7), (tr.6), and (tr.2) to calculate

$$U_r = \int_0^\infty \left[\frac{2\pi}{c}\int_0^\pi I_\nu(\theta)\sin\theta d\theta\right] d\nu = \tag{tr.21}$$

$$\int_0^\infty \left[\frac{2\pi}{c}\int_0^{\pi/2} B_\nu(T)\sin\theta d\theta\right] d\nu = \frac{2\pi}{c}B(T) = \frac{1}{2}aT^4.$$

We obtained only one half of the radiation energy density expected under LTE conditions for the temperature T, because radiation was coming from one hemisphere only. The radiative energy flux may be calculated for our case:

$$F = \int_0^\infty \left[2\pi\int_0^\pi I_\nu(\theta)\sin\theta\cos\theta d\theta\right] d\nu = \int_0^\infty \left[2\pi\int_0^{\pi/2} B_\nu(T)\sin\theta\cos\theta d\theta\right] d\nu = \tag{tr.22}$$

$$\pi B(T) = \frac{ac}{4}T^4 = \sigma T^4.$$

We shall define the effective temperature of a star with a relation

$$\frac{L}{4\pi R^2} = F \equiv \sigma T_{eff}^4. \tag{tr.23}$$

This is a temperature that a black body would have if it radiated just as much energy per unit area as the star does.

The radiation energy density at the surface of a black body is half of the LTE energy density corresponding to the temperature T. We shall adopt an approximation that at the stellar surface, i.e. at $\tau = 0$ the radiation energy density is $aT_{eff}^4/2$, by analogy with a black body case. We find that

$$T_0^4 = \frac{1}{2} T_{eff}^4, \tag{tr.24}$$

and the temperature distribution close to the stellar surface is given as

$$T^4 = \frac{1}{2} T_{eff}^4 + \frac{3F}{ac}\tau = T_{eff}^4 \left(\frac{1}{2} + \frac{3}{4}\tau \right). \tag{tr.25}$$

Therefore, we have $T = T_{eff}$ at $\tau = 2/3$. The optical depth 2/3 corresponds to a **photosphere**. This formula is the origin of the classic $\tau = 2/3$ condition for photospheres.

Note that in the diffusion/Eddington approximation the surface temperature is $2^{1/4}$ times lower than the effective temperature. The exact value is $(4/\sqrt{3})^{1/4}$, when the opacity is independent of frequency.
The radiation diffusion equation may be written as

$$\frac{dP_r}{dr} = -\frac{\kappa\rho}{c} F = -\frac{\kappa\rho}{c} \frac{L_r}{4\pi r^2}. \tag{tr.26}$$

This may be combined with the equation of hydrostatic equilibrium to obtain

$$\frac{dP_g}{dr} = \frac{dP}{dr} - \frac{dP_r}{dr} = -\frac{GM_r}{r^2}\rho + \frac{\kappa\rho}{4\pi c} \frac{L_r}{r^2} = \tag{tr.27}$$

$$-\frac{GM_r}{r^2}\rho \left(1 - \frac{\kappa L_r}{4\pi c G M_r} \right).$$

The emergence of the local Eddington luminosity, $4\pi c G M_r/\kappa$ is not a coincidence and is gratifying.

Near the stellar surface we have $L_r = L$, and $M_r = M$. We shall find later on that when luminosity is very high then density in a stellar atmosphere is very low, and the opacity is dominated by scattering of photons on free electrons. For a fully ionized gas the electron scattering opacity is given as

$$\kappa_e = \frac{n_e}{\rho}\sigma_e = 0.2\,(1+X)\,, \qquad [\,\mathrm{cm}^{\,2}\,\mathrm{g}^{\,-1}], \tag{tr.28}$$

where X is hydrogen content by mass fraction, n_e is number of electrons per cubic centimeter, and σ_e is equal to the Thomson scattering cross-section for scattering photons on electrons

$$\sigma_e = \frac{8\pi}{3}r_e^2 = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 = 0.665\times 10^{-24}\,\mathrm{cm}^{\,2}. \tag{tr.29}$$

Putting the electron scattering opacity into the equation we obtain near the stellar surface

$$\frac{dP_g}{dr} = -\frac{GM}{r^2}\rho\left(1 - \frac{\kappa_e L}{4\pi cGM}\right), \tag{tr.30}$$

while the gradient of the total pressure P is given as

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho. \tag{tr.31}$$

Dividing the last two equations side by side we obtain

$$\frac{dP_g}{dP} = 1 - \frac{\kappa_e L}{4\pi cGM} = \text{const.} \tag{tr.32}$$

This may be integrated to obtain

$$P_g = (P - P_0)\left(1 - \frac{\kappa_e L}{4\pi cGM}\right), \tag{tr.33}$$

where $P_0 = P_{r,(\tau=0)} = 2F/3c$ is a very small radiation pressure at the stellar surface. It is clear, that at a modest depth below the stellar surface the pressure is very much larger than at the surface, and therefore, the equation gives

$$\beta \equiv \frac{P_g}{P} = \left(1 - \frac{\kappa_e L}{4\pi cGM}\right). \tag{tr.34}$$

It is obvious that $0 < \beta < 1$, and, therefore,

$$0 < L < L_{Edd} \equiv \frac{4\pi cGM}{\kappa_e} = \frac{4\pi cG}{0.2\,(1+X)}M = \frac{2.50\times 10^{38}\,\mathrm{erg\ s}^{\,-1}}{1+X}\left(\frac{M}{M_\odot}\right) = \frac{65300 L_\odot}{1+X}\left(\frac{M}{M_\odot}\right), \tag{tr.35}$$

where L_{Edd} is the **Eddington luminosity**. For a normal hydrogen abundance, $X = 0.7$ we have $L_{Edd}/L_\odot = 4\times 10^4\ M/M_\odot$.