# POLYTROPES

Polytropes are self-gravitating gaseous spheres that were, and still are, very useful as crude approximation to more realistic stellar models. Properties of polytropes are thoroughly described in a classical, and very old, textbook: *An Introduction to the Study of Stellar Structure* by S. Chandrasekhar (1939, Dover editions: 1958, 1967) .

We assume that a spherical star is in a hydrostatic equilibrium. Its structure is described by two ordinary, first order differential equation:

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho, \tag{poly.1a}$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho. \tag{poly.1b}$$

These may be combined to a single, second order Poisson equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{poly.2}$$

We assume that there is a polytropic relation between pressure and density:

$$P = K\rho^{1+\frac{1}{n}}, \tag{poly.3}$$

where $K$ and $n$ are real, positive constants, and $n$ is called a polytropic index. We introduce dimensionless variables:

$$\rho = \rho_c \theta^n, \qquad P = P_c \theta^{n+1}, \qquad r = \alpha\xi, \tag{poly.4}$$

where $\rho_c$ is the central density, $\theta$ is "polytropic temperature", $\alpha$ is a length constant defined as

$$\alpha^2 = \frac{K\,(n+1)\,\rho_c^{\frac{1-n}{n}}}{4\pi G}, \tag{poly.5}$$

and $\xi$ is a new radius like variable. Combining equations (poly.2-5) we obtain the Poisson equation in dimensionless variables:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{poly.6}$$

This is known as the Lane-Emden equation for polytropic stars.

The two boundary conditions for the eq. (poly.6) are at the center:

$$\theta = 1, \qquad \frac{d\theta}{d\xi} = 0, \qquad\qquad \text{at} \quad \xi = 0. \tag{poly.7}$$

As both conditions are at the same point, they are in fact initial conditions. Therefore, for every value of a polytropic index $n$ there is only one solution of eq. (poly.6) . The solution has a physical meaning as long as $\theta >= 0$. The surface of a polytropic star is at $\xi = \xi_1$, where $\theta = 0$, and according to eqs. (poly.4) density and pressure go to zero.

There are three analytic solutions known:

$$
\begin{aligned}
&n = 0, \qquad \theta = 1 - \frac{\xi^2}{6}, && \xi_1 = \sqrt{6} \approx 2.45, \\
&n = 1, \qquad \theta = \frac{\sin\xi}{\xi}, && \xi_1 = \pi \approx 3.14, \\
&n = 5, \qquad \theta = \left(1 + \frac{\xi^2}{3}\right)^{-1/2}, && \xi_1 = \infty.
\end{aligned}
\tag{poly.9}
$$

The case $n = 0$ corresponds to incompressible fluid, i.e. $\rho = \rho_c = const$, $P = P_c\theta$, and it requires rewriting eq. (poly.6) in a different form. In this case pressure vanishes at the surface, but density is the same throughout the "star". This solution is a crude approximation to the interior structure of a planet like Earth. The case $n = 5$ is also special, as the radius of this "star" is infinite. It is possible to show that all polytropes with $n > 5$ have infinite radii. This means that only solutions with $n < 5$ have a surface. The two cases most interesting for real stars have $n = 1.5$ and $n = 3$, and unfortunately these do not have analytic solutions.

A solution of eq. (poly.6) depends on one parameter only, the polytropic index $n$. If stellar structure is approximated with a polytrope with a given index, then two scaling parameters are needed to express the structure in physical units, like c.g.s. For example, the two parameter may be $K$ (which is related to entropy) and central density; or stellar mass and stellar radius; or $K$ and the stellar mass; and so on. If we know the numerical or analytical solution of the Lane-Emden equation (poly.6) then we may express the total stellar radius $R$, and the total stellar mass $M$ as follows:

$$
R = \alpha\xi_1 = \left[\frac{K}{G}\frac{n+1}{4\pi}\right]^{1/2} \rho_c^{\frac{1-n}{2n}} \xi_1, \tag{poly.10a}
$$

$$
\begin{aligned}
M &= \int_0^R 4\pi r^2 \rho dr = 4\pi\alpha^3\rho_c \int_0^{\xi_1} \xi^2\theta^n d\xi = \\
&= 4\pi\alpha^3\rho_c \int_0^{\xi_1} \left[-\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)\right] d\xi = \\
&= 4\pi\left[\frac{K}{G}\frac{n+1}{4\pi}\right]^{3/2} \rho_c^{\frac{3-n}{2n}} \left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_1}.
\end{aligned}
\tag{poly.10b}
$$

The last two equations may be combined to obtain

$$
R^{\frac{3-n}{n}} M^{\frac{n-1}{n}} = \frac{K}{GN_n}, \tag{poly.11a}
$$

where

$$
N_n \equiv \frac{(4\pi)^{1/n}}{n+1} \left(\left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_1}\right)^{\frac{1-n}{n}} \xi_1^{\frac{n-3}{n}}, \tag{poly.11b}
$$

is a dimensionless number that depends on the polytropic index only. We may also calculate the average density of a star to be

$$
\rho_{av} \equiv \frac{3M}{4\pi R^3} = \rho_c \frac{3}{\xi_1^3} \left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_1}, \tag{poly.12}
$$

and therefore

$$
\frac{\rho_c}{\rho_{av}} = \frac{\xi_1^3}{3\left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_1}}. \tag{poly.13}
$$

Finally, we may calculate the central pressure to be

$$P_c = K\rho_c^{\frac{n+1}{n}} = W_n \frac{GM^2}{R^4}, \tag{poly.14a}$$

where

$$W_n \equiv \left( \frac{3}{4\pi} \frac{\rho_c}{\rho_{av}} \right)^{\frac{n+1}{n}} N_n. \tag{poly.14b}$$

The equations (poly.11a) and (poly.14a) can be obtained with a much simpler approximate analysis. We replace the differential equations (poly.1a) and (poly.1b) with the corresponding algebraic equations:

$$\rho \approx \rho_{av} \approx \frac{M}{R^3}, \qquad \frac{P}{R} \approx \frac{GM}{R^2}\rho \approx \frac{GM^2}{R^5}, \qquad P \approx \frac{GM^2}{R^4}. \tag{poly.15}$$

Eqs. (poly.15) may be combined with (poly.3) to obtain the mass-radius relation:

$$R^{\frac{3-n}{n}} M^{\frac{n-1}{n}} \approx \frac{K}{G}. \tag{poly.16}$$

Of course, with this approximate analysis we cannot obtain the numerical values of $N_n$ and $W_n$. We must have the solution of the Lane-Emden equation in order to have those coefficients. The results of numerical integrations are given by Chandrasekhar in his textbook. We shall frequently need all those coefficients for the two important cases: for $n = 1.5$, which corresponds to an adiabatic star supported by pressure of non-relativistic gas, and for $n = 3$, which corresponds to an adiabatic star supported by pressure of ultra-relativistic gas. We have

| $n$ | $\rho_c/\rho_{av}$ | $\xi_1$ | $N_n$ | $W_n$ | $\frac{kT_c}{\mu H}\frac{R}{GM}$ |
|---|---|---|---|---|---|
| 0.0 | 1.0 | 2.4494 | · | $3/8\pi$ | · |
| 1.0 | $\pi^2/3$ | $\pi$ | 0.6366 | 0.3927 | · |
| 1.5 | 5.99 | 3.65 | 0.4242 | 0.7701 | 0.539 |
| 3.0 | 54.1825 | 6.8968 | 0.3639 | 11.0507 | 0.854 |

Most of the symbols in the table are self-explanatory. The last column gives the dimensionless number for polytropes supported by pressure of perfect gas, with $P = \frac{k}{\mu H}\,\rho T$, and allows a calculation of the central temperature $T_c$. Notice, that $kT_c$ is roughly thermal energy per particle, while $\mu HGM/R$ is gravitational energy per particle, and the dimensionless numbers in the last column, 0.539 and 0.854, are equal to the ratios of these two energies for the two polytropic models.

The mass radius relation for a star with $n = 1.5$ is given as

$$RM^{1/3} = \frac{K}{0.4242\ G}, \qquad n = 1.5, \tag{poly.17a}$$

while for $n = 3$ we have

$$M = \left( \frac{K}{0.3639\ G} \right)^{1.5}, \qquad n = 3. \tag{poly.17b}$$

These mean that the radius of a star with $n = 1.5$ is smaller if the star is more massive, while a star with $n = 3$ has its mass uniquely determined by the value of $K$ constant, while its radius is not restricted by either $K$ or $M$. Clearly, $n = 3$ is a very special case, and it has many astrophysical applications.

There is an interesting and useful expression for a gravitational potential energy of a polytropic star:

$$\Omega \equiv - \int_0^M \frac{GM_r dM_r}{r} = - \frac{3}{5-n} \frac{GM^2}{R}. \tag{poly.18}$$

We shall derive it using the condition of hydrostatic equilibrium (eq. poly.1a) and the polytropic relation given with eq. (poly.3), and also in the form:

$$\frac{dP}{\rho} = K\frac{n+1}{n}\rho^{\frac{1}{n}-1}d\rho = (n+1)\,d\left(\frac{P}{\rho}\right). \qquad \text{(poly.19)}$$

Gravitational potential energy is defined as energy required to remove all stellar mass, shell after shell, all the way to infinity.

We shall drive the eq. (poly.18) integrating by parts and using eqs. (poly.1a), (poly.1b), (poly.3), and (poly.19). As we shall be changing integration variables many times we shall use symbols $c$ and $s$ to indicate center and the surface, i.e. the limits of the integrals.

$$\begin{aligned}
\Omega \equiv -\int_c^s \frac{GM_r dM_r}{r} &= -\frac{1}{2}\int_c^s \frac{G\,d\left(M_r^2\right)}{r} = \\
&= -\left[\frac{GM_r^2}{2r}\right]_c^s - \frac{1}{2}\int_c^s \frac{GM_r^2}{r^2}dr = \\
&= -\frac{GM^2}{2R} + \frac{1}{2}\int_c^s M_r\frac{1}{\rho}dP = \\
&= -\frac{GM^2}{2R} + \frac{n+1}{2}\int_c^s M_r d\left(\frac{P}{\rho}\right) = \\
&= -\frac{GM^2}{2R} + \left[\frac{n+1}{2}M_r\frac{P}{\rho}\right]_c^s - \frac{n+1}{2}\int_c^s \frac{P}{\rho}dM_r = \\
&= -\frac{GM^2}{2R} - \frac{n+1}{2}\int_c^s \frac{P}{\rho}4\pi r^2\rho dr = \qquad \text{(poly.20)}\\
&= -\frac{GM^2}{2R} - \frac{n+1}{2}\int_c^s P\frac{4\pi}{3}d\left(r^3\right) = \\
&= -\frac{GM^2}{2R} - \left[\frac{n+1}{2}\frac{4\pi}{3}Pr^3\right]_c^s + \frac{n+1}{6}\int_c^s 4\pi r^3 dP = \\
&= -\frac{GM^2}{2R} - \frac{n+1}{6}\int_c^s 4\pi r^3\frac{GM_r}{r^2}\rho dr = \\
&= -\frac{GM^2}{2R} - \frac{n+1}{6}\int_c^s \frac{GM_r dM_r}{r} = \\
= -\frac{GM^2}{2R} + \frac{n+1}{6}\Omega = \Omega.
\end{aligned}$$

The last line of eq. (poly.20) gives the desired answer, i.e. $\Omega = -\frac{3}{5-n}\frac{GM^2}{R}$.

A simple, approximate estimate of the total or average nuclear buring rate in a star can be obtained in the context of polytropes if one assumes that the burning rate is

$$\varepsilon = \varepsilon_c \left(\frac{\rho}{\rho_c}\right)^{u-1} \left(\frac{T}{T_c}\right)^s ,$$

where the subscript refers to the central values, that

$$\theta^n \sim \left(\frac{\rho}{\rho_c}\right) \sim \left(\frac{T}{T_c}\right)^n ,$$

where $n$ is the polytropic index, and that

$$\theta \sim e^{-\frac{\xi^2}{6}} .$$

Plugging in, we derive

$$\int \frac{\varepsilon}{\varepsilon_c} \frac{dM}{M} \sim \frac{3.23}{(3u+s)^{3/2}} \qquad n = 3.0$$

and

$$\int \frac{\varepsilon}{\varepsilon_c} \frac{dM}{M} \sim \frac{2.4}{(1.5u+s)^{3/2}} \qquad n = 1.5 .$$

Note that for large values of $s$ this average nuclear burning rate is correspondingly smaller than the central value. This indicates that the degree of central concentration of the burning is large when the dependence of $\varepsilon$ on temperature is steep. One can show that under such circumstances, core burning is more likely to be convective. Such is the case for the main-sequence burning of stars a bit more massive than the Sun, for which hydrogen burns by the *CNO* cycle and not the *pp* chain.