White Dwarfs, Neutron Stars, & Black Holes

Like **white dwarfs**, **neutron stars** are strongly degenerate compact objects of roughly one solar mass. But whereas white dwarfs are supported against gravity by the zero-point motion of electrons (**electron degeneracy pressure**), neutron stars, as the name suggests, consist mostly of neutrons and are supported by the zero-point motion and **interactions** among the latter particles. The existence of neutron stars was hypothesized by Lev Landau and by Walter Baade and Fritz Zwicky shortly after the discovery of the neutron by Chadwick in 1932—Baade and Zwicky even proposed that neutron stars might form in supernovae, as is now believed. However, the observational history of neutron stars began only in 1967 with the discovery of radio pulsars by Jocelyn Bell and her Ph.D. advisor A. Hewish. In the meantime, the properties of such stars had been elucidated by a few important theorists, including J. R. Oppenheimer and E. Salpeter. For more historical and physical detail than can be fit into these notes, see the book by Shapiro and Teukolsky.

We have seen that white dwarfs have a **maximum mass**

$$\begin{aligned} M_{\rm Ch} &= 3.1\frac{(\hbar c/G)^{3/2}}{m_p^2}\left(2Y_e\right)^2 \approx 1.456\ (2Y_e)^2\ M_\odot \\ &\sim \frac{m_{planck}^3}{m_p^2}\,. \end{aligned} \tag{1}$$

While the electron mass does not appear in eq. (1), it does appear in the mass-radius relation of white dwarfs. When the electrons at the center are marginally relativistic, $x_{F(0)} = p_F(0)/m_e c = 1$, the mass $M \approx 0.5(2/\mu_e)^2 M_\odot$, the central density $\rho_c \approx 2\times10^6(\mu_e/2)\,\mathrm{g\,cm^{-3}}$, and the radius

$$\begin{aligned} R_{x_F(0)=1} &\approx 3.8\frac{(\hbar^3/cG)^{1/2}}{\mu_e m_p m_e} \\ &\approx 0.021\ (2Y_e)\ R_\odot \\ &\approx 1.9\frac{\hbar}{m_e c}\frac{m_{planck}}{m_p}\,(2Y_e) \\ &\sim \frac{m_p}{m_e}\left(\frac{2GM_{Ch}}{c^2}\right)(2Y_e)\ . \end{aligned} \tag{2}$$

Note the appearance of the Schwarzschild radius of a Chandrasekhar mass. Since the radius of a neutron star is some small multiple of this radius, we see that the ratio of the radius of a white dwarf to that of a neutron star is roughly the ratio of the neutron (or proton) and the electron rest masses. Interesting.

For a white dwarf, while electrons supply the pressure, the mass is dominated by nuclei of atomic weight A and atomic number $Z \approx A/2$. However, when the Fermi energy $\varepsilon_F = \sqrt{(p_F c)^2 + (m_e c^2)^2}$ of the electrons is large enough, it becomes energetically favorable for nuclei to undergo inverse beta decay,

$$(A, Z) \ + \ e^- \rightarrow (A, Z-1) \ + \ \nu_e, \tag{3}$$

thereby lowering the energy and pressure of the electron gas. The threshold for this reaction is

$$\epsilon_F \geq M(A, Z-1) - M(A, Z).$$

For example, at normal densities, the stablest of all nuclei is ^{56}Fe: $(A, Z) = (56, 26)$. The threshold for (3) is then 3.695 MeV, which is reached at $p_F(0)/m_e c = 7.174$, $M \approx 1.33 M_\odot$. (Observed white dwarfs are probably made of lighter elements—He, C, O, Mg—for which the threshold is higher. But the progenitors of neutron stars are the degenerate iron cores of evolved massive stars.) With increasing density, nuclei of higher and higher A/Z are favored. Eventually, the nuclei become so neutron-rich that the reaction

$$M(A', Z') \rightarrow M(A'-1, Z') \ + \ n$$

is favored. This is "neutron drip," and in low-temperature equilibrium conditions, it begins at $\rho_{\rm nd} \approx 4 \times 10^{11}\,{\rm g\,cm^{-3}}$. Nuclei dissolve completely at $\rho_{\rm nuc}/2$, where

$$\rho_{\rm nuc} \approx 2.6 \times 10^{14}\,{\rm g\,cm^{-3}} \tag{4}$$

is the density of nuclear matter: $\rho_{\rm nuc}$ corresponds to a volume $\approx (4\pi/3) \times (1.2\,{\rm fm})^3$ per nucleon. Ordinary nuclei have radii $\approx 1.2 A^{1/3}$ fm, where 1 fm $\equiv$ 1 Fermi $\equiv 10^{-13}$ cm is a convenient unit of length.

Nuclear matter is somewhat analogous to a liquid (such as water) in that there is an attraction between neighboring constituent particles, but also a strong resistance to compression; the density increases more slowly with pressure than for an ideal gas.

The exact general-relativistic equations of hydrostatic equlbrium were derived by Oppenheimer & Volkov (1939):

$$\begin{aligned} \frac{dP}{dr} &= -\left(\rho + \frac{P}{c^2}\right)\left(\frac{GM_r}{r^2} + \frac{4\pi G P}{c^2} r\right)\left(1 - \frac{2GM_r}{c^2 r}\right)^{-1} \\ \frac{dM_r}{dr} &= 4\pi \int_0^r \rho(\bar{r})\bar{r}^2 d\bar{r}. \end{aligned} \tag{5}$$

Here $\rho \neq m_n n_n$, which would be the density of rest-mass, but rather $\rho \equiv U/c^2$, where U is the internal energy per unit volume *including* rest mass.

Maximum Mass of a Neutron Star

One might have guessed that an object supported by neutron degeneracy pressure would have a maximum mass similar to M_{Ch}, and for the same reasons. In addition, the generalization to neutron stars would take cognizance of the fact that the Y_e enters the white dwarf discussion as the ratio of the number of pressure-producing particles to mass/weight producing particles. In a white dwarf the pressure comes from the electrons, while the mass comes from the baryons, and $Y_e \sim 0.5$. For a neutron star, since both the pressure and the mass come from the same particle (the neutron), the corresponding "Y_n" is $\sim$1.0. From eq. (1), we see that the associated "Chandrasekhar" mass is four times higher, $\sim$5.8 $M_\odot$. However, this argument is completely incorrect – the maximum mass for a neutron star is determined by the existence of a general-relativistic instability that obtains long before the Chandrasekhar argument would hold sway. In fact, using the relativistically correct equation of hydrostatic equilibrium (eq. (5)), and assuming a non-interacting degenerate gas of neutrons, Oppenheimer & Volkov (1939) derived a maximum neutron star mass of 0.7 $M_\odot$, $\sim$eight times smaller. Observed neutron-star masses are clearly larger than this. The reason is that the strong repulsive nuclear force trumps neutron degeneracy pressure by a wide margin, resulting in less compact and more rigid structures supported by a stiffer EOS. Well-determined masses come from binary systems, especially those containing a pulsar; less accurate mass estimates are sometimes possible for X-ray binaries, which involve a neutron star accreting from a less compact companion. In favorable cases, very precise pulsar timing allows one to detect subtle general-relativistic effects in the binary orbit and thereby constrain more of the system parameters than would be possible in a strictly Newtonian world. Remarkably, all well-determined neutron-star masses are consistent with a very narrow range: $M_{\rm ns} = 1.35 \pm 0.04 M_\odot$.

Equation (5) indicates that gravity is stronger in GR than in the theory of Newton. Moreover, pressure becomes a source of gravitational attraction. This means that there comes a point as the pressure and mass of the star increase when increasing pressure, which would normally act to resist gravity, actually strengthens it. This is the origin of the GR instability which defines the maximum mass of a neutron star. Depending upon the correct neutron EOS, as yet unknown, this maxmimum mass is near $\sim$1.8–2.5 $M_\odot$ gravitational. Near the maximum mass, and for a realistic EOS, $\frac{2GM}{c^2R} \leq 0.5$, not 1.0. (One can show that the maximum possible value of $\frac{2GM}{c^2R}$ is 8/9.)

When a neutron star achieves or exceeds this maximum mass, it collapses to a **black hole**. Therefore, the endpoints of stellar evolution are **white dwarfs, neutron stars, and black holes**.