IONIZATION, SAHA EQUATION

$$\frac{N_A}{N_B} = \frac{g_A}{g_B} \exp\left[-\left(E_A - E_B\right)/kT\right]. \tag{i.1}$$

$$\frac{n_{i+1}}{n_i} = \frac{g_{i+1}}{g_i}\frac{2V_e}{h^3}\int_0^\infty e^{-\left(\chi+p^2/2m\right)/kT} 4\pi p^2 dp = \tag{i.2}$$

$$\frac{g_{i+1}}{g_i}\frac{1}{n_e}\frac{2}{h^3}\left(2mkT\right)^{3/2} e^{-\chi/kT} 2\pi \int_0^\infty e^{-y} y^{1/2} dy =$$

$$\frac{g_{i+1}}{g_i}\frac{1}{n_e}\frac{2}{h^3}\left(2\pi mkT\right)^{3/2} e^{-\chi/kT},$$

where we set $y = p^2/2mkT$, and the value of the last integral was $\pi^{1/2}/2$. χ is the ionization potential.
The last equation is usually written as the Saha equation

$$\frac{n_{i+1}n_e}{n_i} = \frac{\left(2\pi mkT\right)^{1.5}}{h^3}\frac{2g_{i+1}}{g_i} e^{-\chi/kT}. \tag{i.3}$$

Can also use chemical potential balance:

$$\mu_{\rm H} = \mu_{\rm p} + \mu_{\rm e},$$

where $\mu = -kT\log(\frac{(2\pi mkT)^{1.5} g}{h^3 n})$ and $\mu \to \mu - \chi$ in the case of the lower charge ion.

ionization of hydrogen:	$\chi = 13.54\ eV$,	$2g_{i+1}/g_i = 1$,
first ionization of helium:	$\chi = 24.48\ eV$,	$2g_{i+1}/g_i = 4$,
second ionization of helium:	$\chi = 54.17\ eV$,	$2g_{i+1}/g_i = 1$.

Consider now pure, partly ionized hydrogen. Let n_H, n_{HI}, and n_{HII} be number density of all hydrogen, neutral hydrogen atoms, and hydrogen ions, respectively. We have $n_H = n_{HI} + n_{HII}$, $n_e = n_{HII}$, and $x \equiv n_{HII}/n_H$ is the degree of ionization. The density of gas is $\rho = m_p n_H$, where m_p is the mass of a hydrogen atom. We may write the Saha equation as

$$\frac{n_{HII} n_e}{n_{HI}} = \frac{\rho}{m_p} \frac{x^2}{1-x} = \frac{(2\pi mkT)^{1.5}}{h^3} \frac{2g_{i+1}}{g_i} e^{-\chi/kT}. \tag{i.4}$$

The constants in cgs units are: $m_p = 1.673 \times 10^{-24}$, $m = 9.11 \times 10^{-28}$, $k = 1.381 \times 10^{-16}$, $h = 6.63 \times 10^{-27}$. With these constants we have

$$\log \rho + \log \frac{x^2}{1-x} = \tag{i.5}$$

$$= 1.5 \log T - \frac{5040}{T} \chi + \log \left[(2\pi mk)^{1.5} m_p h^{-3} \right] = 1.5 \log T - \frac{68240}{T} - 8.394.$$

Pressure of partially ionized hydrogen gas is simply

$$P_g = (n_H + n_e) kT = (1+x) \frac{k}{m_p} \rho T. \tag{i.6}$$

The internal energy should now include not only kinetic, but also ionization energy:

$$U_g = 1.5 (n_H + n_e) kT + n_e \chi = 1.5P + x \frac{\chi}{m_p} \rho. \tag{i.7}$$