AST 514 Homework Problems – Spring of 2025

- **Due February 13:**

1.

The dynamical timescale of a star, t_{dyn}, the Kelvin-Helmholtz timescale, t_K, and the nuclear timescale, t_{nuc}, are given very approximately by the expressions:

$$t_{dyn} \sim \sqrt{\frac{R^3}{GM}}\,,$$

$$t_K \sim \frac{GM^2}{R}/L\,,$$

and

$$t_{nuc} \sim 0.1 \times \frac{0.007Mc^2}{L}\,,$$

where R, M, and L have there standard meanings.

a. If the luminosity on the main sequence scales with mass as M^3 and the radius of a star on the main sequence scales as $R \propto M^{0.8}$, compare these three timescales for main-sequence stars with masses of 0.5, 1.0, 2.0, 4.0, 6.0, and 10.0 $M_\odot$.

b. What general conclusions (plural) can you draw from what you find?

c. What is the range (in $L_\odot$) for stars on the main sequence?

d. Compare this to what you find in the literature and comment.

2.

The virial theorem for a star in hydrostatic equilibrium states that:

$$-3(\gamma - 1)E_{th} = \Omega\,,$$

where E_{th} and Ω are the total thermal and total gravitational energies, respectively. For a polytrope,

$$\Omega = -\frac{3}{5-n}\frac{GM^2}{R}\text{(where n is the polytropic index).}$$

a. Assuming that:

$$E_{th} = \int \frac{3}{2} N k_B T dV\,,$$

where N is the number density of the gas particles in the star, T is the local temeperature, k_B is Boltzmann's constant, and V is the volume, derive an approximate formula for the average temperature for such a star. Note that we are using the ideal gas law for the matter equation of state.

b. Assuming that the average temperature and central temperature (T_c) are always proportional (not always a good assumption, but ...), that a star's radius on the main sequence is proportional to $M^{0.8}$, that $\gamma = 5/3$, and that $n = 3.0$, derive the central temperature of a main sequence star as a function of total mass. Assume that T_c for the Sun is 1.5×10^7 Kelvin (K).

c. Find T_c for stars of mass 0.3, 1.2, 2.5, 4.0, and 8.0 $M_\odot$ and compare to what you find in the literature. Comment.

3.

a. In a star in hydrostatic equilibrium, the free-fall time and the sound travel time across the star are comparable. Why? Provide analytic arguments, however approximate.

b. How do these times compare with the minimum stable period of stellar rotation and the period of a companion of similar type in Roche lobe contact? For each situation, provide approximate equations to illustrate your conclusions.

c. Calculate the dynamical times for the Sun, a white dwarf, and a neutron star. Comment.

4.

a. Calculate the gravitational energies of Jupiter, the Earth, the Sun, a 10-$M_\odot$ star on the main sequence, and a neutron star. Use for each the appropriate value of the polytropic index.

b. Compare these energies to the corresponding rest-mass energies and comment. What is the relationship between the answer to this question and the gravitational redshifts from the surfaces of these stars? Demonstrate.

5.

a. A brown dwarf with a mass of 0.05 $M_\odot$ has an effective temperature of 900 K. Assuming it is fully-convective and an ideal gas, and using various reasonable physical parameters which apply, what is its Kelvin-Helmholtz timescale? Comment.

b. Perform the same calculation for a two Jupiter-mass object in isolation with an effective temperature of 200 K and comment on the difference between your answer here and in part **a**.

c. How would your results differ if neither the brown dwarf nor the giant planet were ideal gases, but had an effective adiabatic gamma of 2 (more realistic)?

- <u>**Due February 27:**</u>

6.

a. Using the generic scaling relations derived between the luminosity, mass, and radius for a star on the main sequence, and the approximate dependence of the nuclear burning power on mass and radius, derive $R(M)$. Assume that the nuclear burning is due to the CNO cycle, the ideal gas EOS obtains, and that Kramer's opacity dominates. Retain the μ dependence of your result.

b. With this $R(M)$, find the central temperature versus mass and μ. With two quantitative examples, find out how sensitive the central temperature is to μ. Explain using physical arguments the direction of the effect you see.

7.

a. Derive the Lane-Emden equation and solve it analytically for $n = 0$, $n = 1.0$, and $n = 5$.

b. Derive the mass-radius realtionship for a given κ and n for all polytropes.

c. Derive the ratio of the central to the average density for polytropes of index 1, 1.5, and 3.

d. Find the pressure prefactor in the polytropic equation $W_n \frac{GM^2}{R^4}$ for the general n, and quantitatively for $n = 1, 1.5,$ and 3.

8.

Write a computer program to solve the Lane-Emden equation for arbitrary n in the range $0 < n < 5$. Print out $\xi_{\rm max}$ (the eigenvalue) and plot $\theta(\xi)$ versus ξ for many representative values of n in the interval $1 < n < 4.5$ (*i.e.*, in particular non-half-integer numbers, since those results are not widely tabulated).

9.

a. Use MESA with physically "reasonable" default parameters for the Sun to evolve the Sun from the Hayashi phase to beyond the red giant phase. Plot the evolution on a L/$T_{\rm eff}$ HR diagram. Redo these calculations for different values of the mixing-length (plus or minus a factor of two) and include these on the HR plot.

b. For the Sun at its current age of 4.567 Gyrs, according to MESA what is the Sun's radius as a function of mixing length employed?

c. What mixing length is suggested for the Sun from this exercise?

- <u>**Due March 25:**</u>

10.

a. The Earth can be represented roughly as a polytrope of Emden index $n = 1/3$ (i.e., the equation of state for a rocky planet is $P = K\rho^4$) . Assuming this, and given that $M_E = 6 \times 10^{27}$ g and $R_E = 6.4 \times 10^8$ cm, calculate its central density and central pressure. Convert the pressure to atmospheres (1 atm = 1.01×10^6 dyne cm^{-2}).

b. What is the radius of a planet ("super-Earth") with 3 times the mass of the Earth ($M_E \sim 6 \times 10^{27}$ gms.)? Show all your work.

c. Calculate the corresponding quantities for Jupiter, which is composed mostly of quasi-degenerate molecular hydrogen rather than rock and is better approximated by $n = 1$, $M_J = 1.9 \times 10^{30}\,\mathrm{g} = 10^{-3} M_\odot$, $R_J = 7 \times 10^9\,\mathrm{cm} = 0.1 R_\odot$.

11.

Assuming a non-relativistic equation of state, estimate the radius, central density, and central pressure of a cold, solar-mass white dwarf consisting of pure carbon (^{12}C). From your results, check whether the non-relativistic assumption is self-consistent. *[Hint: How does the momentum of an electron at the Fermi surface compare to $m_e c$.]*

12.

a. Derive in the context of mixing-length theory the relationship between the energy flux, F_{conv}, in a convective zone and the mean convective speed, V_{conv}.

b. Calculate this speed for the convection zone of the Sun, making reasonable assumptions concerning the quantities in the resulting formula. You can look up the ρ, T profiles or calculate them using MESA.

c. What is the approximate relationship between the convective Mach number (M) and the quantity $\nabla - \nabla_{ad}$? Are there any stellar contexts in which we can expect M to be near one and what would this mean for the value of $\nabla - \nabla_{ad}$?

13.

The pressure (P)/mass-density (ρ)/temperature (T) relation for an ideal gas depends upon the number density ($N_{i,e}$) of free particles in the gas through the mean molecular weight, $\mu_{i,e}$, via the equation:

$$P_{i,e} = N_{i,e} k_B T = \frac{k_B \rho T}{\mu_{i,e} m_p},$$

where m_p is approximately the proton mass and i, e stand for the ions or the electrons, respectively. (Note that $k_B N_A = R$, the gas constant, and $N_A \sim \frac{1}{m_p}$, where N_A is Avogadro's number). The total gas pressure is $P_i + P_e$ and the total pressure is this pressure, plus P$_{rad}$, where P$_{rad} = \frac{aT^4}{3}$.

a. For a completely ionized gas, derive μ_i and μ_e as a function of the hydrogen, helium, and heavy mass fractions, X, Y, and Z, respectively. Note that the total μ is naturally given by the expression:

$$\frac{1}{\mu} = \frac{1}{\mu_i} + \frac{1}{\mu_e}.$$

b. Calculate and compare all these various pressures at T = 10^7 K and $\rho =$ 100 gcm^{-3} (roughly values at the center of the Sun) for $X = 0.73$, $Y = 0.25$ and $Z = 0.02$ and assemble them in order. Which dominates?

14.

a. Derive the relation between the photospheric pressure, gravity, and $\kappa_{Rosseland}$ in the photosphere that might be used as a boundary condition for stellar evolution calculations.

b. If you assume that the opacity in the photosphere is due H^- absorption, find the relationship between the density in the photosphere (ρ_{ph}) and T$_{\rm eff}$ and gravity, g. Assume a solar mixture and the power-law approximation. Take the opacity due to H^- to be:

$$2.5 \times 10^{-31} \left(\frac{Z}{0.02}\right) \rho^{0.8} T^9 \text{ cm}^2\text{g}^{-1}.$$

c. Derive these densities for main-sequence stars with masses 0.5, 1, 2, 5, 10, 15, 20, and 50 M$_\odot$. What trends do you see?

15.

a. Estimate the "average" opacity in the Sun by making an assumption about the mean temperature and density and using Kramers formula. With this opacity, calculate using a simple formula the diffusion time for a black body photon with thermal properties at this mean position to emerge from the Sun. [Use MESA to get solar profile numbers you can use.]

b. How does this timescale compare with the corresponding Kelvin-Helmholtz time? Explain using *two* arguments any significant differences you may find.

16.

a. The outer ∼2% by mass of the Sun is convective. Using approximate quantities for the temperatures and densities in this region (and, hence, the opacity), show why this is so. You will need to use the condition(s) for convection discussed in the lecture. [Use MESA to make a solar model, which you then query.]

b. How much smaller would the opacity in this region have to be to make this region radiative, all else being equal?

- <u>**Due April 15:**</u>

17. Derive the maximum core temperature achieved by a brown dwarf during its Kelvin-Helmholtz contraction and cooling phase. Use this estimate to determine the "ignition mass" in Jupiter-mass units ($M_J \sim 0.000955\ M_\odot$) for the thermonuclear burning of heavy hydrogen (deuterium) in a brown dwarf. Assume that the deuterium abundance by number is only 2×10^{-5} (i.e., small) and that the core needs to reach a temperature of $\sim 3 \times 10^5$ K to astrate deuterium. Comment on the result.

18.

Calculate in the PPII reaction $^7Be + e^- \rightarrow ^7 Li + \nu_e$ the energy (in MeV) taken away by the ν_e. Note that you need to find the nuclear masses, account for the electron, and then find the ν_e energy.

19.

Calculate the equilibrium ^8Be number density and number fraction in a star undergoing helium burning at temperatures of 10^8 and 2×10^8 Kelvin. Assume that the core is almost pure ^{4}He and that the mass density is 10^4 g cm^{-3}.

20.

Give a rough, but quantitative, explanation for the *ratio* 33.81/152.313, of the arguments of the exponentials in the *pp* and CNO reactions, $-33.81/T_6^{1/3}$ and $-152.313/T_6^{1/3}$, respectively. You need not explain the actual numbers themselves, only the ratio.

21.

Derive the general formula for a nuclear reaction rate:

$$\begin{aligned}\overline{\sigma v}(T) &= \left(\frac{8}{\pi\mu(k_{\rm B}T)^3}\right)^{1/2} \int_0^\infty \sigma(E) E e^{-E/k_{\rm B}T}\, dE, \\ &= \left(\frac{8}{\pi\mu(k_{\rm B}T)^3}\right)^{1/2} \int_0^\infty S(E) \exp\left[-\frac{b}{\sqrt{E}} - \frac{E}{k_{\rm B}T}\right] dE\,. \end{aligned} \tag{1}$$

22.

a. Calculate the positions of the Gamow peak (in keV) for the rate-limiting reaction of hydrogen burning for hydrogen-burning cores in 0.25, 1, 3, 5, and 25 $M_\odot$ stars.

b. What are the approximate "power-law" indices for the temperature-dependence of the nuclear rates in each of these cores on the main sequence?

- **Due April 28:**

23.

a. Use the general (!) zero-temperature equation of state for a degenerate electron gas at any density to derive numerically the mass-radius relationship for a white dwarf with $Y_e = 0.5$. The general expression is:

$$P = \kappa \rho^{4/3} I(x) \tag{2}$$

$$I(x) = \frac{3}{2x^4}(x(1+x^2)^{1/2}(\frac{2x^2}{3} - 1) + \ln(x + (1+x^2)^{1/2}))$$

$$x = p_{\rm F}/m_e c\,, \tag{3}$$

where κ contains the $\frac{1}{\mu_e^{4/3}}$. This involves writing a code to solve the equations of hydrostatic equilibrium. Demonstrate your results with a plot. (*Hint: The Runge-Kutta method for the solution of simultaneous O.D.E.s generally works well.*)

b. Show that the non-relativisitic mass-radius relation for an $n = 1.5$ polytrope ($P \propto \rho^{1+\frac{1}{n}}$) is one limit of your results.

c. Determine how well the approximation to the "Chandrasekhar-mass-adjusted" radius-mass relation for white dwarfs works by calculating the fractional deviation ($\Delta R/R$) of the fit from the correct result.

24.

Using the Oppenheimer-Volkoff equation for relativistic hydrostatic equilibrium:

$$\begin{aligned} \frac{dP}{dr} &= -\left(\rho + \frac{P}{c^2}\right)\left(\frac{GM_r}{r^2} + \frac{4\pi G P}{c^2} r\right)\left(1 - \frac{2GM_r}{c^2 r}\right)^{-1} \\ \frac{dM_r}{dr} &= 4\pi\rho(r) r^2 \end{aligned} \tag{4}$$

and an equation of state: $P = Kn^{2.5}$ and $\rho = m_n n + 1.5P/c^2$, where n is the number density of neutrons, show *numerically* that there is a maximum gravitational mass and what it is (in $M_\odot$). Assume that $K = 2 \times 10^{35} \text{dynes/cm}^2/\left(10^{39}/\text{cm}^3\right)^{2.5}$.

25.

Using MESA, generate an evolutionary model for the Sun through the end of core hydrogen burning, using an initial helium mass fraction of 28% and solar metallicity (Asplund et al. 2009).

a. Plot the evolution after achieving the main sequence (ZAMS) of the core helium mass fraction as a function of time to the end of core hydrogen burning.

How is the core helium fraction correlated with stellar age? How might this correlation be used as a chronometer to determine the age of a solar-like star?

b. Generate another solar model with an initial helium abundance of zero and plot the same evolutionary line. What are the differences you see with the earlier plot and what use might this difference be put to?

c. Plot the corresponding evolution (for the 28% model) of the solar luminosity with time after achieving the main sequence. What might this say about the Earth's surface temperature upon reaching the main sequence, now, and upon leaving the main sequence? [Hint: Assume that the atmosphere and surface of the "Earth" are independent of time and with characteristics we currently witness. Also assume that the Sun-Earth distance has not changed during this interval.]

26.

Using the 28% solar model above, plot the total neutrino power versus time. How might the PPI signal in a Gallium detector sensitive to those neutrinos vary with solar age (from ZAMS to the end of the main sequence). Comment. [Hint: The total log10 power (log L_ν) emitted in neutrinos is an entry in the history_columns.list.]

- **<u>EXTRAS:</u>**

27.

a. Derive the formula for the entropy per particle of an ideal Maxwell-Boltzmann gas of particles with mass m, degeneracy parameter g, temperature T, and number density n. Use the expression for the chemical potential

$$\mu = -kT \log\left(\frac{(2\pi mkT)^{1.5}\, g}{h^3 n}\right) .$$

The result is the Sackur-Tetrode equation.

b. Using your result from part **a.**, calculate the entropy per baryon per Boltzmann's constant in the center of the Sun, in its atmosphere, in the ISM, and in the Big Bang. Include the contribution of the electrons, if important. Assume that free protons are the dominant baryonic constituents in all cases. Ruminate on these numbers.

c. Find the entropy per baryon in the center of the Sun at the stage of quiescent core helium burning and compare this number to that for the Sun's center found above. Which is larger? Comment.

28.

a. Calculate the Q values for all the reactions in the PPI, PPII, and PPIII chains and the energy derived per helium nucleus produced (in ergs g^{-1}). *Hint: You can use a table of mass excesses, perhaps from Clayton. Take care to distinguish atomic from nuclear masses.*

b. Which reaction dominates the PPI chain?

29.

Calculate the Q values for all the reactions in the CNO cycle.

30.

Calculate the Q values for the 3-α, ^{12}C(α,γ)^{16}O, and ^{16}O(α,γ)^{20}Ne reactions.

31.

Calculate the Eddington luminosity in ergs s^{-1} for a neutron star of mass 1.4 $M_\odot$, assuming $\kappa = 0.4$ cm^2 gm^{-1} (Thomson). [Hint: You will need to calculate the radiation pressure force per mass and set it equal to the gravitational force per mass (GM/r^2). The former is proportional to $\kappa L/(4\pi r^2 c)$.]

32.

What is the approximate power-law relationship between luminosity and mass:

a. near the Sun's mass

b. For the most massive stars?

c. For the latter, how does the age of a star on the main sequence depend upon its mass? What characteristic asymptotic main-sequence age emerges for this class of the most massive stars?

33.

Summarize the range of stellar masses, luminosities, and effective temperatures on the hydrogen-burning main sequence.

34.

a. For what range of stellar masses is a star's core convective during the main sequence?

b. At what effective temperature and stellar mass does the outer convective zone of a star disappear?

35.

How does the central density of a ZAMS-mass star vary with mass?

36.

a. Approximately how many stable nuclei are there in Nature?

b. Go through the periodic table and identify the nucleosynthetic source for the major isotope of each stable element.

37.

a. What are the characteristic temperatures for each of the major burning stages of the elements?

b. What are the suggested major reactions that provide neutrons for the s-process? During what stellar stage, and in what stars, are the measured s-process elements made?

38.

a. For what thermonuclear reactions is most of the energy that is available via thermonuclear processes liberated? Why these?

b. How much energy is liberated per nucleus in the neutron-induced fission of ^{238}U?

39.

Calculate and plot the Kelvin-Helmholtz times for stars on the hydrogen-burning main sequence. Comment.

40.

How do the average opacities in the envelope of a solar-mass main-sequence star compare with those in its envelope when a red giant? Explain what you find.

41.

a. Compare the kinetic-energy/gravitational-energy relationship for the stars collectively in a globular cluster with the corresponding comparison in a single star in hydrostatic equilibrium. What general physical principle or relationship binds the two contexts?

b. Calculate the dynamical times in each context and comment.

42.

a. The Rosseland mean opacity weights the roles of those photon frequencies with what interaction characteristics with the matter? How does the result compare with resistors in an electrical circuit?

b. If conduction and radiation both carry energy in a given region of a star, what electrical-circuit analog obtains?

43.

In general, what is the approximate relationship between the thermal energy density and pressure for the general gas?

44.

For a black body of photons emerging from a surface, derive the fraction of the total emergent energy in a given differential log frequency bin, compared with the total σT^4? Dedimensionalize the result as much as possible, to make the result useful.

45.

What is the specific heat of a degenerate electron gas at modest temperature, T? How does this scale with the ratio of temperature to fermi energy?

46.

In the early stages of the big bang near temperatures of 10 MeV, what is the relationship between temperature and total energy density? Make sure to include all relevant components. How does this compare with the corresponding relationship for photons alone?

47.

How would you quickly estimate the rough RMS Mach number of stellar matter in a convective zone? What would you need to know?

48.

Plot the profile of the opacity of the Sun versus interior mass, multiplied by T^3/ρ. What do you notice? T^3/ρ is proportional to the specific entropy of the radiation component in the star and is related to $\beta = 1 - P_{\rm rad}/P_{\rm total}$. Comment.

49.

Why does the effective temperature of a newly-born Sun move away from the Hayashi track towards hotter surface temperatures upon reaching the main sequence?

50.

a. What are the central temperature and density of the core of a massive star just before its Chandrasekhar core collapses?

b. What is the entropy of such a core (per baryon, per Boltzmann's constant) at this time and in this region? How does this comport/jibe with the fact that the core is electron-degenerate?

51.

How does the Gamov peak of the dominant reaction at the core of the Sun shift during its lifetime on the main sequence? Plot the behavior versus time and comment on what you find.

52.

How do the fractions of energy lost to neutrinos for the PP chain and for the CNO cycle compare and what is the reason for the difference?

53.

a. Explain the stellar physics behind the dredge-up phenomenon and why it occurs when it does.

b. Describe the 1st, 2nd, and 3rd dredge-up phases for the stars that experience them and how their role in nucleosynthesis may vary for a given star and across the mass function.

54.

Show how a mass function for a collection of white dwarfs might translate into a luminosity function for that population as a function of time, using the simple Mestel cooling theory. How can this result be used to do astronomy?

55.

When an average white dwarf emerges from its planetary nebula its effective temperature may be ~200,000 K. How does its luminosity then compare with its Eddington limit (assuming for the moment Thomson opacity)? What might you conclude from this comparison?

56.

Describe all the physical corrections to the simple Chandrasekhar argument for the Chandrasekhar mass. What are they and what are the fractional corrections to the classical Chandrasekhar mass due to each?

57.

a. The Eddington model predicts a characteristic mass $\sim 50\mu^{-2}M_\odot$ for the mass at which radiation and gas pressure are equal. Express this mass in terms of fundamental physical constants $\hbar, c, G, k_B, m_p, m_e$, molecular weight μ, and a dimenensionless factor of order unity. Two of the physical constants listed above do not appear in the answer (or appear only as minor corrections); Why? *Hint: You will have to look up an expression for the radiation constant a in terms of the others.*

b. Have you seen this combination of fundamental parameters elsewhere? Comment?

Solution to Eddington maximum mass problem (above):
In the Eddington model, $\beta = 1/2$ for the case $P_{\rm rad} = P_{\rm gas}$. The expressions for T obtained by assuming an ideal gas,

$$T = \frac{\mu m_p \beta P}{k\rho},$$

and for radiation pressure,

$$T^4 = \frac{3P_{\rm rad}}{a} = \frac{3(1-\beta)P}{a},$$

yield $P = K\rho^{4/3}$ with

$$K = \left[\frac{3}{a}\left(\frac{k}{\mu m_p}\right)^4 \frac{1-\beta}{\beta^4}\right]^{1/3}.$$

The Eddington notes repeat the mass formula for the $n = 3$ polytrope obtained by integrating the Lane-Emden equation:

$$M = \left(\frac{K}{0.3639\ G}\right)^{3/2},$$

where this is the same K as above. Let's express a in terms of fundamental constants,

$$a = \frac{4\sigma}{c} = \frac{8\pi^5 k^4}{15h^3c^3},$$

and plug in $\beta = 1/2$ to get

$$K^{3/2} = \left[\frac{360\ \hbar^3 c^3}{\pi^2 \mu^4 m_p^4}\right]^{1/2}$$

So,

$$M = \left(\frac{360}{0.3639^3 \pi^2}\right)^{1/2} \left(\frac{\hbar c}{G}\right)^{3/2} (\mu m_p)^{-2} \approx 27.5 \left(\frac{\hbar c}{G}\right)^{3/2} (\mu m_p)^{-2} \approx \frac{51 M_\odot}{\mu^2}$$

Note that m_e and k do not appear in this equation. We started by canceling out T dependence (think of it as kT energy dependence) so it makes sense that k does not appear. And our argument depends on the ideal gas law, equation for radiation pressure, and the polytrope assumptions, none of which have anything to do with the electron mass. Do you see a connection with the Chandrasekhar mass?

58.

a. Use the general (!) zero-temperature equation of state for a degenerate electron gas at any density to derive numerically the mass-radius relationship for a white dwarf with $Y_e = 0.5$. The general expression is:

$$P = \kappa \rho^{4/3} I(x) \tag{5}$$

$$I(x) = \frac{3}{2x^4}(x(1+x^2)^{1/2}(\frac{2x^2}{3} - 1) + \ln(x + (1+x^2)^{1/2}))$$

$$x = p_{\mathrm{F}}/m_e c\,, \tag{6}$$

where κ contains the $\frac{1}{\mu_e^{4/3}}$. This involves writing a code to solve the equations of hydrostatic equilibrium. Demonstrate your results with a plot. (*Hint: The Runge-Kutta method for the solution of simultaneous O.D.E.s generally works well.*)

b. Show that the non-relativisitic mass-radius relation for an $n = 1.5$ polytrope ($P \propto \rho^{1+\frac{1}{n}}$) is one limit of your results.

c. Determine how well the approximation to the "Chandrasekhar-mass-adjusted" radius-mass relation for white dwarfs works by calculating the fractional deviation ($\Delta R/R$) of the fit from the correct result.