HAYASHI LIMIT

Let us consider a simple "model atmosphere" of a star. The equation of hydrostatic equilibrium and the definition of optical depth are

$$\frac{dP}{dr} = -g\rho, \qquad \frac{d\tau}{dr} = -\kappa\rho, \tag{s2.24}$$

and may be combined to write

$$\frac{dP}{d\tau} = \frac{g}{\kappa}. \tag{s2.25}$$

Assuming $\kappa = const$ we may integrate this equation to obtain

$$P_{\tau=2/3} = \frac{2}{3}\frac{g}{\kappa_{\tau=2/3}}, \qquad g \equiv \frac{GM}{R^2}, \tag{s2.26}$$

where the subscript $\tau = 2/3$ indicates that we evaluate the particular quantity at the photosphere. Let us consider a cool star with the negative hydrogen ion H^- dominating opacity in the atmosphere. When temperature is low we may neglect radiation pressure in the atmosphere. Adopting

$$P = \frac{k}{\mu H}\rho T, \qquad \kappa = \kappa_0 \rho^{0.5} T^{7.7}, \qquad \kappa_0 = 10^{-25} Z^{0.5}, \tag{s2.27}$$

we may write the equation (s2.26) as

$$\frac{k}{\mu H}\rho T = \frac{2}{3}\frac{1}{\kappa_0 \rho^{0.5} T^{7.7}}\frac{GM}{R^2}, \tag{s2.28}$$

which may be rearranged to have

$$\rho^{1.5} T^{8.7} = \frac{2}{3}\frac{\mu H}{k}\frac{G}{\kappa_0}\frac{M}{R^2}. \tag{s2.29}$$

We know that a star with the H^- opacity in the atmosphere becomes convective below optical depth $\tau = 0.775$, i.e. very close to the photosphere. Let us suppose that the convection extends all the way to the stellar center, and let us ignore here all complications due to hydrogen and helium ionization. Convective star is adiabatic, and if it is made of a perfect gas with the equation of state (s2.27) then it is a polytrope with an index $n = 1.5$. Therefore, we expect a polytropic relation all the way from the photosphere down to the center, and we have

$$\frac{\rho}{T^{1.5}} = \frac{\rho_c}{T_c^{1.5}}, \qquad \rho_c = 5.99\rho_{av} = 5.99\frac{3M}{4\pi R^3}, \qquad T_c = 0.539\frac{\mu H}{k}\frac{GM}{R}. \tag{s2.30}$$

Combining equations (s2.29) and (s2.30) we obtain

$$T^{10.95} \approx \rho_c^{-1.5} T_c^{2.25}\frac{\mu H G}{k\kappa_0}\frac{M}{R^2} \approx \frac{0.10}{\kappa_0}\left(\frac{\mu H G}{k}\right)^{3.25} M^{1.75} R^{0.25}. \tag{s2.31}$$

Let us make an approximation that convection begins at the photosphere, i.e. at $T = T_{eff}$, and let us replace stellar radius with the combination of effective temperature and luminosity according to $L = 4\pi R^2 \sigma T_{eff}^4$:

$$T_{eff}^{11.45} \approx \frac{0.07}{\kappa_0 \sigma^{1/8}}\left(\frac{\mu H G}{k}\right)^{3.25} M^{1.75} L^{1/8}, \tag{s2.32a}$$

$$T_{eff} \approx 2 \times 10^3 \left(\frac{M}{M_\odot}\right)^{0.15}\left(\frac{L}{L_\odot}\right)^{0.01}\left(\frac{Z}{0.02}\right)^{-0.04}, \tag{s2.32b}$$

To the right of the Hayashi limit no stars in a hydrostatic equilibrium can exist.