Nuclear Masses and Mass Excess: Q values for Nuclear Reactions

Nuclear masses for a given atomic weight (A) and nuclear charge/atomic number (Z) can be approximated by the **semi-empirical mass formula**

$$\begin{aligned} M(A,Z) &= (A-Z)m_n + Z(m_p+m_e) \\ &\quad - a_1 A + a_2 A^{2/3} + a_3\frac{(A/2-Z)^2}{A} + a_4\frac{Z^2}{A^{1/3}} + a_5\frac{\delta}{A^{3/4}}\,. \end{aligned} \tag{1}$$

Note that implicit in eq. (1) is that the mass density of the nucleus is a constant ($\sim 2.6\times10^{14}$ g cm^{-3}) and that the radius of the spherical nucleus is $\sim$1.2 $\times$ 10^{-13} cm (= 1.2 fermi). The first term ($(A-Z)m_n + Z(m_p+m_e)$) is just the rest masses of the nucleons (and electrons), the second is the **bulk** nuclear term proportional to the number of nucleons in the nucleus, the third term is the bulk **"surface tension"** term proportional to the surface area of the nucleus, the fourth term is the **"symmetry energy"** term that is proportional to the neutron-proton asymmetry in the nucleus, the fifth term is the **Coulomb** term ($\propto Z^2e^2/a$), and the last term is the **pairing** term. This formula was originally inspired by the **"liquid drop"** model of the nucleus and is quite classical and empirical. The original theory of nuclear fission involved the relative values of the surface tension and Coulomb terms. It was determined that a nucleus would fission in twain if Z^2/A was greater than roughly $\sim$45. For values of this *fission parameter* lower than $\sim$37, the mean time for spontaneous fission is longer than the age of the Universe.

The difference between the **atomic mass** and the **nuclear mass** is the atomic binding energy of the electrons, plus the rest mass of the electron. Take care to determine which mass you use from tables in the literature. Generally, the binding energy of the electrons in the atom ($\sim$eVs to keVs) are too small to compete with the nuclear terms ($\sim$MeVs) and you needn't worry, but for nuclei with the largest atomic numbers the atomic correction can be $\sim$50-100 keV. In a strong or electromagnetic nuclear reaction, since total nuclear charge is conserved, the Zm_e term in eq. (1) drops out when determining Q values. For a weak interaction, since the nuclear charge changes, the extra m_e must be accounted for explicitly.

More specifically, the coefficients in eq. (1) have the following approximate values measured in MeV/c^2:

$$a_1 = 15.53,\ a_2 = 17.804,\ a_3 = 94.77,\ a_4 = 0.7103,\ a_5 = 33.6.$$

The nucleon and electron masses (in MeV) are: $m_n \approx 939.57$, $m_p \approx 938.27$, and $m_e \approx 0.511$ MeV/c^2. The term in a_1 represents an increase in the binding energy (*i.e.*, decrease in nuclear mass) due to nearest-neighbor interactions between nucleons: to lowest order, nuclei are rather like drops of liquid, in which the interactions are very short range, and are attractive at low pressure, but are strongly repulsive under compression. Consequently, the liquid prefers to maintain a roughly constant density and the volume of the nucleus $\propto A$. Just as in most liquids, there is a positive energy associated with the surface area of the drop because particles near the surface have fewer neighbors to bond with: this is represented by the positive term $a_2A^{2/3}$. The next term (in a_3) favors comparable numbers of neutrons $(A-Z)$ and protons (Z). However, the term in a_4 is the electrostatic energy of the protons, which causes a shift toward neutron-rich nuclei (*i.e.*, $A - Z > Z$) with increasing atomic number. The last term reflects an attractive pairing between like nucleons: $\delta = -1$ if the number of protons and the number of neutrons are both even, $\delta = 0$ if A or Z but not both is odd, and $\delta = +1$ if both are odd. Nuclei with larger binding energies per nucleon

$$\frac{B}{A} \equiv -\frac{M(A,Z) - (A-Z)m_n - Z(m_p + m_e)}{A} \tag{2}$$

tend of course to be stabler. ^{56}Fe is the most bound common nucleus, ^{56}Ni is the most bound nucleus for symmetric $(A - Z = Z)$ nuclei, and ^{62}Ni is the most bound nucleus of all. Notice that all these “most bound nuclei” are at the **“iron peak.”**

Q Values

The Q value of a nuclear reaction is the difference between the sum of the masses of the initial reactants and the sum of the masses of the final products, in energy units (usually in MeV). This is also the corresponding difference of the binding energies of the nuclei (not per nucleon), since nucleon number is conserved in a reaction. The masses may be provided in a table of **mass excesses** ($\Delta M(A, Z)$), which is the value of $M(A, Z) - Am_u$ (usually in MeV), but relative to the corresponding number for the isotope ^{12}C. ($m_u \equiv m_{amu}$.) A useful table of mass excesses can be found in Clayton's book, p. 289 (his Table 4.1). Hence, the "mass excess" of ^{12}C is defined to be zero. For instance, for the reaction $^{12}\mathrm{C}(\alpha, \gamma)\,^{16}\mathrm{O}$, the Q value is

$$\begin{aligned} Q &= 931.478 \text{ MeV } \left(M(^{12}\mathrm{C}) + M(^{4}\mathrm{He}) - M(^{16}\mathrm{O})\right) \\ &= 7.1613 \text{ MeV}\,, \end{aligned} \tag{3}$$

where the mass excess of ^{16}O is negative (oxygen is more bound than carbon). For the triple-α process, one finds $Q/A = 0.606$ MeV/nucleon, for hydrogen burning it is 26.73 MeV, or $\sim$6.7 MeV per nucleon. Note that 1 MeV/nucleon is equivalent to 0.965×10^{18} erg g^{-1}, a very useful conversion factor. Note also that for hydrogen burning the efficiency of conversion of mass into energy is **$\sim$6.7/931$\approx$0.007**, less than but near 1%. This is the core fact of fusion as the source of energy for stars.

Magic Numbers and Abundances

There is a pronounced tendency for elements with odd Z to be less abundant than those with even Z, reflecting the influence of the last term in eq. (1). Although it is not accounted for in that formula, nuclei whose Z or $A-Z$ is one of the **magic numbers** $2, 8, 20, 28, 50, 82, 126$ are more strongly bound than their neighbors, and doubly magic nuclei such as ^{4}He, ^{16}O ($Z=8$), ^{40}Ca ($Z=20$), and ^{208}Pb ($Z=82$) are particularly strongly bound, because they represent closed shells of nucleons orbiting within the mean-field nuclear potential. These "magic" isotopes are also more abundant than their neighbors. So the binding energy per nucleon clearly has some explanatory power. Note that the dominate stable isotopes tend to be the "α-nuclei" up to ^{40}Ca, after which they trend to more neutron-rich species. This is due to the increasing importance of the Coulomb term, which disfavors protons and requires that there be more neutrons as glue. The region in the (A, Z) plane of stability against nucleon (β) decay is called the **valley of beta stability**.

The reason that the heavier nuclei are less abundant is of course that the Coulomb barrier makes them difficult to form by fusion except at extreme temperatures. In the cores of massive stars at the end of their evolution, the temperature approaches $10^{10}\,\mathrm{K} \approx 1\,\mathrm{MeV}/k_{\mathrm{B}}$, so that elements up to the iron group (meaning $24 \leq Z \leq 28$, chromium through nickel) do form. Since these cores are strongly gravitationally bound, much of the heavier elements remain locked up in neutron stars or black holes, but some is returned to the interstellar medium in supernovae.

In general, the elements and isotopes of Nature are produced predominantly in specific environments, by specific processes. These include the **Big Bang**, **cosmic-ray spallation**, the **triple-α** process and $^{12}\mathrm{C}(\alpha,\gamma)\,^{16}\mathrm{O}$, the **CNO** cycle (e.g., ^{14}N), **supernovae**, the **s-process**, the **r-process**, the **p-process**, the **rp-process**, and the **ν-process**. Can you identifiy which isotopes come from which processes?