SIMPLE ENERGETICS OF STARS: VIRIAL THEOREM, etc.

Gravitational energy and hydrostatic equilibrium

We shall consider stars in a hydrostatic equilibrium, but not necessarily in a thermal equilibrium. Let us define some terms:

$$U = \text{kinetic, or in general internal energy density} \qquad [\text{erg cm}^{-3}], \tag{eql.1a}$$

$$u \equiv \frac{U}{\rho} \qquad \left[\text{erg g}^{-1}\right], \tag{eql.1b}$$

$$E_{th} \equiv \int_0^R U 4\pi r^2 dr = \int_0^M u\, dM_r = \text{thermal energy of a star}, \qquad [\text{erg}], \tag{eql.1c}$$

$$\Omega = -\int_0^M \frac{GM_r dM_r}{r} = \text{gravitational energy of a star}, \qquad [\text{erg}], \tag{eql.1d}$$

$$E_{tot} = E_{th} + \Omega = \text{total energy of a star}\ , \qquad [\text{erg}]\,. \tag{eql.1e}$$

We shall use the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho, \tag{eql.2}$$

and the relation between the mass and radius

$$\frac{dM_r}{dr} = 4\pi r^2 \rho, \tag{eql.3}$$

to find a relations between thermal and gravitational energy of a star. As we shall be changing variables many times we shall adopt a convention of using "c" as a symbol of a stellar center and the lower limit of an integral, and "s" as a symbol of a stellar surface and the upper limit of an integral. We shall be transforming an integral formula:

$$\Omega = -\int_c^s \frac{GM_r dM_r}{r} = -\int_c^s \frac{GM_r}{r} 4\pi r^2 \rho dr = -\int_c^s \frac{GM_r \rho}{r^2} 4\pi r^3 dr = \tag{eql.4}$$

$$\int_c^s \frac{dP}{dr} 4\pi r^3 dr = \int_c^s 4\pi r^3 dP = 4\pi r^3 P \Big|_c^s - \int_c^s 12\pi r^2 P dr =$$

$$4\pi r^3 P \Big|_c^s - 3\int_c^s P 4\pi r^2 dr = \Omega.$$

If we drop the first term and set $u = \frac{P}{\rho(\gamma-1)}$, we obtain the Virial Theorem ($d/dt = 0$):

$$-3\,(\gamma - 1)\, E_{th} = \Omega\,.$$

Note that the term $4\pi r^3 P\Big|_c^s$ involves only the outer boundary when $r_c = 0$. If we set $U = 3/2P$ and $\gamma = 5/3$ or $U = 3P$ and $\gamma = 4/3$, we obtain:

$$\Omega = -2 \int_c^s U 4\pi r^2 dr = -2E_{th}, \qquad \text{(NR)}, \tag{eql.5a}$$

and in the ultra-relativistic limit (UR):

$$\Omega = -\int_c^s U 4\pi r^2 dr = -E_{th}, \qquad \text{(UR)}. \tag{eql.5b}$$

These equations also give

$$E_{tot} = \Omega + E_{th} = \frac{1}{2}\Omega < 0 \qquad \text{(NR)}, \tag{eql.6a}$$

$$E_{tot} = 0 \qquad \text{(UR)}. \tag{eql.6b}$$

In general,

$$E_{tot} = \Omega + E_{th} = \Omega\,(\gamma - 4/3)\,/\,(\gamma - 1) = (4 - 3\gamma)\,E_{th} \tag{eql.6c}$$

and, hence, $\gamma = 4/3$ is special. Also, if $\gamma > 4/3$, the loss of total energy via radiation results in an *increase* in E_{th}. Therefore, stars with $L_n = L_\nu = 0$ have *negative* specific heat - energy loss $(L > 0)$ results in an increase in the average and central temperatures. This result has profound consequences for stellar evolution.

Heat balance in a star

Let us consider now the equation of heat balance for a star. It may be written as

$$\left(\frac{\partial L_r}{\partial M_r}\right)_t = \epsilon_n - \epsilon_\nu - T\left(\frac{\partial S}{\partial t}\right)_{M_r}, \tag{eql.7}$$

where ϵ_n and ϵ_ν are the heat generation and heat loss rates in nuclear reactions and in thermal neutrino emission, respectively [erg g^{-1} s^{-1}], and S is entropy per gram. We shall define nuclear, neutrino, and "gravitational" luminosities of a star as

$$L_n = \int_c^s \epsilon_n dM_r, \tag{eql.8a}$$

$$L_\nu = \int_c^s \epsilon_\nu dM_r, \tag{eql.8b}$$

$$L_g = -\int_c^s T\left(\frac{\partial S}{\partial t}\right)_{M_r} dM_r, \tag{eql.8c}$$

and the total stellar luminosity is given as

$$L = L_n - L_\nu + L_g. \tag{eql.9}$$

According to the first law of thermodynamics we have

$$TdS = d\left(\frac{U}{\rho}\right) - \frac{P}{\rho^2}d\rho = du + Pd\left(\frac{1}{\rho}\right). \qquad \text{(eql.10)}$$

It is convenient to write "gravitational" luminosity as a sum of two terms, $L_g = L_{g1} + L_{g2}$, with

$$L_{g1} = -\int\limits_c^s \left(\frac{\partial u}{\partial t}\right)_{M_r} dM_r = -\frac{d}{dt}\left[\int\limits_c^s u\ dM_r\right] = -\frac{dE_{th}}{dt}, \qquad \text{(eql.11)}$$

$$L_{g2} = \int\limits_c^s \frac{P}{\rho^2}\left(\frac{\partial \rho}{\partial t}\right)_{M_r} dM_r = -\int\limits_c^s P\left[\frac{\partial\left(1/\rho\right)}{\partial t}\right]_{M_r} dM_r. \qquad \text{(eql.12)}$$

In order to modify the last integral we should note the relation

$$\frac{1}{\rho} = \frac{4\pi}{3}\left(\frac{\partial r^3}{\partial M_r}\right)_t. \qquad \text{(eql.13)}$$

Combining equations, we obtain

$$L_{g2} = -\frac{4\pi}{3}\int\limits_c^s P\frac{\partial}{\partial t}\left(\frac{\partial r^3}{\partial M_r}\right)dM_r = -\frac{4\pi}{3}\int\limits_c^s P\frac{\partial}{\partial M_r}\left(\frac{\partial r^3}{\partial t}\right)dM_r = \qquad \text{(eql.14)}$$

$$\left[-\frac{4\pi}{3}P\frac{\partial r^3}{\partial t}\right]_c^s + \frac{4\pi}{3}\int\limits_c^s \frac{\partial P}{\partial M_r}\frac{\partial r^3}{\partial t}dM_r =$$

$$-\frac{4\pi}{3}\int\limits_c^s \frac{GM_r}{4\pi r^4}3r^2\frac{\partial r}{\partial t}dM_r = -\int\limits_c^s \frac{GM_r}{r^2}\frac{\partial r}{\partial t}dM_r =$$

$$\frac{d}{dt}\left[\int\limits_c^s \frac{GM_r dM_r}{r}\right] = -\frac{d\Omega}{dt}.$$

Combining equations (eql.11) and (eql.14) we obtain

$$L_g = -\frac{dE_{th}}{dt} - \frac{d\Omega}{dt} = -\frac{dE_{tot}}{dt}. \qquad \text{(eql.15)}$$

Note that when $L_n = L_\nu = 0$, $L_g = -\int\limits_c^s T\left(\frac{\partial S}{\partial t}\right)_{M_r} dM_r$ equals L, the stellar luminosity.