Thermal Stability of the Eddington Model

Let us apply the results of the last two sections to the Eddington model of a massive star. Throughout the stellar model we assume $\beta \equiv P_g/P = const.$, and we can obtain

$$E_{th} \equiv \int_0^R U 4\pi r^2 dr = \int_0^R (3P_r + 1.5P_g)\, 4\pi r^2 dr = (3 - 1.5\beta) \int_0^R P 4\pi r^2 dr =$$

$$-(1 - 0.5\beta)\Omega. \tag{eq.1}$$

The Eddington model is a polytrope with the index $n = 3$, for which there is a simple analytical formula for the gravitational potential energy

$$\Omega = -1.5\, \frac{GM^2}{R}, \tag{eq.2}$$

and the total energy of the star may be calculated according to

$$E_{tot} = E_{th} + \Omega = 0.5\beta\ \Omega = -0.75\ \beta\ \frac{GM^2}{R}. \tag{eq.3}$$

Combining these equations, the "gravitational" luminosity of the stars is expressed as

$$L_g = -\frac{dE_{tot}}{dt} = -0.75\beta\, \frac{GM^2}{R^2} \frac{dR}{dt}. \tag{eq.4}$$

We know that the luminosity radiated away from the stellar surface depends on its mass only (as long as the Eddington model applies), and it is given as

$$\frac{M}{M_\odot} = \frac{18.1}{\mu^2} \frac{(L/L_{Edd})^{1/2}}{(1 - L/L_{Edd})^2}, \qquad L_{Edd} \equiv \frac{4\pi c G M}{\kappa_e}. \tag{eq.5}$$

Adopting $\epsilon = \epsilon_0 \rho T^\nu$, with $\nu = 16$ we obtain

$$L_n = \int_0^R \epsilon_0 \rho T^\nu dM_r \sim R^{-(\nu+3)}, \tag{eq.6}$$

for a star with a fixed mass. The star is said to be on the Main Sequence when its luminosity (radiative energy losses) is balanced by the heat source due to hydrogen burning. Therefore, we may write

$$L_n = L\ \left(\frac{R_{MS}}{R}\right)^{\nu+3}, \tag{eq.7}$$

where R_{MS} is the main sequence radius of the star (it may be calculated numerically evaluating the associated integral.

Neglecting neutrino luminosity which is never important for main sequence stars, we obtain the differential equation for the time variation of the stellar radius:

$$\frac{dx}{dt} = C\left(1 - x^{\nu+3}\right), \qquad x \equiv \frac{R_{MS}}{R}. \qquad C \equiv \left(\frac{4LR_{MS}}{3GM^2\beta}\right), \tag{eq.8}$$

where the constant C depends on the stellar mass and chemical composition, but not on stellar radius. The constant C has a dimension of $[s^{-1}]$. It is customary to define the Kelvin - Helmholtz (thermal) timescale of a star as

$$\tau_{_{K-H}} \equiv \frac{GM^2}{RL}. \qquad \text{(eq.9)}$$

Apart from dimensionless factor of the order unity the constant C is of the order of $\tau_{_{K-H}}^{-1}$.

Clearly, there is an asymptotic solution of the differential equation: $x = 1$, i.e. $R = R_{_{MS}}$, i.e. the stellar radius is equal to its main sequence value. Now we may ask a question: is the main sequence star thermally stable? If we make a small perturbation, making a star slightly smaller or slightly larger, will this perturbation grow, or will it decrease with time? Let at some time t_0 the dimensionless stellar radius be $x_0 = 1 + \Delta x_0$, with $|\Delta x_0| \ll 1$. The equation for $\frac{dx}{dt}$ may be written as

$$\frac{dx}{dt} = \frac{d\Delta x}{dt} = \frac{4}{3\tau_{_{K-H}}\beta}\left[1 - (1 + \Delta x)^{\nu+3}\right] \approx -\frac{4}{3\tau_{_{K-H}}\beta}(\nu + 3)\Delta x. \qquad \text{(eq.10)}$$

This equation has the solution

$$\frac{R_{_{MS}}}{R} = 1 + \Delta x = 1 + \Delta x_0 exp\left[-\left(\frac{4(\nu + 3)}{3\tau_{_{K-H}}\beta}\right)(t - t_0)\right]. \qquad \text{(eq.11)}$$

We find that the initial disturbance Δx_0 decreases exponentially with time, i.e. the Eddington model of a main sequence star is thermally stable, and the characteristic time scale on which the thermal equilibrium is restored is the Kelvin-Helmholtz timescale.