Convection

Everyone has looked through air wavering over a sun-heated road. This is convection. Air in contact with the tarmac expands, is buoyant, and rises, transferring heat upwards in the process. If water is heated *very* gently on a stove, it transfers heat by conduction, but as one turns up the heat, convection (and eventually boiling) begins. So too in stars: if the heat flux is sufficiently slight, energy transfer occurs by radiative diffusion or conduction, but larger fluxes cause convection.

The phrase "sufficiently slight" in the above paragraph can be quantified by considering the buoyancy of fluid elements in a mean temperature profile

$$\nabla \equiv \frac{d\ln T/dr}{d\ln P/dr} \equiv \frac{d\ln T}{d\ln P}.$$

Suppose a blob of fluid is displaced a small distance δr from its equilibrium position. We assume that this happens slowly enough so that the blob remains in pressure equilibrium with its surroundings, but quickly enough so that temperature equilibrium is not achieved, *i.e.* the process is adiabatic. The change in pressure experienced by the blob is

$$\delta P = \bar{P}(r+\delta r) - \bar{P}(r) \approx \frac{d\bar{P}}{dr}\delta r.$$

(We have marked the undisturbed ambient pressure with an overbar, because later on we will need to consider a turbulent situation where $\bar{P}$ represents a spatial and temporal average.) The change in density is adiabatic,

$$\delta\rho = \left(\frac{\partial\rho}{\partial P}\right)_S \delta P.$$

Because of the ambient density gradient $d\bar{\rho}/dr$, the density contrast between the blob and its new surroundings is not $\delta\rho$ but

$$\Delta\rho \equiv \delta\rho - \frac{d\bar{\rho}}{dr}\delta r.$$

Combining this with the previous two equations, we may write

$$\Delta\rho = \frac{d\bar{P}}{dr}\left[\left(\frac{\partial\rho}{\partial P}\right)_S - \frac{d\bar{\rho}}{d\bar{P}}\right]\delta r.$$

The sign of $\Delta\rho$ is critical. If it has the same sign as the displacement δr, so that a rising blob is overdense and a sinking one underdense, then the force of buoyancy will tend to push the blob back towards its equilibrium position. This case is stable. But if $\Delta\rho/\delta r < 0$, convection spontaneously develops. In fact the vertical acceleration on the blob can be obtained as follows. The ambient fluid feels a gravitational force per unit volume $-\bar{\rho}\boldsymbol{\nabla}V$, and hydrostatic equilibrium requires that this be equal to $\boldsymbol{\nabla}\bar{P}$. The displaced blob feels a total force per unit volume

$$-(\bar{\rho}+\Delta\rho)\boldsymbol{\nabla}V - \boldsymbol{\nabla}\bar{P} = -\underbrace{(\bar{\rho}\boldsymbol{\nabla}V + \boldsymbol{\nabla}\bar{P})}_{=0 \text{ in hyd. eq.}} - \Delta\rho\boldsymbol{\nabla}V = +\frac{\Delta\rho}{\bar{\rho}}\boldsymbol{\nabla}\bar{P}.$$

Since $\Delta\rho$ is already of first order of smallness we may estimate the acceleration on the blob by dividing this force per unit volume by $\bar{\rho}$ (rather than $\bar{\rho}+\Delta\rho$, which would be more accurate). Thus, finally, the net acceleration of the blob is (omitting overbars henceforth)

$$\begin{aligned} \frac{d^2\delta r}{dt^2} &= \left(\frac{1}{\rho}\frac{dP}{dr}\right)^2 \left[\left(\frac{\partial\rho}{\partial P}\right)_S - \frac{d\rho}{dP}\right]\delta r \\ &\equiv -N^2\delta r. \end{aligned} \tag{1}$$

The quantity N is called the Brunt-Väisälä frequency. It is real (*i.e.* $N^2 > 0$) if the fluid is stable to convection, and imaginary ($N^2 < 0$) if unstable. The bars have been dropped from the ambient quantities to save writing. One can write N^2 in many other forms. Since it is proportional to the difference between the actual and adiabatic density gradients, it must also be proportional to $\nabla - \nabla_{\rm ad}$. Henceforth let us assume an ideal gas. Then

$$N^2 = -\frac{P}{\rho}\left(\frac{d\ln P}{dr}\right)^2 (\nabla - \nabla_{\rm ad}) \qquad \text{(for an ideal gas).} \tag{2}$$

Note the sign: a temperature gradient steeper than adiabatic is unstable. Such a gradient is called superadiabatic, $\nabla > \nabla_{\rm ad}$.

Yet another way to write N^2, which does not depend on the ideal-gas assumption, is in terms of the gradient of the entropy per unit mass, S:

$$N^2 = -T\frac{dS}{dr}\frac{d\ln P}{dr}\nabla_{\rm ad}. \tag{3}$$

This is the so-called "Schwarzschild" condition for convection. Thus, since $\nabla_{\rm ad}$ is normally positive, instability results when lower-entropy fluid lies above higher-entropy fluid.

Our analysis is simplified. In particular, we ignored radiative diffusion and conduction within the blob. A very, *very* slightly superadiabatic gradient may be stabilized by these effects.

Mixing length theory (MLT)

The condition $N^2 < 0$ tells us that convection should occur but not how it modifies the ambient temperature profile—though presumably, it tends to reduce the superadiabaticity. For this we need a nonlinear theory. No fully adequate theory exists. Simulations have been made but do not yet provide adequate parametrizations; the problem is very hard, time-dependent and three-dimensional.

For a long time, therefore, stellar modelers have relied on the following heuristic arguments. To some extent these can be calibrated by comparison to real stars, especially the sun, and indeed by comparison to the earth's atmosphere and oceans.

In a convecting fluid, rising blobs are hotter (and therefore less dense) than the average of their surroundings, while falling blobs are cooler. Therefore, there is a positive correlation between temperature contrast and vertical velocity, $\overline{\rho v_r \Delta T} > 0$, the average being taken over position and time, and $v_r \equiv dr/dt$. Since a blob of excess temperature ΔT carries excess internal energy $\rho c_P \Delta T$ per unit volume, the correlation with v_r implies an upward energy flux (note that the rising and falling elements may contribute equally to the upward energy flux):

$$F_{\rm conv} = c_P \overline{\rho v_r \Delta T} \approx c_P \bar{\rho} \langle v_r \Delta T \rangle.$$

Here c_P is the specific heat at constant pressure, which reduces to $(5/2)k_{\rm B}/(\mu m_p)$ for a fully ionized, nondegenerate ideal gas. We have replaced ρ by its average value by replacing the volume average (the overbar) by a mass-weighted average (the angle brackets).

Careful thinkers will realize that there may be another contribution $\overline{\rho v_r^3}/2$ involving the kinetic energy of the blobs. One usually assumes that rising and falling blobs have the same kinetic energy on average (but 3D numerical simulations often disagree), in which case this contribution would vanish; in any case, it is at most of the same order as the heat-flux term.

The total energy flux is divided between convection and radiation $F = F_{\rm rad} + F_{\rm conv}$, and is constrained by the luminosity produced in the core. We ignore conduction for simplicity. It is negligible in the solar convection zone.

After rising or falling a distance δr, the blob acquires kinetic energy per unit mass

$$\frac{1}{2}v_r^2 = \int_0^{\delta r} (-N^2 \delta r')\, d(\delta r') = -\frac{1}{2}N^2(\delta r)^2,$$

if drag forces can be neglected. Thus $v_r = (-N^2)^{1/2}\delta r$ for this blob, and its temperature contrast is

$$\frac{\Delta T}{\bar{T}} = \left[\left(\frac{\partial \ln T}{\partial \ln P}\right)_{\rm ad} - \frac{d\ln T}{d\ln P}\right]\frac{\delta P}{P} = -\frac{d\ln P}{dr}(\nabla - \nabla_{\rm ad})\,\delta r.$$

As the blob accelerates, eventually turbulent drag *cannot* be neglected. The blob will be shredded by shear instabilities that cause it to merge with its surroundings and deposit its excess heat. The process is complicated, but in this simple model one assumes that all blobs accelerate unimpeded over a fixed mixing length ℓ_M before abruptly dissolving. Thus the average energy transport rate is approximately

$$\begin{aligned} F_{\rm conv} &= c_P\bar\rho\langle v_r\Delta T\rangle = c_P\bar\rho\ell_M^{-1}\int_0^{\ell_M} v_r(\delta r)\Delta T(\delta r)\,d\delta r \\ &= \frac{1}{3}c_P\bar T\bar\rho\left(\frac{\bar P}{\bar\rho}\right)^{1/2}\left(\frac{d\ln\bar P}{dr}\right)^2(\nabla-\nabla_{\rm ad})^{3/2}\,\ell_M^2. \end{aligned}$$

It is useful to rewrite the above in terms of the pressure scale height

$$H_P \equiv \left|\frac{d\ln P}{dr}\right|^{-1}, \tag{4}$$

which is the radial distance over which the pressure varies by $\sim e$. Also, since we have assumed an ideal gas, $\rho c_P T = \gamma P/(\gamma-1)$ with $\gamma \equiv c_P/c_V$; if the gas is monatomic, $\gamma = 5/3$. Then

$$F_{\rm conv} = \frac{\gamma}{3(\gamma-1)}\bar P^{3/2}\bar\rho^{-1/2}(\nabla-\nabla_{\rm ad})^{3/2}\left(\frac{\ell_M}{H_P}\right)^2 \quad \text{if } \nabla>\nabla_{\rm ad}\,; \quad \text{else } F_{\rm conv}=0. \tag{5}$$

Kippenhahn & Weigert offer a more complicated formula for $F_{\rm conv}$ that allows for radiative diffusion between the blobs and their surroundings. In view of the handwaving nature of all mixing-length arguments, however, it is not clear whether their result is more accurate than (5). In any case, all such results involve the unknown mixing length ℓ_M. It is usually assumed that $\ell_M = \alpha H_P$ with α a constant dimensionless factor of order unity. (The literature often refers to α as the "mixing length," although that term properly belongs to the dimensional quantity ℓ_M.) But there is no good reason to believe that α is a universal constant.

Equation (5) does make the reasonable prediction that $F_{\rm conv}\to 0$ as $\nabla\to\nabla_{\rm ad}$. Let us use it to investigate the superadiabatic gradient in the convection zone of the sun, which extends from $r\approx 0.713R_\odot$ to $r\approx R_\odot$. At the bottom of that zone, the temperature, density, and pressure are approximately $2.2\times10^6\,$K, $0.19\,{\rm g\,cm^{-3}}$, and $5.7\times10^{13}\,{\rm dyn\,cm^{-2}}$, respectively. Therefore the prefactor in (5) is

$$\frac{5}{6}\bar P^{3/2}\bar\rho^{-1/2}\approx 10^{21}\,{\rm erg\,cm^{-2}\,s^{-1}} \quad \text{at} \quad r=R_{\rm conv}.$$

Compare this to the actual flux $F = L_\odot/4\pi r^2 \approx 10^{11}\,{\rm erg\,cm^{-2}\,s^{-1}}$. Since part of this flux is carried by radiation, we have from (5) the upper bound

$$\nabla-\nabla_{\rm ad}\lesssim\left(\frac{10^{11}}{10^{21}}\right)^{2/3}\alpha^{-4/3}\approx 2\times10^{-7}\alpha^{-4/3}.$$

The moral here is that convection is *extremely* efficient, so that the temperature gradient is *very* close to the adiabatic gradient throughout most of the convection zones of most stars. This leads to the simple prescription stated in the notes on the stellar structure equations: At each radius r or mass fraction M_r, we first calculate ∇ as if $F_{\rm conv}=0$, yielding the result

$$\nabla_{\rm rad} = \frac{\kappa L_r}{16\pi c G M_r}\frac{3P}{aT^4}\,. \tag{6}$$

If $\nabla_{\rm rad}\le\nabla_{\rm ad}$ then we set $\nabla=\nabla_{\rm rad}$, else we set $\nabla=\nabla_{\rm ad}$.

The convective condition

$$\frac{dT}{dr}|_{star} > \left(1 - \frac{1}{\Gamma_2}\right) \frac{T}{P} \frac{dP}{dr}|_{star}$$

can be rewritten by substituting the equation for the radiative flux for the $\frac{dT}{dr}|_{star}$ term. The result is

$$\frac{3}{4ac} \frac{\kappa\rho}{T^3} \frac{L(r)}{4\pi r^2} > \left(1 - \frac{1}{\Gamma_2}\right) \frac{T}{P} \frac{dP}{dr}|_{star}$$

and using the equation of hydrstatic equilibrium and the definition $1 - \beta = P_R/P$ we obtain:

$$L(r) \leq \frac{4\pi GcM(r)}{\kappa(r)} (1 - \beta(r))\, 4 \left(1 - \frac{1}{\Gamma_2}\right) . \tag{7}$$

. This condition looks a lot like what appears in the Eddington model of stars, corrected by the $4\left(1 - \frac{1}{\Gamma_2}\right)$ term. Note the emergence of the "Eddington luminosity" as a prefactor. Importantly, this equation states that for a given luminosity if the opacity is "large," the region is convective and that there is an equation of state dependence through Γ_2. A "small" Γ_2 can help trip convection. Moreover, for a given opacity, if the luminosity is "large," the region will be convective. Equation (7) defines what "large" and "small" mean.