EQUATIONS OF STELLAR STRUCTURE

General Equations

We shall consider a spherically symmetric, self-gravitating star. All the physical quantities will depend on two independent variables: **radius** and **time**, (r, t). First, we shall derive all the equations of stellar structure in a general, non spherical case, but very quickly we shall restrict ourselves to the spherically symmetric case.

Variables: density ρ, temperature T, and chemical composition, i.e. the abundances of various elements X_i, with $i = 1, 2, 3.....$

All thermodynamic properties and transport coefficients are functions of (ρ, T, X_i). In particular we have: pressure $P(\rho, T, X_i)$, internal energy per unit volume $U(\rho, T, X_i)$, entropy per unit mass $S(\rho, T, X_i)$, coefficient of thermal conductivity per unit volume $\lambda(\rho, T, X_i)$, and heat source or heat sink per unit mass $\epsilon(\rho, T, X_i)$.

Using these quantities the first law of thermodynamics may be written as

$$T\,dS = d\left(\frac{U}{\rho}\right) - \frac{P}{\rho^2}d\rho, \tag{1}$$

If there are sources of heat (e.g., thermonuclear), ϵ, and a non-vanishing heat flux $\vec{F}$, then the heat balance equation may be written as

$$\rho T\frac{dS}{dt} = \rho\epsilon - \text{div}\,\vec{F}. \tag{2}$$

The heat flux is directly proportional to the temperature gradient:

$$\vec{F} = -\lambda\nabla T. \tag{3}$$

The equation of motion (the Navier-Stokes equation of hydrodynamics) may be written as

$$\frac{d^2\vec{r}}{dt^2} + \frac{1}{\rho}\nabla P + \nabla V = 0, \tag{4}$$

where the gravitational potential satisfies the Poisson equation

$$\nabla^2 V = 4\pi G\rho, \tag{5}$$

with $V \longrightarrow 0$ when $r \longrightarrow \infty$.

In spherical symmetry these equations may be written as

$$\frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{\partial V}{\partial r} + \frac{d^2 r}{dt^2} = 0;\ \vec{v} = \frac{d\vec{r}}{dt}, \tag{6a}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 4\pi G\rho, \tag{6b}$$

$$F = -\lambda\frac{\partial T}{\partial r}, \tag{6c}$$

$$\frac{1}{\rho r^2}\frac{\partial\left(r^2 F\right)}{\partial r} = \epsilon - T\frac{dS}{dt}, \tag{6d}$$

Also in spherical symmetry, $\nabla V = \frac{\partial V}{\partial r} = \frac{GM_r}{r^2}$, where we have introduced the new variable, M_r :

$$M_r \equiv \int_0^r 4\pi x^2 \rho dx, \tag{7}$$

which is the total mass within the radius r. Another variable, L_r, is defined as :

$$L_r \equiv 4\pi r^2 F, \tag{8}$$

which is the luminosity, i.e. the total heat flux flowing through a spherical shell with the radius r. The Rosseland opacity (per unit mass), κ, is defined through the equation

$$\kappa = \frac{4acT^3}{3\rho}\frac{1}{\lambda}, \tag{9}$$

where c is the speed of light and a is the radiation constant. The last equation is valid if the heat transport is due to radiation.

Using the definitions and relations (7-9) we may write the set of equations (6) in a more standard form:

$$\frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{GM_r}{r^2} + \frac{d^2 r}{dt^2} = 0, \tag{10a}$$

$$\frac{\partial M_r}{\partial r} = 4\pi r^2 \rho, \tag{10b}$$

$$\frac{\partial T}{\partial r} = -\frac{3\kappa\rho L_r}{16\pi acT^3 r^2} \quad ; \; F = -\frac{c\lambda_R}{3}\frac{\partial U_{\rm Rad}}{\partial r}, \tag{10c}$$

$$\frac{\partial L_r}{\partial r} = 4\pi r^2 \rho \left(\epsilon - T\frac{dS}{dt}\right), \tag{10d}$$

This system of equations is written in a somewhat inconvenient way, as all the space derivatives ($\partial/\partial r$) are taken at a fixed value of time, while all the time derivatives (d/dt) are at the fixed mass zones. For this reason, and also because of the way the boundary conditions are specified (we shall see them soon) , it is convenient to use the mass M_r rather than radius r as a space-like independent variable. Therefore, we replace all derivatives $\partial/\partial r$ with $4\pi r^2 \rho \partial/\partial M_r$, and we obtain

$$\frac{\partial P}{\partial M_r} = -\frac{GM_r}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{d^2 r}{dt^2}, \tag{11a}$$

$$\frac{\partial r}{\partial M_r} = \frac{1}{4\pi r^2 \rho}, \tag{11b}$$

$$\frac{\partial T}{\partial M_r} = -\frac{3\kappa L_r}{64\pi^2 acT^3 r^4}, \tag{11c}$$

$$\frac{\partial L_r}{\partial M_r} = \epsilon - T\frac{\partial S}{\partial t}. \tag{11d}$$

The set of equations above describes the time evolution of a spherically symmetric star with a given distribution of chemical composition with mass, $X_i(M_r)$, provided the initial conditions and the boundary conditions are specified. If the time derivative in the equation (11a) vanishes then the star is in **hydrostatic equilibrium**. If the time derivative in equation (11d) vanishes then the star is in **thermal** equilibrium. Notice that we always assume that throughout the star the matter and radiation are in **local thermodynamic equilibrium**, LTE, no matter if the star as a whole is in hydrostatic or in thermal equilibrium. From now on we shall consider stars that are in the hydrostatic equilibrium, i.e. we shall assume that the time derivative in the equation (11a) is negligibly small.

If convection obtains, $\frac{\partial T}{\partial M_r}$ is very close to the adiabatic gradient. One can incorporate this into the equations simply by setting

$$\frac{dT}{dM_r} = \frac{T}{P} \nabla_T \frac{dP}{dM_r}, \tag{12}$$

where

$$\nabla_T = \min\left(\nabla_{rad}, \nabla_{ad}\right), \tag{13}$$

$$\nabla_{rad} \equiv \frac{3\kappa L_r}{16\pi c G M_r} \frac{P}{aT^4}, \tag{14}$$

and

$$\nabla_{ad} \equiv \left(\frac{\partial \ln T}{\partial \ln P}\right)_S = \left(1 - \frac{1}{\Gamma_2}\right). \tag{15}$$

Complications

The fact that stars are luminous, i.e. they are radiating away some energy, implies that they must change in time. Indeed, it is known now that the main energy source for the stars is nuclear, and the nuclear reactions that provide heat also change chemical composition. Therefore, our stellar structure equations are incomplete. They have to be supplemented with a set of equations describing the **nuclear reaction network**, i.e. providing $\partial X_i/\partial t$ as a function of ρ, T, X_j. This will introduce a new time dependence, with its accompanying nuclear time scale.

As soon as convection develops it carries some of the heat flux, and the temperature gradient is modified. It turns out that in the deep interior of a star convection, when present, brings the temperature to the adiabatic value. However, near the surface of a star convection is not very efficient in carrying heat, and there is no good theory to calculate its efficiency. For most practical purposes astronomers use the so called **"mixing length theory"**, which parameterizes our lack of knowledge about convection with one free parameter α, which is equal to the ratio of a characteristic "mixing length" to the pressure scale height, and is usually of the order unity.

In addition to carrying heat convection mixes various stellar layers, with possibly different chemical composition. As a result chemical composition may change not only due to nuclear reactions, but also because of convective mixing. As the **mixing** is a non-local phenomenon, the solution of full stellar structure equations becomes much more complicated.

Still another physical process which is important in some stars is a **diffusion of elements** with different mean molecular weight, or different ratio of electric charge to mass, or different cross section for interaction with radiation. In some cases this process may produce chemical inhomogeneity with important consequences for stellar appearance and/or evolution.

There may be some other processes which lead to some mixing that is not important as the energy transport mechanism, but which may be important for the distribution of chemical composition. This may be **meridional circulation** induced by very rapid **rotation** of a star, or some poorly understood instabilities. A final complication is that stars have **winds** and loss mass. Clearly, stellar evolution in all its true complexity is not simple.

Boundary Conditions

We shall consider now the boundary conditions. At the stellar center the mass M_r, the radius r, and the luminosity L_r, all vanish. Therefore, we have the **inner boundary conditions**

$$r = 0, \qquad L_r = 0, \qquad\qquad \text{at } M_r = 0. \tag{16}$$

In most cases we shall be interested in structure and evolution of a star with a fixed total mass M. At the surface, where $M_r = M$, the density falls to zero, and the temperature falls to a value that is related to the stellar radius and luminosity. The proper outer boundary conditions require rather complicated calculations of a model stellar atmosphere. We shall shall adopt a very simple model atmosphere within the Eddington approximation, which means we shall use the diffusion approximation to calculate the temperature gradient not only at large optical depth, but also at small optical depth. The Eddington approximation also means that the surface temperature is $2^{1/4} \approx 1.189$ times lower than the effective temperature. The **outer boundary conditions** are

$$\rho = 0, \qquad T = T_o = \left(\frac{L}{8\pi R^2 \sigma}\right)^{1/4}, \qquad\qquad \text{at } M_r = M, \tag{17}$$

where σ is the Stefan-Boltzmann constant. Notice, that the so called effective temperature of a star is defined as

$$T_{eff} \equiv \left(\frac{L}{4\pi R^2 \sigma}\right)^{1/4} = 2^{1/4} T_o. \tag{18}$$

At the stellar center we have two adjustable parameters: the central density ρ_c, and the central temperature T_c. At the stellar surface there are other two adjustable parameters: the stellar radius R, and the stellar luminosity L. These four parameters may be calculated when the differential equations of stellar structure are solved. Notice, that only two of those parameters, R and L are directly observable. Also notice, that the equations for spherically symmetric stars (10 or 11) may be derived without considering the general case, but starting with simple geometry of thin, spherically symmetric shells, and balancing mass, momentum and energy across those shells.

SIMPLE ENERGETICS OF STARS: VIRIAL THEOREM, etc.

Gravitational energy and hydrostatic equilibrium

We shall consider stars in a hydrostatic equilibrium, but not necessarily in a thermal equilibrium. Let us define some terms:

$$U = \text{kinetic, or in general internal energy density} \qquad [\text{erg cm}^{-3}], \qquad \text{(eql.1a)}$$

$$u \equiv \frac{U}{\rho} \qquad \left[\text{erg g}^{-1}\right], \qquad \text{(eql.1b)}$$

$$E_{th} \equiv \int_0^R U 4\pi r^2 dr = \int_0^M u \; dM_r = \text{thermal energy of a star}, \qquad [\text{erg}], \qquad \text{(eql.1c)}$$

$$\Omega = -\int_0^M \frac{GM_r dM_r}{r} = \text{gravitational energy of a star}, \qquad [\text{erg}], \qquad \text{(eql.1d)}$$

$$E_{tot} = E_{th} + \Omega = \text{total energy of a star} \;, \qquad [\text{erg}]\,. \qquad \text{(eql.1e)}$$

We shall use the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho, \qquad \text{(eql.2)}$$

and the relation between the mass and radius

$$\frac{dM_r}{dr} = 4\pi r^2 \rho, \qquad \text{(eql.3)}$$

to find a relations between thermal and gravitational energy of a star. As we shall be changing variables many times we shall adopt a convention of using "c" as a symbol of a stellar center and the lower limit of an integral, and "s" as a symbol of a stellar surface and the upper limit of an integral. We shall be transforming an integral formula:

$$\Omega = -\int_c^s \frac{GM_r dM_r}{r} = -\int_c^s \frac{GM_r}{r} 4\pi r^2 \rho dr = -\int_c^s \frac{GM_r \rho}{r^2} 4\pi r^3 dr = \qquad \text{(eql.4)}$$

$$\int_c^s \frac{dP}{dr} 4\pi r^3 dr = \int_c^s 4\pi r^3 dP = 4\pi r^3 P \Big|_c^s - \int_c^s 12\pi r^2 P dr =$$

$$4\pi r^3 P \Big|_c^s - 3 \int_c^s P 4\pi r^2 dr = \Omega.$$

If we drop the first term and set $u = \frac{P}{\rho(\gamma-1)}$, we obtain the Virial Theorem ($d/dt = 0$):

$$-3\,(\gamma - 1)\, E_{th} = \Omega\,.$$

Note that the term $4\pi r^3 P\Big|_c^s$ involves only the outer boundary when $r_c = 0$. If we set $U = 3/2P$ and $\gamma = 5/3$ or $U = 3P$ and $\gamma = 4/3$, we obtain:

$$\Omega = -2 \int_c^s U 4\pi r^2 dr = -2E_{th}, \qquad \text{(NR)}, \tag{eql.5a}$$

and in the ultra-relativistic limit (UR):

$$\Omega = - \int_c^s U 4\pi r^2 dr = -E_{th}, \qquad \text{(UR)}. \tag{eql.5b}$$

These equations also give

$$E_{tot} = \Omega + E_{th} = \frac{1}{2}\Omega < 0 \qquad \text{(NR)}, \tag{eql.6a}$$

$$E_{tot} = 0 \qquad \text{(UR)}. \tag{eql.6b}$$

In general,

$$E_{tot} = \Omega + E_{th} = \Omega\,(\gamma - 4/3)\,/\,(\gamma - 1) = (4 - 3\gamma)\,E_{th} \tag{eql.6c}$$

and, hence, $\gamma = 4/3$ is special. Also, if $\gamma > 4/3$, the loss of total energy via radiation results in an *increase* in E_{th}. Therefore, stars with $L_n = L_\nu = 0$ have *negative* specific heat - energy loss ($L > 0$) results in an increase in the average and central temperatures. This result has profound consequences for stellar evolution.

Heat balance in a star

Let us consider now the equation of heat balance for a star. It may be written as

$$\left(\frac{\partial L_r}{\partial M_r}\right)_t = \epsilon_n - \epsilon_\nu - T\left(\frac{\partial S}{\partial t}\right)_{M_r}, \tag{eql.7}$$

where ϵ_n and ϵ_ν are the heat generation and heat loss rates in nuclear reactions and in thermal neutrino emission, respectively [erg g^{-1} s^{-1}], and S is entropy per gram. We shall define nuclear, neutrino, and "gravitational" luminosities of a star as

$$L_n = \int_c^s \epsilon_n dM_r, \tag{eql.8a}$$

$$L_\nu = \int_c^s \epsilon_\nu dM_r, \tag{eql.8b}$$

$$L_g = -\int_c^s T\left(\frac{\partial S}{\partial t}\right)_{M_r} dM_r, \tag{eql.8c}$$

and the total stellar luminosity is given as

$$L = L_n - L_\nu + L_g. \tag{eql.9}$$

According to the first law of thermodynamics we have

$$TdS = d\left(\frac{U}{\rho}\right) - \frac{P}{\rho^2}d\rho = du + Pd\left(\frac{1}{\rho}\right). \qquad \text{(eql.10)}$$

It is convenient to write "gravitational" luminosity as a sum of two terms, $L_g = L_{g1} + L_{g2}$, with

$$L_{g1} = -\int_c^s \left(\frac{\partial u}{\partial t}\right)_{M_r} dM_r = -\frac{d}{dt}\left[\int_c^s u\, dM_r\right] = -\frac{dE_{th}}{dt}, \qquad \text{(eql.11)}$$

$$L_{g2} = \int_c^s \frac{P}{\rho^2}\left(\frac{\partial \rho}{\partial t}\right)_{M_r} dM_r = -\int_c^s P\left[\frac{\partial\,(1/\rho)}{\partial t}\right]_{M_r} dM_r. \qquad \text{(eql.12)}$$

In order to modify the last integral we should note the relation

$$\frac{1}{\rho} = \frac{4\pi}{3}\left(\frac{\partial r^3}{\partial M_r}\right)_t. \qquad \text{(eql.13)}$$

Combining equations, we obtain

$$L_{g2} = -\frac{4\pi}{3}\int_c^s P\frac{\partial}{\partial t}\left(\frac{\partial r^3}{\partial M_r}\right) dM_r = -\frac{4\pi}{3}\int_c^s P\frac{\partial}{\partial M_r}\left(\frac{\partial r^3}{\partial t}\right) dM_r = \qquad \text{(eql.14)}$$

$$\left[-\frac{4\pi}{3}P\frac{\partial r^3}{\partial t}\right]_c^s + \frac{4\pi}{3}\int_c^s \frac{\partial P}{\partial M_r}\frac{\partial r^3}{\partial t}dM_r =$$

$$-\frac{4\pi}{3}\int_c^s \frac{GM_r}{4\pi r^4}3r^2\frac{\partial r}{\partial t}dM_r = -\int_c^s \frac{GM_r}{r^2}\frac{\partial r}{\partial t}dM_r =$$

$$\frac{d}{dt}\left[\int_c^s \frac{GM_r dM_r}{r}\right] = -\frac{d\Omega}{dt}.$$

Combining equations (eql.11) and (eql.14) we obtain

$$L_g = -\frac{dE_{th}}{dt} - \frac{d\Omega}{dt} = -\frac{dE_{tot}}{dt}. \qquad \text{(eql.15)}$$

Note that when $L_n = L_\nu = 0$, $L_g = -\int_c^s T\left(\frac{\partial S}{\partial t}\right)_{M_r} dM_r$ equals L, the stellar luminosity.