Some of these questions are a bit harder than the average exam question, be it the final or the mid-term. However, if you can do and understand these questions, the final, in particular, should be a breeze.

1. If a spherical planet has a constant density (ρ_0) and is in hydrostatic equilibrium, one can straighforwardly find its pressure profile, and in particular its central pressure. (a) First, what is the relationship between its total mass (M) and (total) radius (R_p)? (b) Derive its pressure−radius profile ($P(r)$). (c) What is its central pressure in terms of M and R_p? (d) For the Earth, what would be the central pressure (in Megabars; 1 Megabar $\equiv 10^6$ bars; 1 bar $\equiv 10^6$ dynes cm^{-2})?

2. A planet has a Bond albedo of $\sim$0.3 and executes a circular orbit of radius (semi-major axis) a around a G2V star. (a) If it reradiates the energy it absorbs from the star isotropically, what is its effective temperature (T_p) as a function of a, if a is in AU? For what range of values of a will T_p be between the boiling and melting points of water? This orbital range is called the "habitable zone."

3. A new telescope can detect an object in the visible down to 30th magnitude in V. If the Sun's absolute V magnitude is 4.6, out to what distance in parsecs can this telescope see a sun-like star?

4. The Kelvin-Helmholtz timescale (τ_{KH}) is an important characteristic time for stellar evolution, and, hence, is worth exploring. (a) If stars along the main sequence obey the relationships: $L \propto M^3$ and $R \propto M^{0.8}$, what is the mass-dependence of τ_{KH}? (b) Calculate τ_{KH} for the Sun. (c) Using the results in (a) and (b), determine τ_{KH} for a 3-solar-mass star on the main sequence. Comment.

5. An interstellar cloud of thickness Δx is comprised solely of atomic hydrogen at a temperature T and number density n_H. We are interested in how opaque the cloud is to Hα photons. Assuming that the absorption optical depth is proportional to the fraction of the atoms in the $n = 2$ state of hydrogen, what is the ratio of the optical depth of such a cloud at $T = 1000$ K and $T = 100$ K? [*Hint: You don't need the Hα absorption cross section to solve this.*]

6. A Jovian-mass planet orbits a sun-like star with a period of 10 days and no eccentricity. (a) If the orbital plane is edge-on to us, what is maximum Doppler shift ($\Delta\nu/\nu_0$) of spectral lines from the planet? (b) Find the same quantity for spectral lines from the star? [*Hint: First determine aspects of the orbit, and then use them to determine the Doppler shift. Note that the mass of the planet is much smaller than the mass of the star, but that the star too executes an orbit.*]

7. In this problem, we will explicitly derive the relationship between scale factor and time in a critical density universe, treating it as a Newtonian problem. The (positive) kinetic energy of a galaxy a distance r from us is equal and opposite to the (negative) gravitational potential energy due to the gravitational pull of the mass within the radius r. Using this fact, write down a differential equation for the relation between radius and time, and solve it. Use this to get an exact expression for the present age of the universe, expressed in terms of the Hubble Constant.

8. We have seen that the luminosity of stars on the main sequence scales roughly as the cube of their mass, $L \propto M^3$. Assume that the number of stars on the main sequence per unit mass (which is called the Initial Mass Function) scales as the -1.5 power of the mass; that is, the number of stars per unit volume in the disk of the Milky Way with mass between M and $M + dM$ is proportional to $M^{-1.5}dM$. Assume that each of these relationships holds true exactly for stars between 0.1 and 60 solar masses, and no stars exist for any other masses. Moreover, assume for simplicity that all stars are main sequence stars.

Calculate the mass to light ratio (i.e., the ratio of total mass of stars per unit volume to total luminosity of stars per unit volume), and express it in solar units (solar masses divided by solar luminosities).

9. Consider two spherical, self-gravitating objects of the same uniform density throughout, and radii r and R, respectively; one is much larger than the other ($r \ll R$). Consider the tidal force (i.e., difference in gravitational force between the near and far side of the object) on the small object due to the gravitational force of the large object.

a. Compare with the self-gravity of the small object, and derive a condition on the distance between the two objects such that the small object remains intact. What you have calculated is termed the "Roche limit."

b. (XXX points) Calculate approximately the radius of the Roche limit around the Earth. Do we, sitting on the surface of the Earth, lie inside the Roche limit? If so, how is it that we remain intact?

c. The star τ Boötes has a mass 1.25 times that of the Sun; the radius is 1.25 times that of the Sun as well. Calculate the radius of its Roche limit. It has a planet, discovered by Marcy and Butler, in an orbit with semi-major axis 0.045 AU. Does this lie inside or outside the Roche limit?

10. a. Calculate the tidal force (i.e., the difference in gravitational force between the near and far side) of an object of length r falling into a black hole of mass M, when that object lies at the Schwarzschild radius R_S ($= 2GM/c^2$). Assume $r \ll R_S$. Is this tidal force larger for a big black hole or a small black hole?

b. Calculate the Schwarzschild radius for a black hole large enough to power a quasar ($10^9\, M_\odot$). Calculate the tidal force on the Sun if were unlucky enough to fall into such a black hole.

c. Would the Sun be torn apart by the black hole's tidal forces before it fell in? Can you conclude therefore that you can survive falling into such a massive black hole?

11. Consider a spherically symmetric star in which the density as a function of radius scales as:

$$\rho(r) = \rho_0 \left(1 - \frac{r}{R_p}\right),$$

where ρ_0 is a constant and R_p is the radius of the star.

a. Calculate the total mass of the star.

b. Calculate the pressure of the star at the center.

12. a. Calculate the rate at which the Sun is losing mass due to the burning of hydrogen in its core (and the subsequent conversion of mass into energy). Over the lifetime of the Sun, what fraction of its mass will be lost to energy?

b. Consider a star of mass 20 times that of the Sun. Will the fraction of its mass lost to energy be larger or smaller than that of the Sun? To answer this, do a crude, order-of-magnitude estimate of the fraction of mass lost as a function of mass.

c. A wind of particles, termed the *solar wind*, is seen streaming radially from the Sun. It is made mostly of protons and electrons, and at the radius of the Earth's orbit, has a characteristic speed of 500 km/s, and a density of 7 protons/cm^3. Assuming that the solar wind is emitted uniformly from the surface of the sun, calculate the rate at which the Sun is losing mass due to this process. Assuming that the rate at which material is lost to the solar wind is constant, calculate the fraction of the mass of the Sun lost over its lifetime. Is this fraction larger or smaller than that you calculated in part (a)?

13. The interstellar medium has a mean density of about 0.5 H atoms/cm^3. What is the radius of the spherical volume of the ISM that you'd need to collapse to form the Sun? If the rotation speed around the Galactic center at the Sun's distance (8.5 kpc) is 220 km/sec, what is the angular momentum of this protosolar cloud due to differential galactic rotation? Compare this to the angular momentum of the Sun and of the Solar System.

14. For the sake of argument, assume that every star has exactly one planet with the same mass and composition as the Earth, and that the planet's orbital distance is equally likely to have any value between 0.01 and 100 AU. Define the inner and outer radii of the habitable zone as the freezing and boiling points of water. Star A has twice the Sun's mass, and star B has half the Sun's mass. (a) Which is more likely to have a habitable planet, A or B ? Why? (b) If the probability that intelligent life arises on a habitable planet is proportional to the lifetime of its star, which of these stars is more likely to support intelligent life? [*Hint: Assume an appropriate dependence of stellar luminosity on mass. Remember that the luminosity of a star increases very rapidly with mass.*]

15. How do we detect the presence of neutral atomic hydrogen in the Universe? Describe the physics of the effect that is used in the detection method in the radio band.

16. Discuss the critical role of quantum effects for the nuclear fusion inside stars.

17. Describe the evolutionary path of a solar-mass star. Discuss which conditions are necessary for the occurence of a helium flash.

18. Describe the physics that goes into setting an upper limit on the mass of a white dwarf.

19. A bright star is observed to orbit a supermassive black hole in a circular orbit with a period of 20 years. The orbital plane is inclined with respect to the plane of the sky by 30°. Observations of various lines in the stellar spectrum show that the maximum line-of-sight velocity exhibited by the star in the course of its orbital motion is 1200 km s^{-1}. The angular distance between the most separated points of the stellar orbit is 0.21 arcseconds. Determine the mass of the black hole and how far away it is from us based on these data.

20. Calculate the amount of mass the Sun loses per second as a result of thermonuclear burning in its interior. Compare this number with the amount of mass the Sun loses per second in the form of solar wind. The solar wind consists of protons and at Earth's orbit it has a number density of 10 particles cm^{-3} and velocity 400 km s^{-1}. Assume that the wind is spherically symmetric.

21. Small dust grains around stars are affected not only by the stellar gravity but also by the pressure exerted by stellar radiation. Determine at what size (radius) a spherical dust grain gets pushed away from a 1 $L_\odot$ star by radiation pressure if it absorbs all infalling radiation with cross section equal to its geometric cross section? The density of the grain material is 2 g cm^{-3}, and the mass of the star is 1 $M_\odot$.

22. The stellar mass function dN/dM is the distribution of the number of stars as a function of their initial mass (number of stars dN per interval of initial mass dM). Observations show that the mass function can be reasonably well fit by the following expression:

$$\frac{dN}{dM} \propto M^{-2.35}$$

for 0.5 $M_\odot$ < M < 100 $M_\odot$ and $dN/dM = 0$ outside of this mass range (the so-called Salpeter mass function). (a) What is the mean stellar mass for such a distribution? (b) Let's assume that all stars with initial mass 10 $M_\odot$ < M < 30 $M_\odot$ end up producing a neutron star at the end of their life. Calculate the total number of neutron stars in our Galaxy if all the stars in it were born 4 billion years ago with this mass function. Assume that the initial number of stars was 10^{11}.

23. Calculate the parallax of a star located 50 pc away from us.

24. A giant star expands increasing its radius by a factor of 10. Assuming that its effective temperature stays the same at all times, what is the change in the apparent magnitude of the star during expansion (it is important that you get not only the magnitude but also the sign of the change right!)?

25. What would be the orbital period of a planet moving around the Sun if its semimajor axis were 4 AU?

26. Assuming that the current rate of hydrogen fusion in the Sun remains constant, what fraction of the Sun's mass will be converted into helium over the next 5 billion years? How will this affect the overall chemical composition of the Sun?

27. The star Krüger 60B in the constellation Cepheus has an apparent bolometric magnitude of 11.3 and a parallax of 0.25 arcseconds.

a. Determine its absolute magnitude.

b. What is the ratio of the luminosity of Krüger 60B to the Sun's luminosity?

28. Consider a planet of constant density (mass per unit volume) ρ and radius R.

a. What is the mass of the planet?

b. What is the gravitational acceleration at distances $r > R$ (measured from the center of the planet)?

c. What is the gravitational acceleration at distances $r < R$?

d. What is the period of an orbit that skims the surface of the planet? Express your answer in terms of Newton's G and ρ. Assume the orbiting object is a "test particle" of negligible mass.

e. Imagine a tunnel that makes a circle around the planet with radius (of the circle, not the tunnel passageway, which is very thin) $r < R$. What is the orbital period inside such a tunnel?

29. a. Calculate the nuclear energy released in a Type Ia supernova. The energy released from nuclear burning of Carbon to the Iron peak is about $0.001 m_p c^2$ per baryon (p^+ or n), where m_p is the proton mass. You can assume the WD is $1.3 M_\odot$ of pure Carbon and burns completely.

b. Calculate the gravitational binding energy of a $1.3 M_\odot$ white dwarf. Assume that the radius of the white dwarf is 4000 km.

c. Is there enough energy from nuclear burning to unbind the white dwarf?

30. Translate the peak bolometric luminosity of a Type Ia, M = -20.5, into ergs/s.

31. This problem is to gain familiarity with the expression for the spectrum of black-body radiation,

$$B_\lambda(T) = \frac{2\,hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \,.$$

a. (XXX points). Calculate the wavelength at which the black-body function peaks, as a function of temperature. [*Hint: This will require solving one equation numerically; you can do so with a pocket calculator, and only one or two decimal places of precision are required.*]

b. (XXX points). Derive the corresponding spectrum $B_\nu(T)$, the flux per unit *frequency*. Calculate the frequency at which this spectrum peaks.

c. (XXX points). Are the peak wavelength λ_{max} and peak frequency ν_{max} you've calculated in parts (a) and (b) related by $\lambda_{max}\nu_{max} = c$? Why or why not?

d. (XXX points). Consider two black-bodies of the same size, one radiating at a higher temperature than another. Prove that the hotter one gives off more black-body radiation per unit wavelength than the cooler one, *at any wavelength.* [*Hint: Consider the derivative of the black-body function with respect to temperature.*]

e. (XXX points). The Moon shines brightly in the night sky, clearly radiating profusely in the visible part of the spectrum. From this statement, what can you infer about its surface temperature? Explain your answer in full. [*Hint: This is a bit of a trick question.*]

f. (XXX points). Calculate approximately the rate at which your body loses energy due to black-body radiation. Explain your assumptions clearly. Express your answer in watts (joules/sec); remember that a typical lightbulb consumes 100 watts of power.

32. The average person eats of order 2000 (kilo)calories of food a day; one (kilo)calorie is roughly 4200 joules. What, therefore, is your power intake? Do you eat enough to keep yourself at a constant temperature? Explain your answer.

33. The formula for a black-body spectral distribution is:

$$B_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} \,.$$

However, if $h\nu/kT >> 1$, we have the Wien distribution. Assuming this for all νs (an approximation, to be sure), what is the average energy (!) of a photon in terms of kT? [*Hint: B_ν is an energy spectrum. What is the number spectrum? How do you define the average energy? Do various constants drop out?*]

34. Argue that the luminosity of a star is given by its thermal energy content divided by the random walk time for a photon to diffuse from the center to the edge of the star. Write a general expression for this luminosity in terms of the radius and mean temperature of the star, assuming (very roughly) that the temperature is uniform throughout. For the Sun, assume a mean interior temperature of $T = 4.5 \times 10^6\,^\circ$K, and a mean free path between absorptions of $l = 0.5$cm. Calculate the luminosity of the Sun, and compare with the observed value of $4 \times 10^{33}\,\mathrm{erg\,s^{-1}}$.

35. Use general scaling arguments to determine how the internal pressure of a star depends on its mass and radius. Only proportionalities are needed here; there is no need to work out the constants. [*Hint: A pressure is a force per unit area.*] Similarly, determine how the mean density of the star scales with mass and radius.

36. For stars of low to high mass, the perfect gas law gives a relationship between the pressure, temperature and density. For very high mass stars, radiation pressure dominates; this pressure is proportional to the energy density of photons. Use this fact, and the proportionalities you've derived from the equations of stellar structure to show that for main-sequence stars:

a. $L \propto M^{5.5}/R^{0.5}$ for stars with low to medium mass;

b. $L \propto M^3$ for stars with high mass;

c. $L \propto M$ for stars with very high mass.

37. An O star lights up at the center of an HI region with particle number density $n_H = 5 \times 10^3$ cm^{-3}. The star produces $\dot{N}_{OB} = 3 \times 10^{48}$ ionizing photons per second, and the recombination coefficient (α) is 3.1×10^{-13} cm^3 s^{-1}. A photoionization front propagates into the HI region until the size of ionized region reaches equilibrium. What is this equilibrium size? How long does it take for the ionized region to reach the equilibrium size after the emergence of an O star? [*Hint: Before the ionized region reaches equilibrium, all UV flux goes into ionization of hydrogen atoms.*]

38. Suppose you have a white dwarf of mass 1 $M_\odot$, radius the same as the Earth's, and composed of pure hydrogen. If its initial temperature is 100,000 K, calculate the time it takes for the star to cool to a temperature of 5,000 K. (Assume: 1. the star stays *isothermal*, i.e. the whole star has the same temperature at any given time: 2. the star is composed of a perfect gas: 3. its radius stays constant: and 4. the star radiates like a perfect black body).

39. Follow the simple heuristic derivation of the radius-mass relationship for a white dwarf to derive the radius-mass relationship for a neutron star. Treat the neutrons as free particles, i.e. as a perfect gas of fermions, and ignore the difference between the masses of protons and neutrons. Also, assume that you can use the classical equation for hydrostatic equilibrium. For a given mass (less than the Chandrasekhar mass limit) find an expression for the ratio of the radii of a neutron star and a white dwarf. Numerically it's about 10^{-3}. Why? (one sentence, qualitative answer).

40. a. I just observed a pulsar with a period of 5 milliseconds. On the assumption that I'm seeing the rotation period of a 1 $M_\odot$ object, give simple physical arguments to show that the pulsar cannot be a normal star or even a white dwarf, but must be a neutron star.

b. The Sun's rotation period is about one rotation per 27 days. If the Sun were to shrink to a neutron star (losing no mass in the process) what would be its rotation period? Is this physically possible?

41. The smallest period known for a pulsar is 1.6 milliseconds. Assume the pulsar is a neutron star of $M = 1.4 M_{\odot}$, and radius 10 km. Can gravity hold the pulsar together? That is, can gravity supply the centrifugal force to keep material on the surface from flying off?

42. Consider the strength of the tides due to the neutron star. Imagine a cube of iron 1 cm on a side held just above the surface of the neutron star. Iron has a density $\rho \sim 8\,\mathrm{g\,cm^{-3}}$, and will rupture (be torn to shreds) if the stress (force per cross-sectional area) on it is greater than $1.5 \times 10^9\,\mathrm{dyne\,cm^{-2}}$; for stresses greater than $4.2 \times 10^8\,\mathrm{dyne\,cm^{-2}}$, it will be permanently stretched. Consider the *difference* between the gravitational force on the top and bottom of the cube, and describe what will happen to it. [*Hint: Consider the mass of the cube concentrated into two halves: one on the top of the cube and one on the bottom.*] Would the cube be more or less strong (i.e., more or less susceptible to stretching and rupture) if it were made larger?

43. Calculate the ratio of the Newtonian gravitational potential energy of this neutron star (i.e., that released when the neutron star first forms) to its rest-mass energy. You will find that the gravitational collapse of a star to form a neutron star releases a substantial fraction of its rest-mass energy. Compare this fraction with that released in the thermonuclear fusion of Hydrogen. Compare this fraction with that released in the thermonuclear fusion of all elements heavier than Hydrogen. [*Hint for study: See Fig. 10.9 of the textbook.*] Compare the energy released in the formation of a neutron star with the 10^{51} ergs of energy released in photons in a typical Type II supernova, and speculate on other avenues by which energy might be released.

44. The pulsar in the Crab nebula completes 30 revolutions per second. It is observed to be slowing down at a rate, here assumed to be constant, which would bring its spin to a halt in about 2,500 years, due to magnetic braking. It is, therefore, releasing rotational energy $I\omega^2/2$. Calculate the associated power output of the pulsar, and compare with the observed luminosity of the Crab nebula (3×10^{38} erg/sec). The moment of inertia of a uniform-density sphere, which you may assume the pulsar to be, is $I = 2MR^2/5$.

45. The rotation curves of spiral galaxies are flat as far as they can be measured, implying the existence of a dark matter halo. It is not known how far this dark matter halo extends. Imagine that the dark matter halo of the Milky Way extends half-way to the Andromeda Galaxy (which is itself 2 million light years away), so that the halos of the two galaxies just touch each other. We assume that the rotation curve remains flat to this point, so that the rotation speed at that radius is equal to that at our radius.

a. What is the total mass you infer for the Milky Way, in solar masses?

b. The mean distance between massive galaxies in the universe is 10 million light years. Assuming that they all have the same mass that you just calculated above, and they represent all the mass of the universe, what is the mean density of the universe? What is the value of the Cosmological Density Parameter Ω?

c. What volume of space at the density you calculated in part b would contain the mass of the Sun? Express your answer in terms of the side of a cube of this volume, in light-years.

46. The temperature of the microwave background is 2.73 K and it is a nearly perfect black body. Suppose the motion of the Sun around the galactic center at 220 km/sec were our only motion with respect to the microwave background. What would you measure as the temperature of the microwave background in the direction of the Sun's motion, and in the opposite direction? The observed effect is much larger than this. What might be going on?

47. Suppose the Galaxy's mass (let's say 8×10^{10} $M_\odot$) is all located at the center of the Galaxy and the Sun was 8 kpc from the center when the Galaxy formed at the beginning of the Universe. Suppose that the Galaxy has a central black hole which is turning mass into radiation at the rate of 1 $M_\odot$ yr^{-1} and the Universe is 1.3×10^{10} years old. How far from the center of the Galaxy will the Sun be today? How does this increase in the Sun's distance compare with the expansion of the Universe? (Think carefully about what is conserved in this problem, and state your assumptions carefully.)

48. The Galaxy and the Andromeda galaxy are now moving *towards* each other with a relative velocity of -130 km/sec, and are about 640 kpc apart. Assuming that the two galaxies are moving along the line joining them (i.e. have no angular momentum about each other) and that they are equal point masses (is this reasonable?), what is the total mass of the Galaxy/M31 system? If the luminosity of the Galaxy is 2×10^{10} solar luminosities, what is their mass to light ratio? State *all* your assumptions *very* carefully.

49. Suppose we live in an $\Omega > 1$ (and hence k = +1) universe, and the present radius is R. Calculate the volume of the universe [*Hint: Remember that spheres are spheres, even in curved spaces*]. Show that, if Λ is zero, no photon can circumnavigate the universe. [*Hint: Find the total comoving distance a photon moves between two times after the big bang and before the big crunch.*]

50. A cluster of galaxies has density distribution that is well approximated as a "Jaffe sphere," which has a density distribution

$$\rho(r) = \frac{\rho_\circ a^4}{r^2(r+a)^2},$$

where $\rho_\circ$ and $a = 5$ Mpc are constants. The total mass of the cluster is 10^{15} $M_\odot$. Two clouds of gas each of mass 10^8 $M_\odot$ freely fall into this cluster (from very large distance $\gg a$) on parabolic orbits and collide with each other in such a way that the resulting cloud is initially at rest at a distance of 2.5 Mpc with respect to the cluster center (obviously, the resultant cloud would subsequently fall into the cluster center, but do not worry about this stage). Determine the total amount of energy that gets emitted in the form of radiation as a result of this collision of clouds.

51. In the 1930's, Zwicky measured the velocities of galaxies moving in the Coma cluster to be $\simeq 1000\mathrm{km\,s^{-1}}$. The radius of the Coma cluster is $\simeq$ 3Mpc. Estimate the mass of the coma cluster. A rough estimate of the mass of stellar material in the cluster is $5 \times 10^{12} M_\odot$. What does this mean?

52. Consider a cloud of neutral hydrogen gas with all of the atoms in the ground state. The gas cools by collisional excitation of electrons. What is the interaction rate for *one* hydrogen atom in the cloud? Show that the cooling function (luminosity per unit volume) for the *entire* cloud is proportional to n^2, where n is the number density of atoms.

53. Consider an elliptical galaxy in equilibrium. Suppose you decrease the velocity of all of the stars by 10% and then allow the system to relax to a new equilibrium. What is the final radius of the stellar system? The final velocity dispersion?

54. A cloud of gas can be characterized by three timescales: The *dynamical time* $\tau_{dyn} \propto 1/\sqrt{G\rho}$ giving the timescale over which gravitational effects become important; the *sound crossing time* $\tau_{sct} \propto R/C_s$ (where R is the size of the system and C_s is the sound speed) giving the timescale over which sound (pressure) effects become important; and the *cooling time* $\tau_{cool} \propto nkT/\Lambda$ (where Λ is the collisional cooling luminosity per unit volume) giving the time required for a parcel of gas to radiate away all of its thermal energy.

What happens if:

a. $\tau_{dyn} \simeq \tau_{sct} \ll \tau_{cool}$

b. $\tau_{sct} \ll \tau_{dyn}, \tau_{cool}$

c. $\tau_{dyn} \ll \tau_{cool}, \tau_{sct}$

d. $\tau_{cool} \ll \tau_{sct}, \tau_{dyn}$

(Recall the dependence of the Jeans mass on temperature)

55. The velocity dispersion of galaxies in the Coma cluster is $\simeq 1000\mathrm{km\,s^{-1}}$. Find the temperature of the hot intracluster gas.

56. Derive the relation between the scale height of stars in a thin galactic disk and their vertical velocity dispersion. You'll need to know that $\phi(z) = 2\pi G\Sigma z$ where ϕ is the gravitational potential, Σ is the mass surface density of the disk and z is the height above the disk.

57. Consider a flat radiation-only universe. This describes the early evolution of our own universe. $\Omega_{r0} = 1$ and $\Omega_{m0} = \Omega_{\Lambda 0} = \Omega_{k0} = 0$. Find the evolution of the scale factor with time and the comoving size of the past light cone.

58. Consider the Milne universe, which contains no material in the form of matter, radiation, or vacuum energy. $\Omega_{r0} = \Omega_{m0} = \Omega_{\Lambda 0} = 0$, so $\Omega_{k0} = 1$. If there were no cosmological constant, then eventually our universe would evolve into the Milne universe because $\Omega_{m0} < 1$. Find the evolution of the scale factor with time and the comoving size of the past light cone.

59. Consider the anti-de Sitter universe. The AdS universe contains only vacuum energy, but the Hubble constant today is negative. Find the evolution of the scale factor with time and the comoving size of the past light cone.

60. Suppose that the universe has a large scale positive charge. Derive the Friedmann equation in a quasi-Newtonian way, like we did in class.

61. We have been saying that the timescale for the evolution of the universe is so long compared to human timescales that as far as cosmology is concerned, "today" is the same thing as "tomorrow" or "100 years from now." How long would you have to wait for the redshift of a given galaxy to change by one part in 10^6? You'll need to compute dz/dt for a given galaxy.

62. Consider a matter-only universe where $\Omega_{m0} = 1.5$. What is the eventual fate of such a universe? Are there any limits on the value of the scale factor? If so, what happens at those limits? What do they mean?

63. What is the comoving distance to the surface of last scattering at recombination, assuming a flat matter-only universe and that $z_{ls} = 1100$. What was the proper distance when recombination happened?

64. Assume that the Universe consists of only Helium. Use the Saha equation to find the redshift at which recombination occurs.

Useful Constants and Facts

Radius of Earth $= 6.4 \times 10^8$ cm

Radius of Moon $= 1.7 \times 10^8$ cm

Radius of Sun $= 6.96 \times 10^{10}$ cm

Mass of Earth $= 6 \times 10^{27}$ g

Mass of Sun $= 1.9892 \times 10^{33}$ g

Mass of Jupiter $= 1.9 \times 10^{30}$ g

Surface Temperature of Sun $= 5777$ K

1 Astronomical Unit $\sim 1.5 \times 10^{13}$ cm

1 light year $\sim 1 \times 10^{18}$ cm

1 year $\sim 3 \times 10^7$ sec

1 parsec $= 3.26$ light years

Radius of Moon's orbit around the Earth $= 3.8 \times 10^{10}$ cm

Luminosity of Sun $= 4 \times 10^{33}$ erg/s

Newton's constant $G = 6.67 \times 10^{-8}\,\mathrm{cm^3\,s^{-2}\,g^{-1}}$

Planck's Constant $h = 6.63 \times 10^{-27}$ erg·s

Boltzmann Constant $k_B = 1.38 \times 10^{-16}$ erg/K

Stephan-Boltzmann Constant $\sigma = 5.67 \times 10^{-5}\,\mathrm{erg/cm^2/s/K^4}$.

1 eV $\equiv 1.602 \times 10^{-12}$ ergs ; Mass of proton $m_p = 1.67 \times 10^{-24}$ g

Speed of light: c $= 2.99792458 \times 10^{10}$ cm s^{-1}

Hubble Constant $H_0 \approx 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$

Critical Density of Universe $\rho_{crit} \approx 10^{-29}\,\mathrm{g\,cm^{-3}}$

Temperature of Cosmic Microwave Background $= 2.735$ K

Rotation speed of the Milky Way at our radius $= 220$ km s^{-1}

Distance to the center of the Milky Way $= 8$ kpc

$[M_{Bol} = 0] \equiv [L_{Bol} = 3 \times 10^{35}$ ergs/s$]$

1 year $= 3.15 \times 10^6$ s

1 parsec $= 3.08 \times 10^{18}$ cm

1 km/s $\simeq$ 1 pc/Myr

Solar Mass $M_\odot = 1.9892 \times 10^{33}$ g

Hubble Time $(1/H_0) = 13.7$ Gyr

Hubble Radius $(c/H_0) = 4.2$ Gpc

$$\int_0^\infty x^n e^{-x} dx = n!\,;\ \frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

$$\frac{dM}{dr} = 4\pi r^2\,;\ L = 4\pi r^2 \sigma T_{\rm eff}^4$$

$$N_i \propto g_i e^{-(E_i/k_B T)}\,;\ B_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$

$$\omega^2 a^3 = G(M_1 + M_2)$$

$$\tau_{KH} = \frac{3}{10}\frac{GM^2}{RL}$$

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/kT}$$

$$\tau_{MS} = \eta\varepsilon \frac{M_\star c^2}{L_\star}$$

$$E_{th} = \frac{3}{2} N k_B T$$

$$2K + W = 0$$

$$\tau = \frac{1}{n\sigma v}$$

$$H = \frac{1}{R}\frac{dR}{dt} \quad \rho_c = \frac{3H^2}{8\pi G} \quad \Omega = \frac{\rho}{\rho_c} \quad R = \frac{1}{1+z}$$

$$H^2 = \left(\frac{1}{R}\frac{dR}{dt}\right)^2 = H_0^2\left(\frac{\Omega_{k0}}{R^2} + \frac{\Omega_{m0}}{R^3} + \frac{\Omega_{r0}}{R^4} + \Omega_{\Lambda 0}\right)$$

$$ds^2 = -c^2 dt^2 + R^2(t)(d\varpi^2 + S_k^2(\varpi)(d\,\theta^2 + \sin^2\theta\, d\phi^2))$$

$$S_k(\varpi) = D_0 \sin(\varpi/D_0) \quad (\Omega_k < 0)$$

$$S_k(\varpi) = \varpi \quad (\Omega_k = 0)$$

$$S_k(\varpi) = D_0 \sinh(\varpi/R) \quad (\Omega_k > 0)$$

$$\frac{N_{II}}{N_I} = \frac{2Z_{II}}{n_e Z_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-E_i/kT}$$