Math Tips for AST 203, Spring 2009

Appendix C of our textbook, *Cosmic Perspective*, gives a primer of some of the basic mathematics techniques we will be using in this course. It covers powers of ten and scientific notation, and how to do arithmetic with units. We urge you to study this Appendix carefully, and come to office hours if there are aspects of it which are unclear; you are responsible for all this material. In this writeup, we cover a few additional topics that will be useful in this course.

1 Significant Figures

In mathematics courses that you have taken in high school and at Princeton, you were taught to think of numbers as absolute and precise quantities. Thus, for example, if you were asked to divide 10 by 3, the correct answer is $3.3333\ldots$, or $3.\bar{3}$, where the $^-$ sign over the 3 means that the 3's continue on forever.

However, in science (and especially astronomy), numbers are often not known precisely. We refer to the number of significant figures a number has: this means the number of digits it is written with in scientific notation. For example, the number 5.2987×10^{-11} has 5 significant figures, and the number 4 ($= 4 \times 10^0$) has a single significant figure. In writing a number with a certain amount of significant figures, we are making a statement about the precision with which we know the number. That is, when we write a number like 5.2987×10^{-11}, we are saying that we have confidence it isn't as large as 5.2988×10^{-11}, or as small as 5.2986×10^{-11}. This degree of certainty or uncertainty carries through in the calculations we do with this number.

For example, suppose we are told that the nearest star is about 4 light years away, and are asked to convert this number to kilometers. Appendix A of the textbook tells us that 1 light year is about 9.46×10^{12} km, so the calculation seems straightforward:

$$4 \text{ light years} = 4 \text{ light years} \times \frac{9.46 \times 10^{12}\,\text{km}}{1\,\text{light year}} = 3.785 \times 10^{13}\,\text{km}.$$

This is what your calculator would state, but this is *not* correct. The problem is that the number of 4 light years is given to you with only a single significant figure. That is, from the statement of the problem, we only know that the distance to the nearest star is between 3.5 and 4.5 light years (i.e., between 3.3×10^{13} km and 4.3×10^{13} km). So it is misleading and wrong to give an answer with four significant figures; it implies that you know the answer much more precisely than you actually do. In this class, *points will be taken off* for giving too many significant figures.

The right way, then, to handle any calculation involving multiplication and division is to limit the precision of your answer to that of those of the input numbers with the least number of significant figures. In this case, 4 light years has exactly one significant figure, and thus the answer should have a single significant figure. The correct answer in this case is 4×10^{13} km. When you are doing arithmetic to one significant figure (as we will do a lot of in this course), things really become easier, and you are allowed to make approximations that would horrify your high-school math teacher, such as $4/3 \approx 1, 3 \times 3 \approx 10$, and so on.

Note that if I had told you that the distance to the nearest star was 4.00 light years, I would be giving you three significant figures, and you should give the resulting distance in kilometers to three significant figures.

We will find in doing astronomy that often our numbers are known rather imprecisely, to only one or two significant figures. This makes doing calculations quite a bit easier than it would be otherwise. Again, take the example of 4 light years. We know already that the final answer will have only one significant figure, so we are justified in rounding the number of kilometers per light year to a single significant figure, 1 light year $= 1 \times 10^{13}$ km. Now the calculation becomes so easy that we can do it in our head, without a calculator:

$$4 \text{ light years} = 4 \text{ light years} \times \frac{1 \times 10^{13}\,\text{km}}{1\,\text{light year}} = 4 \times 10^{13}\,\text{km}.$$

This is easier than the calculation above, less prone to error, and moreover gives us an answer with the right number of significant figures. In the solution sets, we will give many examples of doing such arithmetic without a calculator. However, for longer calculations it sometimes is a good idea to keep one extra significant figure during the calculation, and only round at the end. To take a trivial example, you might be tempted, in calculating 2.4×4, to round the first number down to 2, giving the result of 8. Doing the calculation exactly gives 9.6, and then rounding gives 10.

Let's think a bit more deeply about what the point of significant figures are: they really express the uncertainty we have in a given number. Thus suppose we're asked to add 8 and 6. The single significant figure in the two cases imply uncertainties (roughly) of about one in each case. The sum of 8 and 6 is 14; should we round the result down to a single significant figure of 10? In this case, no, because we know the uncertainty in the sum is roughly the sum of the uncertainties of the two individual numbers, i.e., about two[1]. Rounding from 14 to 10 is a change substantially larger than the uncertainty in the sum, so in this case, it is best to keep two significant figures.

We won't always write numbers in scientific notation in this course, but usually by context you can understand the number of significant figures. For example, if we tell you that a star has a surface temperature of 10,000 degrees, you can assume that this number is known to one or perhaps two significant figures, unless we explicitly tell you otherwise.

Thus as rules of thumb:

- When doing calculations with numbers with one or two significant figures, you can do

[1]There is a mathematically more precise way of combining uncertainties which gives a somewhat smaller quantity than this, but that is beyond the scope of this course.

your arithmetic without a calculator as illustrated above, and quote your answer to one (or occasionally two) significant figures. Most of the calculations you will do in this class will be of this type.

- When you have to keep track of more significant figures, it may be easiest to do the calculation "exactly" on a calculator, and then round to the appropriate number of significant figures in the end.

The above rules about significant figures are a bit trickier when dealing with subtraction. We will occasionally see problems in which two large numbers need to be subtracted from one another:

$$4000.001 - 4000.000 = 0.001$$

In this example, if we rounded the numbers off before doing the subtraction, we would find ourselves with an answer of zero, which may miss the point of the problem. In another example, near the end of this course, we will find ourselves dealing with objects moving *very* close to the speed of light, c, and we write their speed as, say, $v = 0.999999999999c$. You will of course be tempted to round this number to c, but as we'll see, the physically significant quantity (and in an interesting sense, the one we'll actually measure) is the difference between the c and the actual speed v (i.e., $10^{-12}c$ in this example), which we actually know to a single significant figure. We will give hints on how to handle such problems when they come up in the class.

2 Doing Arithmetic with Ratios

A general rule which will hold you in good stead in this course is to simplify calculations as much as possible using the techniques of algebra, *before* doing any arithmetic. Here is an example which we will see later in the course. You are asked to take the ratio of the fourth root of the luminosities of two stars, whose values are 3.2×10^{27} and 2×10^{26} Joules/sec, respectively:

$$\frac{(3.2 \times 10^{27}\,\text{Joules/sec})^{1/4}}{(2 \times 10^{26}\,\text{Joules/sec})^{1/4}}$$

At this point, you may be tempted to pull out your calculator, and find that:

$$(3.2 \times 10^{27}\,\text{Joules/sec})^{1/4} = 7.52 \times 10^{6}\,(\text{Joules/sec})^{1/4};$$

$$(2 \times 10^{26}\,Joules/sec)^{1/4} = 3.76 \times 10^{6}\,(Joules/sec)^{1/4},$$

then take their ratio, worrying terribly all along whether you're dealing with significant figures correctly, and what these strange units of $(\text{Joules/sec})^{1/4}$ mean... But life is much simpler, when you realize that the ratio of powers is the power of the ratio; that is, you can write the calculation as:

$$\left(\frac{3.2 \times 10^{27}\,\text{Joules/sec}}{2 \times 10^{26}\,\text{Joules/sec}}\right)^{1/4} = 16^{1/4} = 2.$$

Look Ma, no calculator! And the units cancelled as well. We will often set up problems to be done easily with tricks like this. Please come talk to us if the mathematical manipulations here are not clear.

In this course, we will often use ratios to do calculations with quantities that are proportional to one another. Suppose we tell you about two stars that have equal surface temperature, but star A has twice the radius of star B. We ask you to calculate the ratio of their luminosities. We will learn later that the luminosity L of a star of a given temperature is proportional to the square of its radius R:

$$L \propto R^2$$

where the $\propto$ symbol means "proportional to". What this really means is that there is a constant C, such that:

$$L = CR^2.$$

Let us use this to determine the ratio of the two:

$$\frac{L_A}{L_B} = \frac{CR_A^2}{CR_B^2} = \frac{R_A^2}{R_B^2} = \left(\frac{R_A}{R_B}\right)^2.$$

Note that the constant C cancels out here; we don't need to know it. But writing down the constant as C allowed us to remember what the proportionality meant. Also note the trick, like that described above, of turning a ratio of squares into the square of a ratio. Now we are already told that $R_A/R_B = 2$, so the calculation is easy: the ratio of the luminosities is the square of this, or 4.

There will be other such mathematical techniques we will bring up through the semester when we need them.