Homework #6, AST 203, Spring 2009

Due in class (i.e., by 4:20 pm), Thursday April 30 (last lecture of the course)

- To receive full credit, you must give the correct answer *and* show that you understand it. This requires writing your explanations in full, complete English sentences, clearly labeling all figures and graphs, showing us how you did the arithmetic, and being explicit about the units of all numbers given. All relevant mathematical variables should be explicitly defined. And please use your best handwriting; if we can't read it, we can't give you credit for it! Please staple together the sheets of paper you hand in.
- Most of the calculations in this course involve numbers that are only approximately known. The result of such a calculation should reflect this imprecision. In particular, it is *wrong* to simply write down all the digits that your calculator spits out. Your final answer should have the same number of significant figures as the least precise number going into your calculation. In many (but not all!) cases, it's best to do the problems without a calculator.
- Feel free to work with your classmates on this homework, but your write-up and wording should be your own. Answer all questions.

100 total points

1. Astronaut aging. (45 points)

In addition to the time dilation due to an object moving at a finite speed that we have learned about in special relativity, there is an effect in general relativity, termed "gravitational redshift," caused by gravity itself. To understand this latter effect, consider a photon escaping from the Earth's surface to infinity. It loses energy as it climbs out of the Earth's gravitational well. As its energy E is related to its frequency ν by Einstein's formula $E = h\nu$, its frequency must therefore also be reduced, so observers at a great distance $r = \infty$ must see clocks on the surface ticking at a lower frequency as well.

Therefore an astronaut orbiting the Earth ages differently from an astronomer sitting still far from the Earth for two reasons; the effect of gravity, and the time dilation due to motion. In this problem, you will calculate both these effects, and determine their relative importance. In parts a through c we will derive analytical expressions for each effect, simplify them in part d, and plug in numbers in part e.

a. The escape speed from an object of mass M if you are a distance r from it is given by

$$v_{escape} = \sqrt{\frac{2\,GM}{r}}.$$

That is, if you are moving this fast, you will not fall back to the object, but will escape its gravitational field entirely.

Karl Schwarzschild's solution to Einstein's field equations of general relativity shows that a stationary, non-moving clock at a radius $r \geq r_\oplus$ from the Earth will tick at a rate that is

$$\sqrt{1 - \frac{1}{c^2}\frac{2GM_\oplus}{r}} = \sqrt{1 - \frac{v_{escape}^2}{c^2}}$$

times as fast as one located far away from the Earth (i.e. at $r = \infty$). Note how much this expression looks like the equivalent expression from special relativity for time dilation. Here $r_\oplus$ is the radius of the Earth, and $M_\oplus$ is its mass. Using the formula above, calculate the rate at which a stationary clock at a radius r (for $r > r_\oplus$) will tick relative to one at the surface of the Earth. Is your rate greater or less than 1? If greater than 1, this means the high altitude clock at $r > r_\oplus$ ticks faster than one on the surface; if less than one, this means the high altitude clock ticks slower than one on the surface. (10 points)

b. Now consider an astronaut orbiting at $r > r_\oplus$. What is her orbital velocity as a function of r? Because she is moving with respect to a stationary observer at radius r, special relativity says that her clock is ticking slower. Calculate the ratio of the rate her clock ticks to that of a stationary observer at radius r. (10 points)

c. Now multiply the results of parts (a) and (b) to determine an expression for the ratio of the rate at which the orbiting astronaut's clock ticks to a stationary clock on the surface of the Earth, as a function of the radius r at which she orbits. You may ignore the small velocity of the clock on the surface of the Earth due to the Earth's rotation. (5 points) *Hint: the answer is an ugly expression, but have no fear – we will use approximations next!*

d. We'll now simplify this expression using tricks similar to those we found in Homework 5 (problem # 1). There we found that if $x \ll 1$, then $\sqrt{1-x} \approx 1 - \frac{1}{2}x$. Demonstrate that, if $x \ll 1$ and $y \ll 1$:

$$\frac{1}{1-x} \approx 1 + x$$

Hint: Multiply both sides by $1 - x$*...*
and

$$(1-x)(1-y) \approx 1 - (x+y).$$

(5 points)

e. Use the above approximations to derive a final expression, of the form $1 - \alpha$, for the relative rate of a clicking clock on the surface of the Earth and the orbiting astronaut. Demonstrate that your quantity α is indeed much smaller than 1. (8 points)

f. Using your result from (e), calculate the radius r at which the clock of the orbiting astronaut ticks at the same rate as a stationary one on the surface of the Earth; express your result in Earth radii and kilometers. Will an astronaut orbiting at a smaller radius age more or less than one who stayed home? Therefore, do astronauts on the Space Shuttle (orbiting 300 km above the Earth's surface) age more or less than one staying home? (7 points)

2. A Hitchhiker's Challenge (25 points)

> "A full set of rules [of Brockian Ultra Cricket, as played in the higher dimensions] is so massively complicated that the only time they were all bound together in a single volume they underwent gravitational collapse and became a Black Hole."
> *Chapter 17 of Life, the Universe and Everything, the third volume of the Hitchhiker's Guide to the Galaxy series (1982, Douglas Adams)*

A quote like that above is crying out for a calculation. In this problem, we will answer Adams' challenge, and determine just how complicated these rules actually are.

An object will collapse into a black hole when its radius is equal to the radius of a black hole of the same mass; under these conditions, the escape speed at its surface is the speed of light (which is in fact the defining characteristic of a black hole!). We can rephrase the above to say that an object will collapse into a black hole when its *density* is equal to the density of a black hole of the same mass.

a. Derive an expression for the density of a black hole of mass M. Treat the volume of the black hole as the volume of a sphere of radius given by the Schwarzschild radius. As the mass of a black hole gets larger, does the density grow or shrink? (5 points)

b. Determine the density of the paper making up the Cricket rule book, in units of kilograms per cubic meter. Standard paper has a surface density of 75 grams per square meter, and a thickness of 0.1 millimeters. (5 points)

c. Calculate the mass (in solar masses), and radius (in AU) of the black hole with density equal to that of paper. (10 points)

d. How many pages long is the Brockian Ultra Cricket rule book? Assume the pages are standard size ($8.5'' \times 11''$). For calculational simplicity, treat the book as spherical (a common approximation in this kind of problem). What if the rule book were even longer than you have just calculated? Would it still collapse into a black hole? (5 points)

3. Falling into a black hole (30 points)

The tides near black holes can be so extreme that a process informally called "spaghettification" occurs in which a body falling towards a black hole is strongly stretched due to the difference in gravitational force at different locations along the body (this is called a tidal effect). In the following, imagine that you are falling into a 3 solar mass black hole.

a) What is the Schwarzschild radius of this black hole (in km)? (5 points)

b) You are 1.5 meters tall and 70 kg in mass and are falling feet first. At what distance from the black hole would the gravitational force on your feet exceed the gravitational force on your head by 10kN (kN stands for kilo-Newtons; Newton is an MKS unit of force)? Express this distance in km and in Schwarzschild radii of the black hole. (10 points)

Hint: Remember that Newton's law of gravitation tells that the force between bodies of mass m_1 and m_2 a distance r apart is given by $F = Gm_1m_2/r^2$, where G is Newton's gravitational constant. You may also find useful the following mathematical simplification:

$$\frac{1}{r^2} - \frac{1}{(r+\Delta)^2} \approx \frac{2\Delta}{r^3},$$

which is valid when $\Delta \ll r$. Proving that this approximation is correct will get you extra 5 points – this is extra credit, but will not lead to your total exceeding 100 points for this homework.

c) To appreciate if this amount of force is enough to "spaghettify" and kill you, imagine that you are suspended from a ceiling of your room (on Earth) with a steel plate tied to your feet. Calculate the mass of the plate (in kg) that will give you a nice tug of 10kN (you can ignore the weight of your body here). Do you think this pull will kill you? (5 points)

d) Now consider a trip toward the supermassive black hole at the center of our Galaxy, which has an estimated mass of 4 million solar masses. How does this change the distance at which you will be "spaghettified" by the differential gravity force of 10kN? Express your answer in km and in Schwarzschild radii of the black hole. (5 points)

e) Find the smallest mass of the black hole for which you would not die by "spaghettification" before falling within its Schwarzschild radius. (5 points)