Homework #5, AST 203, Spring 2009

Due in class (i.e., by 4:20 pm), Thursday April 16

- To receive full credit, you must give the correct answer *and* show that you understand it. This requires writing your explanations in full, complete English sentences, clearly labeling all figures and graphs, showing us how you did the arithmetic, and being explicit about the units of all numbers given. All relevant mathematical variables should be explicitly defined. And please use your best handwriting; if we can't read it, we can't give you credit for it! Please staple together the sheets of paper you hand in.
- Most of the calculations in this course involve numbers that are only approximately known. The result of such a calculation should reflect this imprecision. In particular, it is *wrong* to simply write down all the digits that your calculator spits out. Your final answer should have the same number of significant figures as the least precise number going into your calculation. In many (but not all!) cases, it's best to do the problems without a calculator.
- Feel free to work with your classmates on this homework, but your write-up and wording should be your own. Answer all questions.

100 total points

1. Length Contraction (50 points)

In this problem, we will explore the nature of the function:

 $y = \sqrt{1 - \frac{v^2}{c^2}}$. This quantity, sometimes called the "Lorentz Factor" is the factor in Special Relativity by which an astronaut moving by at speed v ages. That is, I age 1 year while I observe that the astronaut ages y years.

We start by exploring the behavior of this function for small values of v, and those close to the speed of light. To do this, we'll need to develop a few mathematical tools. If we define x = v/c, the Lorentz Factor can be written $y = \sqrt{1 - x^2}$.

- a. (10 points) For very small velocities, $v \ll c$ and we expect y to be very close to (but slightly less than) 1. Thus we write $y = 1 - \epsilon$, where $\epsilon \ll 1$. Our exercise will be to determine ϵ . Solve the equation above for ϵ in terms of x: start by squaring both sides of the equation, and then recognize that if ϵ is small, ϵ^2 is tiny, and additive terms involving ϵ^2 can be neglected. With the value of ϵ in terms of x in hand, now express the Lorentz Factor y in terms of v and c, for $v \ll c$.
- b. (10 points) Now let's take the opposite limit, namely speeds very close to the speed of light. This time, we'll write $x = v/c = 1 \alpha$, where now $\alpha \ll 1$. Plug into the equation for the Lorentz Factor; similar to part a, if α is small, then α^2 is really tiny and can be neglected. Thus write an expression for the Lorentz Factor in this case.

- c. (5 points) Now we're ready to plug in some numbers. Draw a graph of the Lorentz Factor as a function of velocity, where the x-axis ranges from 0 to the speed of light c, and the y-axis ranges from 0 to 1. Plug in many values of v and calculate the value of the Lorentz Factor, and plot them up. Describe the shape of this function in words. Next, plot on the same graph the two approximations you calculated, for small velocities and large velocities, in parts (a) and (b). Over what range of velocities does each do a decent job of approximating the Lorentz value? That is, over what range does each approximation give answers within 10% of the correct value?
- d. (10 points) What is the value of y for v = 0.001c, v = 0.6c, v = 0.8c, v = 0.99995c; v = 0.999995c? Use the approximations you've just developed, as appropriate. If you find yourself rounding off and saying that y = 1 or y = 0, you've rounded too much!
- e. (15 points) A muon is a particle very much like an electron, only with more mass. Unlike electrons, muons are unstable; they have a half-life of only 2.2×10^{-6} seconds, after which they decay into an electron, a muon neutrino, and an anti-electron neutrino. Very fast-moving muons are produced in the upper atmosphere of the Earth (i.e., 100 km above the Earth's surface) in collisions with high-energy cosmic rays (produced ultimately from distant supernova explosions), and come whizzing down to be detected on the surface. A typical atmospheric muon is moving at 0.9999995 the speed of light; i.e., $(1 - 5 \times 10^{-7})c$. Taking into account time dilation, how far could such a muon travel before decaying (i.e., before one half-life is over)? Now do the same calculation, not taking into account time dilation. Discuss: does the fact that we observe muons that were produced 100 km away give any support to Einstein's prediction of time contraction?

2. What's at the center of our Galaxy? (10 points)

You observe a star orbiting the center of our Galaxy at a speed of 1,000 km/s in a circular orbit with a radius of 20 light-days. We infer stellar velocity from the Doppler shift of the absorption lines. This gives the velocity along the line of sight, or "radial velocity". For some stars we can also detect the proper motion on the sky, if the star is moving fast – this is actually seen near the Galactic center. You can learn more about this in Chapter 19.4 of the book. Based on your measurement, what would be your estimate for the mass of the object that the star is orbiting? What kind of an object do you think this is? Give arguments to support your conclusion.

3. Expanding Universe, Expanding Oceans (10 points)

The expansion of the universe discovered by Hubble really only takes place on physical scales larger than galaxies, but for the purposes of this exercise, imagine that the proportionality between distance and recession velocity holds on scales comparable to the size of the Earth. Calculate by what distance the Atlantic Ocean (width 7000 km) would grow in a year under the expansion of the universe. Compare with the actual growth, due to plate tectonics (a completely different physical mechanism), which adds up to several inches a year. Which is the faster mechanism on these scales: plate tectonics, or the expansion of the Universe?

4. The Shining Sun (30 points)

- a. Four hydrogen atoms (total mass = 6.693×10^{-27} kilograms) fuse to produce one helium atom (mass = 6.645×10^{-27} kilograms). If one gram of hydrogen is burned in the sun into helium, how much energy will be produced? (5 points)
- b. The sun has a luminosity of 4×10^{26} Joules per second. It has a mass of 2×10^{30} kilograms. If three quarters of the sun's mass is hydrogen, using the results of question a) above, make a rough estimate of the maximum time the sun could shine at its present luminosity by burning hydrogen into helium. How does this estimate compare with its actual main sequence lifetime? Explain any discrepancies you might find. (15 points)
- c. When we talk about hydrogen being burned in the Sun, it is tempting to think of the Sun as getting less massive with time. If the Sun is shining at 4×10^{26} Joules per second, and all the energy comes from nuclear burning, calculate what percentage of the Sun's mass will actually be converted into energy after 10 billion years. (5 points)
- d. The Sun is also losing mass through a solar wind a spherical outflow of charged particles that escape from the outermost layers of the Sun. You can think of these particles as boiling off the Sun (there is a very tenuous but hot layer outside the Sun's atmosphere, called the "corona" from which these particles escape). The rate of particle loss in the solar wind is 10³⁶ protons per second. What is the rate of mass loss in solar masses per year? Compute how much mass the Sun is going to lose to the Solar wind over its 10 billion year life time (in solar masses). How does this compare to the mass lost during the fusion? (5 points)