Solutions to Homework #3, AST 203, Spring 2009

Due on March 5, 2009

*General grading rules: One point off per question (e.g., 1a or 2c) for egregiously ignoring the admonition to set the context of your solution. Thus take the point off if relevant symbols aren't defined, if important steps of explanation are missing, etc. If the answer is written down without *any* context whatsoever take off 1/3 of the points. One point off per question for inappropriately high precision (which usually means more than 2 significant figures in this homework). However, no points off for calculating a result to full precision, and rounding appropriately at the last step. No more than two points per problem for overly high precision. Three points off for each arithmetic or algebra error. Further calculations correctly done based on this erroneous value should be given full credit. However, if the resulting answer is completely ludicrous (e.g.,* 10^{-30} *seconds for the time to travel to the nearest star, 50 stars in the visible universe), and no mention is made that the value seems wrong, take three points further off. Answers differing slightly from the solutions given here because of slightly different rounding (e.g., off in the second decimal point for results that should be given to two significant figures) get full credit. One point off per question for not being explicit about the units, or for not expressing the final result in the units requested. Two points off for the right numerical answer with the wrong units. Leaving out the units in intermediate steps should be pointed out, but no points taken off. Specific instructions for each problem take precedence over the above. In each question, one cannot get less than zero points, or more than the total number the question is worth.*

100 total points

1. Solar power in space *25 total points*

NASA's Mars Reconnaissance Orbiter (MRO) is currently in orbit around Mars. The spacecraft carries six instruments to study the Martian atmosphere and surface, including a ground-penetrating radar to search for water beneath the surface. You can find more information (not necessary for this problem) at `http://www.nasa.gov/mro` .

MRO communicates with Earth using a 10-foot diameter dish antenna and a transmitter powered by 100 square feet of solar panels. A very good modern conversion efficiency for solar panels is 30%, i.e. the panel converts 30% of the sunlight incident upon it into electrical power - the other 70% is lost.

a. Take the Sun to be a blackbody with a surface temperature of 6000 K. The Sun's radius is 7.0×10^5 km. Calculate the Sun's luminosity, in watts (Joules/second). *(6 points)*

   Answer: This is a straightforward application of the Stefan-Boltzmann formula, which states that the luminosity of a spherical blackbody of temperature T and radius R is:

   $$L = 4\,\pi R^2 \sigma T^4.$$

   Here, $\sigma = 5.6 \times 10^{-8}\,\mathrm{Joules/sec/m^2/K^4}$ is the Stefan-Boltzmann constant. So this is just a matter of plugging in the numbers, which we will try without a calculator:

   $$L = 4 \times 3 \times (7 \times 10^8\,\mathrm{m})^2 \times 5.6 \times 10^{-8}\mathrm{Joules/sec/m^2/K^4} \times (6000\,\mathrm{K})^4 \approx$$

$$\approx 12 \times 50 \times 5.6 \times 1300 \times 10^{16-8+12}\,\text{Joules/sec} \approx 4.5 \times 10^{26}\,\text{Watts}.$$

Any reasonable value to one or two significant figures gets full credit.
This is a bit higher than the true value of 3.8×10^{26} watts, as the input temperature of 6000 K we used is a bit higher than the true value of 5800 K.
No credit for simply stating the result (which is, e.g., available in the book), without explanation. 2 points for writing down the Stefan-Boltzmann relation, and going no further. 2 points for forgetting the $4\,\pi R^2$ term.

b. Mars is 1.5 AU from the Sun. Calculate the brightness of the Sun at Mars' distance (i.e., the solar flux on Mars' surface) expressed in watts per square meter. *(6 points)*
Answer: This is an exercise in the inverse square law. The brightness is simply given by the luminosity, divided by $4\,\pi d^2$, where $d = 1.5$ AU is the distance of Mars from the Sun:

$$b = \frac{L}{4\,\pi d^2} = \frac{4.5 \times 10^{26}\,\text{watts}}{12 \times (1.5 \times 1.5 \times 10^{11}\,\text{meters})^2} = 750\,\text{watts/m}^2.$$

2 points for writing down the inverse square law, and getting no further.

c. Using the approximation (good to about 10%) that 1 meter = 3 feet, calculate how much electrical power will be available to the MRO transmitter, assuming that the MRO solar panels are facing the Sun and that they have a conversion efficiency of 20%. Express your result in watts. *(6 points)*
The above-calculated power is incident on every square meter of the solar panels. We first need to calculate the number of square meters 100 square feet are. 1 foot = 1/3 meter, so 1 foot^2 = 1/9 meter^2. So 100 foot^2 $\approx$ 10 meters^2, to one significant figure. Thus the total power hitting the solar arrays is 10 times the value we calculated above, or 7500 Watts. However, only 30%, or roughly 1/3, of that is available to the transmitter, namely 2500 watts.
Three points off for forgetting the factor of 20%.

d. Suppose now that a spacecraft identical to MRO were launched (presumably from a bigger rocket!) to observe Saturn's moon Titan, rather than to observe Mars. How much power (in watts) would the MRO at Saturn have available to its transmitter? Saturn orbits at 9.6 AU from the Sun. For comparison, a typical light bulb in your home has a total power output of about 60 watts. What would be the ratio of MRO transmitter power at Saturn to that of a 60 watt light bulb? This is why missions to the outer Solar System so often rely on power generated from the radioactive decay of a plutonium isotope, rather than from solar panels. *(7 points)*
Answer: The inverse square law says that the power available to the same spacecraft at a larger distance is down by the inverse square of the ratio of the distances. That is, if we went through the calculation we did in parts (b) and (c), and used the distance to Saturn rather than Mars, everything would be identical but the distance itself, and the results would be proportional to the inverse square of the distance. So we would find that the power available to the spacecraft at Saturn would be:

$$\text{Power at Saturn} = \text{Power at Mars} \times \left(\frac{\text{Distance to Mars}}{\text{Distance to Saturn}}\right)^2 = 2500\,\text{watts}\left(\frac{1.5\,\text{AU}}{9.6\,\text{AU}}\right)^2$$

The ratio of distances is roughly 1/6.5, whose square is about 1/40. So the available power at Saturn's distance is about 60 watts, or close to that of an ordinary light bulb. Scarcely enough to run a spacecraft on!
Full credit, of course, for doing the calculation all over again without scaling. Full credit for results consistent with (c) and the inverse square law. The calculation of the power available to the spacecraft is worth 5 points, and the calculation of the ratio to the lightbulb is worth 2 points.

2. Human energetics *25 total points*

You are sitting down to study for the mid-term in the library. Let's find out what limits the time you are able to spend studying.

a) Calculate how much power you are radiating, in watts. While your skin temperature is about 33° Celsius, the outside of your clothing is at about 28° Celsius. In the absence of perspiration, black body radiation at this temperature is your main radiation mechanism. You can take your surface area to be 2 m^2. (*5 points*)

Solution: In this problem "you" implies the system of "your body + your clothes," because we are interested in how much power gets radiated into your surroundings. The heat that your body emits at 33° is partially reabsorbed from being trapped in the clothes (that's why clothes generally make you warmer!). So, in order to know your heat loss, we should consider the black body radiation at the temperature of the outer layer of your clothes, or 28° Celsius. The radiated power is then $\mathrm{P_{loss}} = \sigma \mathrm{T_{clothes}^4} \times (\mathrm{Area}) = 5.6 \times 10^{-8} \mathrm{Joules/sec/m^2/K^4} \times ((28 + 273)\,\mathrm{K})^4 \times 2\,\mathrm{m}^2 = 920\mathrm{W}$. Note that we converted from Celsius to Kelvin by adding 273K which corresponds to zero degrees Celsius.
2 points off for not converting to Kelvins, 2 points off for using the body temperature, instead of the clothes temperature.

b) What is the wavelength of the peak of your spectrum? Comment why we cannot see each other in a dark room. (*5 points*)
Solution. From Wein's law: $\lambda_{\mathrm{peak}} = 3\mathrm{mm/T(K)} = 10^{-5}\mathrm{m} = 10\mathrm{micron}$. This is the wavelength of infrared radiation. Our eyes are sensitive in the visible range (0.4-0.7 micron), so we cannot see this emission with our eyes in a dark room. A camera with an infrared detector will see us glowing in the dark, however!
Two points off for not explaining that this is infrared emission, which is not visible to us.

c) Your body is constantly emitting thermal radiation and is also absorbing radiation from the surroundings. The library is unusually chilly today, and the ambient temperature is 15° Celsius. How much power are you absorbing from the room, in watts? You can use the black body formula at the temperature of the room. (*5 points*)
Solution: The black body radiation from the room will be absorbed by you, so the this will provide the input of heat into the system of "you+your clothes." The incoming radiation is $\mathrm{P_{input}} = \sigma \mathrm{T_{room}^4} \times (\mathrm{Area}) = 5.6 \times 10^{-8} \mathrm{Joules/sec/m^2/K^4} \times ((15 + 273)\,\mathrm{K})^4 \times 2\,\mathrm{m}^2 = 770\mathrm{W}$.

d) What's your net energy loss rate, in watts? Compare this to a typical 60W light bulb. How much energy will you lose per day, in mega-Joules? (*5 points*)
Solution: The net energy loss is the difference between P_{loss} and P_{input}, or $P_{net} = P_{loss} - P_{input} = 920W - 770W = 150W$. This output is 2.5 larger than the 60W lightbulb. So, you can now think of yourself as a big infrared light bulb!
A day is 24 hours, so the total energy loss is (remembering that Watt is Joules/sec): 150 Joules/sec × 24 hours × 3600 sec/hour = 13 megaJoules.
1 point off for not comparing with the light bulb. 2 points off for conversion mistakes to find mega joules.

e) On a normal dorm diet, your daily energy intake from food should be close to 2000 kilo-calories (kcal), where one calorie is defined as 4.2 Joules. How long will you be able to study in this library before your daily energy intake is exhausted? (*5 points*)
Solution: Of course, in order to be able to radiate energy, you need to keep supplying it, otherwise you would not last very long. That's where the food we eat comes in. A significant fraction of our daily energy intake actually goes to maintaining the body temperature! The situation is exacerbated by being in a cold room. The dorm diet provides you with 2000×10^3 calories × 4.2Joules/calorie = 8.4×10^6 J = 8.4 MegaJoules. So, your daily energy supply will be exhausted in (8.4/13)×24 hours = 15 hours. Now you have an official excuse why you can't study in a cold library for 24 hours!
1 point off for conversion mistakes, 2 points off for incorrect logic in calculating the time to use up all the daily energy intake.

3. Extrasolar planet *25 total points*

There are currently 340 known extrasolar planets, i.e., planets around other suns. Most of them we infer only by their gravitational pull on their host stars, and a handful appear to dim the light of their host star as they transit in front of the star. In November of last year, however, astronomers announced the detection of a planet which not only was resolved (which is already amazing) but whose orbital motion was also confirmed by direct imaging. In other words, in two pictures made by Hubble Space Telescope, the planet was observed to have shifted position. This Jupiter-like planet is called Fomalhaut b, orbiting a star Fomalhaut, which is 25 light years from Earth (the name Fom al-Haut comes from Arabic and means 'the mouth of the fish', as the star is in the constellation Southern Fish – Piscis Austrinus). The host star has a mass of 2.1 $M_\odot$ and its radius is 1.8 $R_\odot$. Its surface temperature is 8700 K. The planet orbits with the semimajor axis of 115 AU around the star, has small eccentricity, and the plane of the orbit is in the plane of the sky (in other words we are seeing the orbit face-on). If you want to see the picture of the star, the planet and the "debris disk" left over after the planet formation in this system, look at `http://apod.nasa.gov/apod/ap081114.html`

a) Determine the angular separation between the star and the planet, in arc seconds. (*5 points*)
Solution: The small-angle formula tells us that an angle p subtended on the sky by two objects that are 1 AU apart and are located at a distance of 1 parsec away is 1 arc second. So, we can obtain the angular separation between the planet and the star

by scaling this relation. Remembering that 1 parsec is 3.3 light years, we convert 25 light years to 7.5 parsecs (abbreviated as pc).

$$p(arcsec) = \frac{115\ \text{AU}}{\text{d(pc)}} = \frac{115}{7.5} = 15 \text{ arcseconds.}$$

Of course, a long but sure way to obtain this answer is directly from the geometry. Imagine a long thin triangle consisting of the Earth, the star Fomalhaut and the planet Fomalhaut b. The angle between the long sides is then the short side of the triangle divided by the long side, or

$$p = \frac{115\ \text{AU}}{d} = \frac{115 \times 1.5 \times 10^8\ \text{km}}{25 \text{ light years} \times 9.5 \times 10^{12}\ \text{km/light year}} = 7.27 \times 10^{-5},$$

but now p is in radians. To convert radians to arc seconds remember that there are roughly 200000 arc seconds in a radian (a more precise number is 206265 arc seconds), so p $= 7.27 \times 10^{-5} \times 2 \times 10^5 \approx 15$ arc seconds

b) Determine the period of the planet, in years. Hubble imaged the system in 2004 and 2006. What is the distance that the planet traveled between the images, in AU? (*5 points*)
Hint: if you are reaching for your lecture notes to find the formula for the velocity of the planet, you are doing it the hard way.
Solution: To find the period, we can use Newton's version of Kepler's third law:

$$a^3 = \frac{GMP^2}{4\pi^2},$$

where a is the semimajor axis, P is the period and M is the mass of the star. Now we can either plug in all the numbers and find the period P, or we can simplify the computation by scaling the relationship for the Earth and the Sun.

$$\left(\frac{a_{\text{Fb}}}{a_{\text{Earth}}}\right)^3 = \left(\frac{M_*}{M_\odot}\right)\left(\frac{P_{\text{Fb}}}{P_{\text{Earth}}}\right)^2.$$

Here, subscript Fb stands for the planet, and $*$ for the star Fomalhaut. This equation means that if measured in AUs and years, the semimajor axis and the period of Formalhaut b are related as:

$$a_{\text{Fb}}(\text{in AU})^3 = M_*(\text{in M}_\odot)P_{Fb}(\text{in years})^2.$$

Expressing the period, we get

$$P_{Fb}(\text{in years}) = M_*(\text{in M}_\odot)^{-1/2}a_{\text{Fb}}(\text{in AU})^{3/2} = 2.1^{-1/2} \times 115^{3/2} = 850 \text{ years.}$$

In 2 years between the observations, the planet must have traveled (2/850) fraction of the full circle, or $(2\pi a_{\text{Fb}}) \times 2/850 = 1.7$ AU.
2 points off for errors in the period calculation, 1 point off for not calculating the distance; 2 points for writing the formula without numbers.

c) What is the angular separation between the position of the planet in 2004 and 2006, in arc seconds? Can the Hubble telescope, with the angular resolution of 0.04 arc seconds, detect this motion? (*5 points*)

Solution The easiest way is to rescale the answer we got in part a. If 115 AU subtended the angle of 15 arc seconds,1.7 AU subtend an angle 1.7/115 times smaller, or 0.2 arc seconds. This is larger than the angular resolution of Hubble, which means HST can detect this motion (and it did!).

2 points off for not comparing with HST resolution

d) What is the equilibrium temperature on Fomalhaut b? You can ignore the effects of albedo and greenhouse effect. Can liquid water exist on Fomalhaut b? (*5 points*)

Solution: The equilibrium temperature is defined as the balance between the incoming radiation from the star and the black body radiation from the planet. In lecture, we derived this temperature as

$$T_{\text{planet}} = T_*(1-A)^{1/4}\left(\frac{R_*}{2d}\right)^{1/2},$$

where A is the albedo, and T_* and R_* are the temperature and the radius of the host star. Ignoring the albedo effect (i.e., saying that the planet is not reflecting any light, thus setting A=0), we get: $T_{\text{planet}} = 8700K(1.8R_\odot/(2\times 115\text{AU}))^{1/2} = 52K$. To simplify the math you can use the fact that 1AU is about 200 radii of the Sun. At 52 K the temperature of the planet is rather frigid for any liquid water to exist!

-1 point for not commenting on liquid water

e) At what distance from the star Fomalhaut would the planet have the same equilibrium temperature as on Earth? Again, ignore albedo and greenhouse effects. (*5 points*)

The equilibrium temperature of the Earth (without albedo or greenhouse effects) is: $T_{\text{Earth}} = T_\odot(R_\odot/2AU)^{1/2}$ (note that there is $2d$ in the formula, hence 2 AU). We want to equate this temperature to the equilibrium temperature near Fomalhaut.

$$T_*(R_*/2d)^{1/2} = T_\odot(R_\odot/2AU)^{1/2},$$

or expressing the distance d, we find:

$$d(\text{in AU}) = \left(\frac{R_*}{R_\odot}\right)\left(\frac{T_*}{T_\odot}\right)^2 = 1.8\times\left(\frac{8700K}{5800K}\right)^2 = 4$$

Hence, the planet has to be at 4 AU from the star to have the same equilibrium temperature as the Earth. This makes sense, because Fomalhaut is a more luminous star, so to have the same temperature as the Earth you want to be farther away from Fomalhaut.

Answers taking the equilibrium temperature of the Earth to be 300K from the outset were also accepted.

4. Spinning too fast *25 total points*

The rate of rotation of astrophysical objects that are held together by gravity (e.g., stars or planets) cannot be larger than a certain maximum. Rotating faster than this rate will tear the star apart. Let's find the expression for this maximum rotation rate.

a) You are sitting on the equator of a star of radius R that is spinning about its axis with period P. What is the rotation speed that you have on the equator? (*3 points*)
Solution. The point fixed on the equator goes around the rotation axis of the star in a circle of radius R. The period of this rotation is P, so the speed of the rotation is $v_{\rm rot} = 2\pi R/P$.

b) Using your expression for the rotation speed, what is the centrifugal acceleration that you experience on the equator? (*3 points*)
Solution: In lecture we discussed that the centrifugal (or centripetal) acceleration is given by the formula $a_c = v^2/R$.

c) The star has mass M. What is the gravitational acceleration that you feel on the surface? (*3 points*)
Solution: From Newton's law of gravitation, the force acting on a body of mass m at a distance R from the center of the star is $F = GMm/R^2$. From Newton's second law, the force is mass times acceleration, of $F = ma$. Dividing by m, we get $a_g = GM/R^2$.

d) Now equate the centrifugal and gravitational acceleration, and find the period of rotation when they are equal. What happens if the star rotates faster than this period? (*10 points*)
Solution: Equating a_g and a_c, we have

$$\frac{v^2}{R} = \frac{GM}{R^2}.$$

We now substitute for the rotation velocity the expression we derived in part a):

$$\frac{4\pi^2 R^2}{RP^2} = \frac{GM}{R^2}.$$

Expressing the period P, and simplifying, we get:

$$P_{\rm limit} = \frac{2\pi}{(GM)^{1/2}} R^{3/2}.$$

Why is this period special? As you see, at this period of rotation, the centrifugal acceleration equals to the gravitational pull. So, if the period is any shorter, the body on the equator of the star will not be feeling the pull of the star anymore, and it will fly away! Thus, the star (or a planet) spinning faster than this rate will not be able to stay together, and will break itself apart.

Interestingly, there is nothing in the formula that we derived that seems to break. Instead, what breaks is the original assumption of the balance between gravitation and the centrifugal force. Right now, for instance, you are spinning around the axis of the Earth, but the centrifugal acceleration is small compared to gravity, so instead you have to balance the gravity by pushing against the floor. If the Earth were spinning at the period you calculate in part e, the floor would no longer be necessary to support you. And instead of sitting on the Earth you would essentially be in orbit around it! That's why astronauts are weightless, by the way.

6 points for derivation, 4 points for explanation

e) Calculate the limiting rotation periods for the Earth and for the Sun. Write them in the most appropriate units (e.g., seconds, minutes, hours, days, years, etc.). Find the ratio of the critical rotation period for the Earth to the current rotation period. The Sun is rotating with a period of 25 days. FInd this ratio for the Sun as well. (*6 points*)
Solution: We are now in a position to plug in the numbers. For the Earth, the mass is 6×10^{24} kg, and the radius is 6400 km, so the limiting period is $P_{limit} = 5100$ seconds, or about 90 minutes. This, by the way, is the orbital time to make one revolution in the low earth orbit – no wonder at this spin the planet can't hold itself together! The ratio is 1.5 hours/24 hours $= 6 \times 10^{-3}$. For the Sun, the mass is 2×10^{30} kg, and the radius is 7×10^{5}km. Therefore, the limiting period is 168 minutes, or close to 3 hours. The ratio with the current rotation period is 5×10^{-3}. So, neither Earth, nor Sun are in danger of being blown apart by centrifugal forces. However, other astrophysical objects are! We will discuss this later in the course.
1 point off for not converting to reasonable units of time. 2 points off for not calculating the ratios