# Solutions to Homework #2, AST 203, Spring 2009

Due on February 24, 2009

General grading rules: One point off per question (e.g., 1a or 2c) for egregiously ignoring the admonition to set the context of your solution. Thus take the point off if relevant symbols aren't defined, if important steps of explanation are missing, etc. If the answer is written down without \*any\* context whatsoever (e.g., for 2b, writing "once a month", and nothing else), take off 1/3 of the points. One point off per question for inappropriately high precision (which usually means more than 2 significant figures in this homework). However, no points off for calculating a result to full precision, and rounding appropriately at the last step. No more than two points per problem for overly high precision. Three points off for each arithmetic or algebra error. Further calculations correctly done based on this erroneous value should be given full credit. However, if the resulting answer is completely ludicrous (e.q.,  $10^{-30}$  seconds for the time to travel to the nearest star, 50 stars in the visible universe), and no mention is made that the value seems wrong, take three points further off. Answers differing slightly from the solutions given here because of slightly different rounding (e.g., off in the second decimal point for results that should be given to two significant figures) get full credit. One point off per question for not being explicit about the units, or for not expressing the final result in the units requested. Two points off for the right numerical answer with the wrong units. Leaving out the units in intermediate steps should be pointed out, but no points taken off. Specific instructions for each problem take precedence over the above. In each question, one cannot get less than zero points, or more than the total number the question is worth.

#### 100 total points

#### **1. Dropping out of the sky** 30 total points

MIR was a Soviet (and later Russian) space station that operated from mid-1980s in circular low Earth orbit, at an altitude of 400 km above the surface of the Earth. In 2001 MIR was successfully deorbited and crashed in a remote region of the Pacific Ocean. In this problem we will calculate

a) Let's find out how fast MIR was moving in its orbit. Derive an algebraic expression for the speed v of an object in a circular orbit around the Earth, in terms of the mass of the Earth  $M_{\oplus}$ , orbital radius r, gravitational constant G, and numerical constants. You can do this either by starting with Newton's form of Kepler's Third Law, or by using Newton's Law of Gravitation and his second law of motion. (10 points)

**Solution**: Let's begin from Newton's form of Kepler's Third Law. It relates the semimajor axis a of an orbit about a body of mass M to the period P as:

$$a^3 = \frac{GMP^2}{4\,\pi^2}.$$

In our case the central body is the Earth  $M = M_{\oplus}$  ( $\oplus$  is the symbol for Earth), and the orbit is a circle of radius r (remember that for a circle the semimajor axis and the radius are the same). The period of the circular orbit is connected to the orbital velocity v as  $P = 2\pi r/v$ . Substituting this into the Kepler's Third Law we get:

$$r^3 = \frac{4\,\pi^2 G M_\oplus r^2}{4\,\pi^2 v^2}.$$

Cancelling the common  $4\pi^2$  term, and solving for v gives:

$$v = \sqrt{\frac{GM_{\oplus}}{r}}.$$

Alternatively, let's argue from Newton's laws. The force on the satellite of mass m is given by Newton's Law of gravity,  $GM_{\oplus}m/r^2$ . It is equal to its mass times its acceleration; the latter is given by the standard expression for circular motion, i.e.,  $v^2/r$ . Equating the two gives:

$$m\frac{v^2}{r} = \frac{GM_{\oplus}}{r^2}$$

Cancelling the common factor of r and solving for v again gives:

$$v = \sqrt{\frac{GM_{\oplus}}{r}}$$

Full credit for either derivation. Only 4 points for writing down the answer with no intermediate steps given. No points off for giving no English.

b) Now plug in the numbers and calculate MIR's orbital speed, in kilometers per second. *Hint: Think hard what you want to use for the radius of the orbit! (5 points)* 

**Solution:** We can use the equation we just derived to determine the speed. The one tricky bit is to figure out what radius we want to use. 400 km is the height of the satellite above the Earth's *surface*, but what counts here is the distance from the *center* of the Earth, 6400 km away. Thus we use  $r = (6400+400) \text{ km} \times 10^3 \text{ m/km} = 7 \times 10^6 \text{ m}$ , to a single significant figure. Putting in numbers:

$$v = \sqrt{\frac{GM_{\oplus}}{r}} = \sqrt{\frac{2/3 \times 10^{-10} \mathrm{m}^3 \,\mathrm{s}^{-2} \,\mathrm{kg}^{-1} \times 6 \times 10^{24} \,\mathrm{kg}}{7 \times 10^6 \,\mathrm{m}}} \approx \sqrt{\frac{1}{2} \times 10^8 \,\frac{\mathrm{m}^2}{\mathrm{sec}^2}} \approx 7000 \,\mathrm{m/sec}.$$

The satellite is moving at 7 kilometers per second; that's fast!

3 points off for using r = 400 km as the radius of the orbit. 1-3 significant figures get full credit.

c) The space station had a mass of 125 tons. What was its orbital kinetic energy, in Joules?

*Hint:* A joule is the MKS unit of energy, and is equal to one kilogram meter<sup>2</sup>/second<sup>2</sup>. (5 points)

**Solution:** The kinetic energy of an object of mass m moving at speed v is  $KE = \frac{1}{2}mv^2$ :

$$KE = 0.5 \times 125 \times 10^3 \,\mathrm{kg} \times (7 \times 10^3 \,\mathrm{m/sec})^2 \approx 3 \times 10^{12} \,\mathrm{Joules},$$

to a single significant figure. The next problem will put this amount of energy in context.

d) The kinetic energy of the station was dissipated during the burning up in thick layers of the atmosphere and at final impact in the ocean. Let's get a sense for how much energy that was. The energy per kilogram of the high explosive TNT is  $1.5 \times 10^7$  joules/kg. Calculate how many tons of TNT release the same amount of energy as the deorbiting of MIR. For comparison, a typical aircraft bomb is 500 pounds of explosives, and the first nuclear test was equivalent to 18 kilotons of TNT. Which one is more destructive: a kilogram of TNT or a kilogram thrown with MIR's orbital velocity? Calculate the ratio of the released energies. (10 points)

**Solution:** The amount of TNT needed to release the same amount of energy as during the deorbiting of MIR is:  $3 \times 10^{12}$  Joules/ $1.5 \times 10^{7}$  Joules/ $\text{kg} = 2 \times 10^{5}$  kg = 200 tons of TNT. This is equivalent to about 1000 aircraft bombs, or 1% of a nuclear blast.

To calculate the ratio of released energies, we write down the kinetic energy of an object of mass m moving at speed v as  $\frac{1}{2}mv^2$  (this is the energy released upon burning up), and the explosive energy of the same mass of TNT as mp, where  $p = 1.5 \times 10^7$  joules per kilogram is the explosive energy per kilogram. The ratio of the two is thus:

$$\frac{\frac{1}{2}mv^2}{mp} = \frac{1}{2}\frac{v^2}{p}.$$

Plugging in numbers (with v from part b), and being careful with our units, we find:

$$\frac{1}{2} \frac{(7 \times 10^3 \,\mathrm{m/sec})^2}{1.5 \times 10^7 \mathrm{kg} \,\mathrm{m}^2/\mathrm{sec}^2/\mathrm{kg}} = \frac{0.5 \times 5 \times 10^7}{1.5 \times 10^7} \approx 1.5.$$

First, note that the mass dropped out; the answer here did not depend on that in part (c). Second, notice the work done on the units; we plugged in the definition of a Joule, and by the end, all the units dropped out (as they had to for a meaningful ratio). Finally, marvel at the answer: a collision from a piece of space debris releases about 50% more energy than an equivalent amount of TNT! No wonder why people are so concerned about uncontrolled re-entries of space hardware!

Just last week a Russian and American satellites collided at an orbit of 800 km. The debris generated by the collision pose a large risk to other satellites in low earth orbit, mainly because they carry large amounts of kinetic energy.

1 point off for not comparing the mass of TNT with the weight of the bombs; 3 points off for not finding the ratio

# **2. Solar eclipse** 40 total points

a) What would the orbit of the Moon have to be in order to have one total solar eclipse each month? Explain and sketch a diagram. (5 points)



Figure 1: This figure shows the inclination of the orbit of the Moon with respect to the ecliptic, and the configuration that would result in one solar eclipse per month. For this to happen, the orbit of the Moon has to lie in the plane of the ecliptic.

**Solution:** It is easiest to think about this problem by first understanding why we *do not* have one solar eclipse per month right now. A solar eclipse occurs when the Moon blocks the Sun, so the Earth, the Moon and the Sun have to lie along one line. This can happen only when the Moon is in the plane of the ecliptic, i.e., the plane which contains the Sun and the orbit of the Earth. The orbital plane of the Moon is inclined with respect to the ecliptic by 5 degrees, so the Moon can potentially block the Sun only when it happens to cross the ecliptic plane at the phase of the new Moon. This happens roughly every half a year. In order to have a solar eclipse every month, i.e., once every orbit of the moon around the Earth, the orbit of the Moon would have to be in the plane of the ecliptic, i.e., the inclination of the Moon would have to be 0 degrees. The diagram below shows the orbit of the Moon now and when the eclipse happens every month.

2 points off for correct answer with no explanation, 1 point off for diagrams that are not correct.

b) After the orbital change in part a), how often would lunar eclipses be visible from the Earth? Explain. (5 points)

**Solution:** With the Moon in the plane of the ecliptic, every month would bring a solar eclipse when the Moon is in the new phase (i.e., between the Earth and the Sun), and a lunar eclipse when the Moon is full (i.e., the Earth is between the Moon and the Sun). So, the answer is "once a month."

In the following we will calculate how long a typical solar eclipse can last on Earth. During the eclipse, the shadow of the Moon sweeps over the face of the Earth. The motion of the shadow is due to Moon's orbital motion. Let's find out how fast the shadow is moving.

c) You know that the period of rotation of the Moon about the Earth is approximately 27 days. Using Newton's version of the Kepler's third law, find the orbital velocity of the Moon about the Earth, in kilometers per second (you already derived part of the answer in problem 1a). The mass of the Earth is  $6.0 \times 10^{24}$  kg, and you can neglect the eccentricity of Moon's orbit. (15 points)

Solution: Start with the Newton's form of Kepler's Third Law:

$$a^3 = \frac{GMP^2}{4\,\pi^2}.$$

The orbital velocity is related to semimajor axis (which is the radius of a circular orbit) by  $v = 2\pi a/P$ , or  $a = vP/(2\pi)$ . Substituting this we get:

$$\left(\frac{vP}{2\pi}\right)^3 = \frac{GM_{\oplus}P^2}{4\,\pi^2}$$

Solving for the orbital velocity of the Moon, we get:

$$v = \left(\frac{2\pi GM}{P}\right)^{\frac{1}{3}}.$$

You could have obtained the same expression starting with  $v = \sqrt{GM_{\oplus}/r}$  derived in problem 1a, and substituting  $r = vP/(2\pi)$ . Now, we can obtain the numerical value, if we express Moon's period in seconds (P = 27 days = 27 days × 24 hours/day × 60 minutes/hour × 60 sec/minute = 2 × 10<sup>6</sup> sec):

$$\mathbf{v} = \left(\frac{2\pi \ 6.7 \times 10^{-11} \mathrm{m}^3 \ \mathrm{kg}^{-1} \mathrm{s}^{-2} \ \times 6.0 \times 10^{24} \mathrm{kg}}{2 \times 10^6 \mathrm{sec}}\right)^{\frac{1}{3}} = 1000 \ \mathrm{m/s} = 1 \mathrm{km/s}.$$

#### 5 points off for not deriving the formula, but quoting it to get the result

d) Since the Moon is very close to the Earth compared to the distance to the Sun (check this!), the shadow of the Moon is a circle with the Moon's diameter, and the shadow moves over the Earth with the orbital velocity of the Moon. How long does it take for the shadow of the Moon to completely sweep across the diameter of the Earth (in hours)? Earth's radius is ~ 6400km, Lunar radius is ~ 1700km.

Hint: you are looking for the time between the moment the Moon's shadow begins to cross the Earth to the time the shadow completely leaves the Earth. Draw a sketch to help you along. (15 points)



Figure 2: Illustration of the shadow of the Moon crossing the face of the Earth. The distance the shadow travels includes the entry and exit parts of the eclipse – the diameter of the Moon has to be added to the diameter of the Earth.

**Solution:** The distance to the moon is 400000 km, while the distance to the Sun is 150 million km. Thus, the Moon is really close to the Earth, and the shadow will be a circle of the size very close to the size of the Moon. If the Moon were further away from the Earth, the size of the shadow would have been larger. Imagine putting a piece of paper close to a wall and looking at its shadow from a bright light far away. The further the paper is from the wall, the larger the shadow. Ok, now the circular shadow of the Moon is moving across the face of the Earth with the speed of 1 km/sec (orbital velocity) and has the diameter  $2R_{Moon}$ . In order to completely cross the face of the Earth, the shadow will have to move a total distance of  $2R_{\oplus} + 2R_{Moon} = 2(6400 \text{ km} + 1700 \text{ km}) = 16000 \text{ km}$ . At the speed of 1 km/s this will take 16000 seconds, or, roughly 4.5 hours. This is the total time the shadow of the moon will be *anywhere* on the face of the Earth. The duration of the eclipse as seen from any given location on the Earth will be much shorter – typical totality during solar eclipses lasts less than 10 minutes.

5 points off for neglecting to add the diameter of the Moon to the calculation

## **3. Radioactive decay** 20 total points

a) Cesium-137 (<sup>137</sup>Cs; Cs has atomic number 55), is unstable isotope of Cesium which decays through the "beta-decay," or the reaction that turns one of the neutrons in the nucleus into a proton and also emits an electron ("beta-particle") and an antineutrino. After beta-decay, what type of atom has <sup>137</sup>Cs become? Give the symbol of the resulting element, its atomic mass and atomic number. You will need to refer to the periodic table, e.g. in Appendix D of the text, to answer this problem. (10 points) Solution: During beta decay one neutron in the nucleus turns into a proton, which increases the charge of the nucleus (to conserve the total charge, a negatively charged electron, or beta-particle, escapes). Thus, atomic number (also known as proton number) will increase by one to 56, and element will be called Barium. The total atomic mass remains the same at 137, as electron takes away negligible amount of mass (for contrast, during alpha decay the atomic mass drops by 4 – the mass of

two protons and two neutrons). The answer is then: Barium-137, atomic number 56, atomic mass 137.

2 points off for getting the element wrong, 2 points off for missing either atomic number or atomic mass.

b) The Chernobyl nuclear disaster was a notorious nuclear accident that happened in 1986 in Ukraine. A fission reactor melted, and part of its core, containing nuclear fuel was vaporized and contaminated a very large area (from Ukraine to Sweden). Significant amount of the fallout consisted of Cesium-137, whose half-life is 30 years. Due to its location in the periodic table, Cesium has very similar chemical properties to Potassium, and is readily taken up by the human body, where it damages the tissues as it emits electrons (and gamma rays) during the decay. This element is one of the main health hazards of the Chernobyl fallout. What is the fraction of Cesium-137 that has decayed away by now? *Hint: The mass of the material undergoing radioactive decay follows the following formula with time:*  $M(t) = M_0(\frac{1}{2})^{(t/t_{1/2})}$ , where  $M_0$  is the initial mass of the material, and  $t_{1/2}$  is the half-life. (10 points)

**Solution:** We know that the mass of the radioactive material which had initial mass  $M_0$  is decreasing with time as  $M(t) = M_0(\frac{1}{2})^{(t/t_{1/2})}$ . The fraction that has decayed away by time t is then

$$f_{decay} = \frac{M_0 - M(t)}{M_0} = 1 - \left(\frac{1}{2}\right)^{t/t_{1/2}}$$

We know that the accident happened in 1986, or 23 years ago. The half-life is 30 years, so  $f_{decay} = 1 - 0.5^{(23/30)} = 0.41 \approx 0.4$ , or 40% to one significant figure. Note that we knew the half-life to one significant figure.

3 points off for calculating the fraction that has not decayed away – the mass that is left.

## 4. Light and power 10 total points

Choose the answer, and explain why this answer is correct. Provide formulas that support your choice.

1) (5 points)

A star has the same radius as the Sun, but its surface temperature is twice as large. The luminosity of this star is:

- a) twice as large as the Sun's;
- b) four times as large as the Sun's;
- c) eight times as large as the Sun's;
- d) sixteen times as large as the Sun's;
- 2) (5 points)

The wavelength of the peak of black body emission from this star is:

- a) two times longer than the Sun's;
- b) same as the Sun's;
- c) half that of the Sun's;
- d) one quarter that of the Sun's;

**Solution:** 1) The luminosity of the star is described by black-body emission formula  $L = \sigma T^4 \times (\text{SurfaceArea}) = \sigma T^4 (4\pi R^2)$ , where  $\sigma$  is the Stefan-Boltzman constant, T is the surface temperature, and R is the stellar radius. In order to compare with the Sun's luminosity, let's form a ratio:

$$\frac{L_*}{L_\odot} = \frac{4\pi\sigma T_*^4 R_*^2}{4\pi\sigma T_\odot^4 R_\odot^2} = \left(\frac{T_*}{T_\odot}\right)^4 \left(\frac{R_*}{R_\odot}\right)^2,$$

where  $\odot$  is the symbol commonly used for the Sun. We have  $T_* = 2T_{\odot}$ , and  $R_* = R_{\odot}$ , hence, the ratio is  $L_*/L_{\odot} = 2^4 = 16$ , or answer "d".

2) The Wein Law relates the wavelength of the peak of the black-body emission to the temperature of the black body:  $\lambda_{max} = (2.9/T)$  mm, where T is measured in Kelvin. The temperature of the star in question is twice that of the Sun, hence the wavelength of the peak is two times shorter, or answer "c".

If formulas are given but the answer is incorrect, take 2 points off