General grading rules for calculational problems (in both Sections A and C): 4 points off for each arithmetic or algebraic error. 2 points off per problem for excess significant figures in a final result (i.e., more than 2 sig figs). In computational problems in this exam, there is no requirement to write in full sentences. Explanations should be clear, even if they use a minimum of English; if the context is unambiguous, no points off for undefined symbols. However, take off 2 points per problem if a student gives absolutely no context for their calculation (this should be relatively rare). Not giving units or not giving results in the units asked for, is 2 points off. Giving the correct number with incorrect units is 4 points off. In problem 1b (and depending how they do it, 3b), the answer depends on the previous parts; in these cases, you should give full credit for consistent results. Take an additional 3 points off in cases in which the answer is absurd (e.g., the Milky Way has a mass much less than a solar mass, the speed is thousands of times the speed of light, etc.), without any comment that something is not quite right.

In many of the problems, there is an easy way and a hard way to do it; full credit for doing problems the hard way (and getting the right answer). The exam is worth a total of 300 points. Do all problems.

Section A: Analytic Questions (100 points)

1. Distant Stars (40 points)
In this problem, we will consider stars in two galaxies, A and B.

   a. (10 points) An emission line is seen in the spectrum of galaxy A at a wavelength of 4590 Ångstroms. You recognize this emission line as one that has a rest wavelength of 4500 Ångstroms. Calculate the redshift, and use it to calculate the recession velocity of the galaxy in kilometers per second.

   Answer: The redshift is given by:

   \[
   z = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{90 \text{ Å}}{4500 \text{ Å}} = 0.02.
   \]

   This is equal to \(v/c\), solving for \(v\) gives 6000 km/s.

   5 points for calculating the dimensionless redshift, \(z = 0.02\), and going no further.

   4 points off for not knowing what a redshift is, and thinking it’s just \(\lambda - \lambda_0\), or not explicitly calculating \(z\), but getting the velocity correctly.

   b. (10 points) Use Hubble’s Law to calculate the distance to galaxy A, in Megaparsecs. One significant figure is adequate.

   Answer: The Hubble Law relates the redshift \(v\) and the distance \(d\) of a galaxy:

   \[
   v = H_0d
   \]
Solving for \( d \), and putting in our value of the Hubble Constant gives:
\[
d = \frac{6000 \text{ km/s}}{70 \text{ km/s/Mpc}} \approx 90 \text{ Mpc}.
\]

3 points for writing down the Hubble law and going no further. Full credit for a distance of 100 Mpc.

c. (10 points) The star in Galaxy B has four times the luminosity of the star in Galaxy A. Galaxy A has a redshift two times that of Galaxy B. Neither star is affected by dust. What is the ratio of the brightness of star A to star B?

**Answer:** Redshift is proportional to distance, so Galaxy A is twice the distance of Galaxy B. The brightness is related to luminosity as:
\[
b = \frac{L}{4 \pi d^2},
\]
so
\[
\frac{b_A}{b_B} = \frac{L_A}{L_B} \left( \frac{d_B}{d_A} \right)^2 = \frac{1}{4} \times \left( \frac{1}{2} \right)^2 = \frac{1}{16}.
\]
Star B is 16 times brighter than star A.

3 points for recognizing the relevance of the inverse square law and going no further.
2 points for recognizing that Galaxy B is half the distance of Galaxy A.

d. (10 points) Consider these two stars as black bodies. From their spectra, you find that the peak wavelength of star A is 5000 Å, and the peak wavelength of star B is 2500 Å. What is the ratio of the radius of star A to that of star B?

**Answer:** For blackbodies, \( L \propto T^4 R^2 \). Therefore:
\[
\frac{R_A}{R_B} = \left( \frac{T_B}{T_A} \right)^2 \left( \frac{L_A}{L_B} \right)^{1/2}.
\]
As the peak wavelength is inversely proportional to temperature, the ratio of temperatures is the inverse of ratios of peak wavelengths. The ratio of luminosities was given to us in part (c). So we get:
\[
\frac{R_A}{R_B} = 2^2 \left( \frac{1}{4} \right)^{1/2} = 2.
\]

3 points for recognizing the relevance of the black-body formula and going no further.
2 points for correctly inferring the ratio of temperatures from the peak wavelengths.

2. Orbiting Galaxy (40 points)
The Large Magellanic Cloud (LMC) is a small nearby galaxy in orbit around the center of the Milky Way. It lies at a distance of 150,000 light years from the center of the Milky Way. Its proper motion (i.e., its motion across the sky) has recently been measured to be \( 1 \times 10^{-3} \) arcseconds per year.
a. (15 points) Calculate the orbital speed of the Large Magellanic Cloud, in kilometers per second. You may assume its orbit to be circular, and make the approximation that the Sun lies at the center of the Milky Way (as the distance between the center of the Milky Way and the Sun is much less than 150,000 light years).

**Answer:** This is a problem in unit conversion and the small-angle formula. The distance the object moves in one year is:

$$1 \times 10^{-3} \text{arcsec} \times \frac{1 \text{ radian}}{2 \times 10^5 \text{ arcsec}} \times 1.5 \times 10^5 \text{ light years} \times \frac{10^{13} \text{ km}}{1 \text{ lightyear}} = 7.5 \times 10^9 \text{ km/yr}.$$

Converting to km/sec, we find

$$7.5 \times 10^9 \text{ km/year} \times \frac{1 \text{ year}}{3 \times 10^7 \text{ sec}} = 250 \text{ km/s}.$$

Rounding to 300 km/s is OK. 3 points for recognizing the small-angle formula, and going no further. 7 points for correctly calculating the velocity based on Kepler’s third law and assuming that the mass enclosed is the same as in the solar circle – this misses a lot of the mass beyond the solar circle!

b. (10 points) Calculate the time it will take the LMC to make one orbit around the Milky Way, in years. *Hint: there is a way to do this without using the results of part (a).*

**Answer:** The easy way to do this is to realize that one orbit is 360 degrees = 360×3600 arcsec. Thus to go all the way around is

$$\frac{360 \times 3600 \text{ arcsec}}{1 \times 10^{-3} \text{ arcsec/year}} = 1.3 \times 10^9 \text{ years}.$$

Another way to do this is to calculate the circumference of the circle, and divide by the speed. This is much more work, but of course gets full credit if done correctly.

*Full credit for being consistent with the results of part a, even if these are wrong. Starting with part b, and working backwards to part a, gets full credit if done correctly.*

c. (15 points) Calculate the mass of the Milky Way out to the radius of the LMC from this information; express your result in solar masses. *Hint: The information on the cheat sheet about the orbit of the Sun around the Milky Way may be useful.*

**Answer:** The easiest way to do this is to scale directly from the equivalent result at the Solar Circle. The mass scales as $a^3/P^2$ by Kepler’s Law, which gives a factor of $6^3/5^2 = 8$. Thus the mass out to the LMC is eight times that at the solar circle, or $8 \times 10^{11} \text{ M}_\odot$.

*Full credit for assuming that for flat rotation curves, $M \propto R$, and thus getting $6 \times 10^{11} \text{ M}_\odot$ (but this requires an explanation of what they’re doing; without this explanation, take 5 points off). Most people will use Newton’s form of Kepler’s equation; full credit for doing it right. That’s much more work. Some people may do it incorrectly by assuming a constant volume or areal density; give 3 points credit for*
this if the arithmetic looks right, otherwise no credit at all. Note that the precision of
the numbers on the cheat sheet are such that different approaches using Kepler’s law
will get somewhat different numbers, between 5 and $8 \times 10^{11} \text{M}_\odot$; those are all fine.
Simply stating Kepler’s law and going no further gets 4 points.

3. Accreting black hole (20 points)
Cygnus X-1 is a black hole in our Galaxy, accreting matter from a companion star. The
matter in the accretion disk heats up to tremendous temperatures and produces a luminosity
equal to $4 \times 10^4$ times that of the Sun. This emission is well characterized as a black body
emitted by a sphere with radius equal to the Schwarzschild radius of the black hole, and a
surface temperature 2000 times that of the Sun.

a. (5 points) In what wavelength band (e.g., optical, infrared, radio) does the peak of
the black body spectrum of Cygnus X-1 lie? Can it be observed from the ground?

**Answer:** The peak of the black body spectrum at the temperature of $2000 \times 6000 K =
1.2 \times 10^7 K$ is $\lambda_{\text{max}} = 3\text{mm}/1.2 \times 10^7 K = 2.5 \times 10^{-10} = 2.5 \text{ Å}$. This is the
wavelength of X-rays, and cannot be observed from the ground, because X-rays are
absorbed by the atmosphere.

*Full credit for recognizing that X-1 stands for an X-ray source and not doing any
calculations. 2 points for correctly finding the peak wavelength, 2 points for recognizing
which band it is in and 1 point for recognizing it’s not observable from the ground. If
someone says that it is also emitting optical light due to the secondary and therefore
will be visible from the ground, give them the point, and a ticket to AST 204.*

b. (15 points) What is the mass of the black hole in Cygnus X-1? Express your answer
in solar masses.

**Answer:** In order to determine the mass, we need to find the Schwarzschild radius of
the black hole. We know that a sphere with Schwarzschild radius at the temperature
of $T_{\text{BH}} = 2000 T_\odot$ produces the same luminosity as Cygnus X-1, or $4 \times 10^4 L_\odot$. So,

$$4\pi R_{\text{BH}}^2 \sigma T_{\text{BH}}^4 = 4 \times 10^4 L_\odot = 4 \times 10^4 \times (4\pi R_\odot^2 \sigma T_\odot^4)$$

Here, we expressed the solar luminosity through the black body formula. Now, we
can divide one side by the other:

$$\left(\frac{R_{\text{BH}}}{R_\odot}\right)^2 \left(\frac{T_{\text{BH}}}{T_\odot}\right)^4 = 4 \times 10^4,$$

or

$$R_{\text{BH}} = 2 \times 10^2 R_\odot \left(\frac{T_\odot}{T_{\text{BH}}}\right)^2 = 2 \times 10^2 \frac{T_\odot}{2000^2} R_\odot = 5 \times 10^{-5} R_\odot = 35\text{km}.$$  

One solar mass black hole has the Schwarzschild radius of 3 km. Since the Schwarzschild
radius is proportional to mass, Cygnus X-1 must have a $35/3 \approx 12$ solar mass black
hole.
Section B: Essay Questions (100 points)

This section asks you to write two essays. To give full answers, each essay will have to be several paragraphs long at least. Please use equations, diagrams, and numbers, if it helps you make your point. And write in full sentences, of course!

4. (50 points) You meet a person at a party who is convinced that the Universe is 5000 years old. Utilizing the scientific method, outline the arguments that you can use to disprove this point of view. Discuss what you know about the ages of mass extinctions of life on Earth, the Earth itself, the Solar system, the Sun, star clusters, the Galaxy, and the Universe as a whole. Most importantly, discuss how these ages are measured or inferred in each case.

Answer
- We determine the ages of rocks using radioactive dating. (6 points, plus 3 points for explanation of how it works, plus 2 points for mentioning Potassium/Argon).
- Ages of mass extinctions are determined by looking at radioactive dating of rocks in craters (5 points). Extinction of the dinosaurs was thus dated to be 65 million years ago (5 points + 2 points for mentioning Iridium layer).
- Radioactive dating of fossils also gets 5 points.
- From radioactive dating, we determine the ages of oldest rocks on the Earth to be 4.5 billion years – this is the age of the Earth (5 points).
- Age of the Solar system is determined from radioactive dating of meteorites, which yields a very similar age as the Earth (5 points).
- Age of the Sun is determined from modeling stellar evolution: knowing the mass and luminosity of the Sun and the energy production in nuclear reactions of hydrogen fusion, we can calculate the age of the Sun to be 4.5 billion years – very close to the age of the planets in the Solar System (8 points). This way we can also calculate how long the sun will live on the main sequence (10 billion years) (5 points).
- Ages of star clusters are determined by looking at the main sequence turnoff in the HR diagram (5 points). Younger stars have evolved off the main sequence, so knowing how long stars live in different parts of the HR diagram, we can determine the age of the star cluster. The oldest clusters are close to 13 billion years old (5 points).
- The Milky Way Galaxy is as old as the oldest star clusters (globular clusters), which are 13.5 billion years old (8 points).
- We see objects in the galaxy and other galaxies that are further than 5000 light years away, and hence, the Universe must be older than 5000 yrs (8 points).
- The expansion of the Universe, as measured from redshifts of galaxies (5 points), allows us to calculate the age when the expansion started. This gives 13.7 billion years (10 points). It is calculated as the inverse of the Hubble constant (5 points).

Ages have to be right to a factor of 5 for the mass extinction, and to factor of 2 for the Universe. Creative answers based on Gott’s Copernican principle get half the credit for the
corresponding age. Any other argument deemed convincing by the grader gets 8 points each. Two points off for every egregiously incorrect statement.

5. (50 points) The Hertzsprung-Russell (H-R) diagram is a basic tool that astronomers use to understand the properties of stars. Explain what the H-R diagram is, sketch it (be sure to label your axes!), and explain how the placement of stars on the diagram relates to the masses, temperatures, radii, lifetimes, and evolutionary stages of stars. Your explanation should clearly explain the nature of main sequence, red giant, and white dwarf stars, and the evolutionary relationship between the three.

25 points for drawing the figure correctly; 5 points off for no axes, 7 points off for forgetting to indicate which are main sequence, giants, and dwarfs.

- 10 points for explaining correctly that more luminous stars at a given temperature must be larger, by \(L \propto T^4 R^2\). But give 5 points out of 10 for correctly describing all the characteristics of giants and dwarfs \((L, T, R)\), without explicitly stating that \(L \propto T^4 R^2\) is the reason why high \(L\) implies large \(R\).
- 5 points for explaining the relation between temperature (or luminosity) and mass on the main sequence.
- 5 points for a brief description that stars evolve from main sequence to giants to dwarfs (no details about hydrogen burning, etc., are required).
- 5 points for pointing out the depletion of hydrogen and switching to helium burning at the end of main sequence.
- 5 points for correct description of the mass-lifetime relation, and pointing out that high mass stars are in the upper left, and mass decreases going lower right. 3 points only for the latter description only (without the lifetime relation) or the former only (without pointing out where on the HR diagram high mass and low mass stars lie).

5 points off for each egregiously incorrect statement.

The most common mis-statement here will involve mixing up the concepts of mass, luminosity, and size. For example, some students will erroneously state that red giants are big because they are massive.

10 points off for not writing in full sentences.
Section C: Relativity, Cosmology, and Everything (100 points)

This section has a mixture of computational problems and short response questions. The short response questions can be answered in a paragraph or two; in some cases (e.g., #13), the response can be shorter than that.

6. (15 points) Rank the following objects in order of their distance from Earth starting with the closest: Uranus, Kuiper belt, Andromeda Galaxy (M31), WMAP satellite, Cosmic Microwave Background, Hubble Space Telescope, Alpha Centauri, the Sun, the Moon, the center of the Milky Way, Small Magellanic Cloud, the gamma ray burster announced during the last lecture, Jupiter, Sirius, the Orion Nebula, the Virgo cluster.

Answer:
- HST
- Moon
- WMAP
- Sun
- Jupiter
- Uranus
- Kuiper belt
- Alpha Centauri
- Sirius
- Orion Nebula
- Center of the Milky Way
- Small Magellanic Cloud
- Andromeda
- Virgo cluster
- gamma ray burster
- Cosmic Microwave Background.

To grade this, go down their list in pairs. Take the first two entries: if they are in the correct order, i.e., the first entry is closer than the second, then give them 1 point. Then consider the second and third entries, and again give 1 points if they are correctly ordered. Continue down; there are ten such comparisons, and therefore 15 points possible. Full credit for any abbreviations of the different objects, as long as they are clear (e.g., they can write just ‘M31’, instead of ‘M31, the Andromeda Galaxy’.)

7. (10 points) How did observations of Supernova 1987a confirm our theory of supernova explosions? 10 points for any hint of the following:
- We saw the progenitor of the supernova before it exploded – it was a massive star, confirming that supernovae come from massive stars
- We saw neutrinos from this supernova, suggesting the collapse of a core of the star, and formation of neutrons from protons and electrons.
- We saw gamma-ray lines from the radioactive Cobalt, confirming that nucleosynthesis of heavy elements takes place in supernovae.

3 points off for egregiously incorrect statements.

8. (10 points) An astronaut passes you and you see his clock is ticking at 80% the rate of your clock. How fast is the astronaut going with respect to you?
\[
\sqrt{1 - \left(\frac{v}{c}\right)^2} = 0.8, \text{ therefore, } \left(\frac{v}{c}\right)^2 = 0.36, \text{ or } v = 0.6c = 1.8 \times 10^8 \text{m/s}. \text{ Full credit for leaving the answer in units of } c.
\]

9. (10 points) Describe how inflation explains 1) how the Big Bang explosion got started, 2) why the Cosmic Microwave Background is so uniform in temperature over the whole sky.

Answer

1) During inflation, the Universe expanded exponentially due to the repulsive vacuum energy. This began the Big Bang expansion, which slowed down when matter and radiation formed in the Universe. 2) The regions that inflated were on the Planck length scale \((10^{-35}\text{m})\), and were in thermal equilibrium before inflation. This equilibrium was maintained in different regions of space when it expanded, so the CMB is uniform in temperature. 

*Each subquestion is worth 5 points*

10. (10 points) Match the following with something they did

(Write the answers in your exam book as numerical pairs, e.g., 5-9, 7-2, corresponding to entries from the left and right column, respectively)

1) Alan Guth 1) Wrote down equations of Electromagnetism
2) Richard Gott and Li-Xin Li 2) Discovered expansion of the universe
3) Adam Riess 3) Discovered that black holes have finite entropy
4) James Maxwell 4) Proposed wormhole time machines
5) Albert Einstein 5) Developed theory of gravity using curved spacetime
6) Jacob Beckenstein 6) Found that black holes eventually evaporate
7) Stephen Hawking 7) Discovered that expansion of universe is accelerating
8) Edwin Hubble 8) Proposed Big Bang model
9) Alexander Friedmann 9) Proposed that universe started with a time loop
10) Kip Thorne 10) Proposed theory of inflation

Answer:

1-10, 2-9, 3-7, 4-1, 5-5, 6-3, 7-6, 8-2, 9-8, 10-4.

*Give a point for each correct pair*

11. (10 points) Which of the following statements about black holes is not true? No explanation is needed; simply state which statement it is.

(a) If you watch someone else fall into a black hole, you will never see him or her cross the event horizon. However, he or she will fade from view as the light he or she emits (or reflects) becomes more and more redshifted.
(b) If we watch a clock fall toward a black hole, we will see it tick slower and slower as it falls nearer to the black hole.
(c) A black hole is truly a hole in spacetime, through which we could leave the observable universe.
(d) If the Sun magically disappeared and was replaced by a black hole of the same mass, Earth would soon be sucked into the black hole.
(e) If you fell into a black hole, you would experience time to be running normally as you plunged rapidly across the event horizon.

Answer: d. *If more than one statement is called false, take off 2 points for each. No partial credit otherwise.*
12. (10 points) Describe Prof. Gott’s time machine for visiting the past. What does it consist of? How does it work (only a brief answer is necessary)? If you were to make such a time machine today, could you use it to visit Cleopatra in ancient Egypt? Why or why not?

**Answer**

Gott’s time machine involves two cosmic strings, moving past each other close to the speed of light. (5 points) The spacetime is warped in such a way that it is possible to outrun a light beam while going around the strings, and visit yourself in the past. (2 points). You cannot visit the time before the time machine was created, though, so ancient Egypt is off limits. (3 points)

13. (5 points) What color is Einstein’s house in Princeton? (Remember the assignment to go see it!)

**Answer:** White. No partial credit.

14. (10 points) If you converted all the mass of a bowling ball into energy it would make an explosion like which of the following: a typical firecracker, enough to destroy a city, enough to destroy the Earth? State your reasoning.

**Hint:** 1 ton of TNT releases roughly $10^9$ Joules when it explodes.

**Answer:** This is an estimate, so take the bowling ball to be 2 kg. $E = mc^2 = 2\text{kg} \times (3 \times 10^8 \text{m/s})^2 \approx 10^{17} J = 10^8 \text{tons of TNT} = 100 \text{ megatons of TNT}$. This is larger but close to the largest thermonuclear bomb – definitely enough to destroy a city, but not the Earth!

This is an estimate, so take points off for egregious estimates of the mass of the ball – .5 to 10 kg should be ok. 7 points for getting the number in TNT units, but not knowing whether it’s large or small.

15. (10 points) Assuming that your location in the history of the United States is not special, give 95% confidence limits for the future longevity of the United States – i.e., there is a 95% chance that the United States government will last at least _______ more years, but less than _______ more years. Please show your reasoning.

**Hint:** 95% confidence means that you estimate that there is a 2.5% chance of the result being less than the lower limit and a 2.5% chance that it will exceed the upper limit.

**Answer.** The USA is 233 years old, so we estimate it can stay together at least $1/39$th, or 6 years, to 39 times as long: 9000 years.

5 points for writing $1/39$ and $39x$ but going no further.

May your world-lines visit only happy regions of spacetime.

Chris Chyba, J. Richard Gott, Michael A. Strauss, and Anatoly Spitkovsky