RADIATIVE ENVELOPES - CONVECTIVE ENVELOPES

Let us consider a model of a stellar envelope, i.e. outer part of a star, located below the photosphere. Let the opacity be given as a power law

$$\kappa \approx \kappa_0 \rho^{\kappa_\rho} T^{\kappa_T}, \qquad \kappa_0, \kappa_\rho, \kappa_T = \text{const.} \tag{s2.11}$$

For simplicity we shall assume that pressure is dominated by gas pressure, i.e.

$$P \approx \frac{k}{\mu H}\rho T, \qquad \frac{k}{\mu H} = \text{const.} \tag{s2.12}$$

We shall write the equations of hydrostatic and radiative equilibria as

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho, \tag{s2.13}$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi acT^3 r^2}. \tag{s2.14}$$

The radiative temperature gradient is given as

$$\nabla_{rad} \equiv \left(\frac{d\ln T}{d\ln P}\right)_{rad} = \frac{3L_r}{16\pi acGM_r}\frac{\kappa P}{T^4} \approx \frac{3L}{16\pi acGM}\frac{\kappa P}{T^4}, \tag{s2.15}$$

as near the surface $M_r \approx M$ and $L_r \approx L$.

Let us take a logarithmic derivative of equation (s2.14) with respect to optical depth:

$$\frac{d\left(\ln \nabla_{rad}\right)}{d\tau} = \frac{d\ln\kappa}{d\tau} + \frac{d\ln P}{d\tau} - 4\frac{d\ln T}{d\tau}. \tag{s2.16}$$

The equations (s2.11) and (s2.12) may be differentiated to obtain

$$d\ln P = d\ln\rho + d\ln T = \frac{1}{\nabla_{rad}} d\ln T, \tag{s2.17a}$$

$$d\ln\kappa = \kappa_\rho d\ln\rho + \kappa_T d\ln T = \kappa_\rho d\ln P + (\kappa_T - \kappa_\rho)\, d\ln T = \tag{s2.17b}$$

$$\left[\frac{\kappa_\rho}{\nabla_{rad}} + (\kappa_T - \kappa_\rho)\right] d\ln T,$$

and the equation (s2.16) may be written as

$$\frac{1}{\nabla_{rad}}\frac{d\nabla_{rad}}{d\tau} = \left[\frac{\kappa_\rho + 1}{\nabla_{rad}} + (\kappa_T - \kappa_\rho - 4)\right]\frac{d\ln T}{d\tau}. \tag{s2.18}$$

Within Eddington approximation the variation of temperature with optical depth is approximated as

$$T^4 = T_{eff}^4\left(\frac{1}{2} + \frac{3}{4}\tau\right), \qquad \frac{d\ln T}{d\tau} = \frac{3}{8+12\tau}. \tag{s2.19}$$

Combining equations (s2.18) and (s2.19) we obtain

$$\frac{d\nabla_{rad}}{(\kappa_\rho + 1) + (\kappa_T - \kappa_\rho - 4)\nabla_{rad}} = \frac{3\,d\tau}{8+12\tau}. \tag{s2.20}$$

All radiative opacities either increase or stay constant with density, i.e. $\kappa_\rho >= 0$. At the surface we have $\tau = 0$, $T = 2^{-1/4} T_{eff}$, $\rho = 0$, and hence $\kappa = 0$. Therefore, at the surface $\nabla_{rad} = 0$ and we may integrate equation (s2.20) to obtain

$$\frac{1}{\kappa_T - \kappa_\rho - 4} \ln\left(1 + \frac{\kappa_T - \kappa_\rho - 4}{\kappa_\rho + 1}\nabla_{rad}\right) = \frac{1}{4}\ln\left(1 + 1.5\tau\right),$$

$$1 + \frac{\kappa_T - \kappa_\rho - 4}{\kappa_\rho + 1}\nabla_{rad} = \left(1 + 1.5\tau\right)^{(\kappa_T - \kappa_\rho - 4)/4},$$

$$\nabla_{rad} = \frac{\kappa_\rho + 1}{\kappa_T - \kappa_\rho - 4}\left[\left(1 + 1.5\tau\right)^{(\kappa_T - \kappa_\rho - 4)/4} - 1\right]. \qquad \text{(s2.21)}$$

As $\kappa_\rho > 0$ then $\kappa_\rho + 1 > 0$, but the sign of $(\kappa_T - \kappa_\rho - 4)$ may be either positive or negative. If $(\kappa_T - \kappa_\rho - 4) < 0$ then the expression in the square bracket in equation (s2.21) approaches -1 while $\tau \longrightarrow \infty$, and we have

$$\nabla_{rad} \longrightarrow -\frac{\kappa_\rho + 1}{\kappa_T - \kappa_\rho - 4}, \qquad \text{while} \quad \tau \longrightarrow \infty, \quad \text{and} \quad \kappa_T - \kappa_\rho - 4 < 0. \qquad \text{(s2.22)}$$

In the opposite case we have

$$\nabla_{rad} \longrightarrow \infty, \qquad \text{while} \quad \tau \longrightarrow \infty, \quad \text{and} \quad \kappa_T - \kappa_\rho - 4 > 0. \qquad \text{(s2.23)}$$

Adiabatic temperature gradient $\nabla_{ad} = 0.4$ for a perfect gas equation of state (s2.12) . If $(\kappa_T - \kappa_\rho - 4) > 0$ then envelope becomes convective at the optical depth

$$\tau_{conv} = \frac{2}{3}\left[\left(0.4\frac{\kappa_T - \kappa_\rho - 4}{\kappa_\rho + 1} + 1\right)^{4/(\kappa_T - \kappa_\rho - 4)} - 1\right]. \qquad \text{(s2.23)}$$

We shall consider three common cases of opacity in the stellar envelopes: electron scattering, Kramers opacity, and negative hydrogen ion (H^-) opacity. In the first case we have $\kappa_\rho = 0$, $\kappa_T = 0$, $(\kappa_T - \kappa_\rho - 4) = -4 < 0$, and $\nabla_{rad} \longrightarrow 1/4 < 0.4$ while $\tau \longrightarrow \infty$, i.e. electron scattering envelope stays radiative at any optical depth. For Kramers opacity we have $\kappa_\rho = 1$, $\kappa_T = -3.5$, $(\kappa_T - \kappa_\rho - 4) = -8.5 < 0$, and $\nabla_{rad} \longrightarrow 1/4.25 < 0.4$ while $\tau \longrightarrow \infty$, i.e. such envelope is also radiative. However, for the H^- opacity $\kappa_\rho = 0.5$, $\kappa_T = 7.7$, $(\kappa_T - \kappa_\rho - 4) = 3.2 > 0$, and $\nabla_{rad} = 0.4$ for $\tau = \tau_{conv} = 0.775$, i.e. the envelope becomes convective just under the photosphere. Notice, that a transition from radiative atmosphere to convective envelope does not depend on the value of opacity, κ_0, but on the rate at which the opacity changes with density and temperature. Negative hydrogen ion becomes the dominant opacity source at a temperature lower than $10^4\,\mathrm{K}$. For this reason cool stars have convective envelopes.