Kunz Lecture Notes for the 2025 SCEECS/NSF Summer School

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CONTENTS

PART I

Introduction to astrophysical plasmas

I.1. What is a plasma?

Astrophysical plasmas are remarkably varied, and so it may appear difficult at first to provide a definition of just what constitutes a "plasma". Is it an ionized, conducting gas? Well, the cold, molecular phase of the interstellar medium has a degree of ionization of $\lesssim 10^{-6}$, and yet is considered a plasma. (Indeed, plenty of researchers still model this phase using ideal magnetohydrodynamics!) Okay, so perhaps a sufficiently ionized, conducting gas (setting aside for now what is meant precisely by "sufficiently")? Well, plasmas don't necessarily have to be good conductors. Indeed, many frontier topics in plasma astrophysics involve situations in which resistivity is fundamentally important.

Clearly, any definition of a plasma must be accompanied by qualifiers, and these qualifiers are often cast in terms of dimensionless parameters that compare length and time scales. Perhaps the most important dimensionless parameter in the definition of a plasma is the *plasma parameter*,

$$\Lambda \doteq n_{\mathrm{e}} \lambda_{\mathrm{D}}^{3}, \tag{I.1}$$

where n_{e} is the electron number density and

$$\lambda_{\mathrm{D}} \doteq \left(\frac{T}{4\pi e^2 n_{\mathrm{e}}} \right)^{1/2} = 7.4 \left(\frac{T_{\mathrm{eV}}}{n_{\mathrm{cm^{-3}}}} \right)^{1/2} \ \mathrm{m} \tag{I.2}$$

is the Debye length. We'll derive this formula for the Debye length and discuss its physics more in § III.1 of these notes, but for now I'll simply state its meaning: it is the characteristic length scale on which the Coulomb potential of an individual charged particle is exponentially attenuated ("screened") by the preferential accumulation (exclusion) of oppositely- (like-) charged particles into (from) its vicinity.[1] Thus, Λ reflects the number of electrons in a Debye sphere. Its dependence upon the temperature T suggests an alternative interpretation of Λ:

$$\Lambda = \frac{T}{4\pi e^2/\lambda_{\mathrm{D}}} \sim \frac{\text{kinetic energy}}{\text{potential energy}}. \tag{I.3}$$

Indeed, if the plasma is in thermodynamic equilibrium with a heat bath at temperature T, then the concentration of discrete charges follows the Boltzmann distribution,

$$n_\alpha(\boldsymbol{r}) = \overline{n}_\alpha \exp\left(-\frac{q_\alpha \phi(\boldsymbol{r})}{T} \right), \tag{I.4}$$

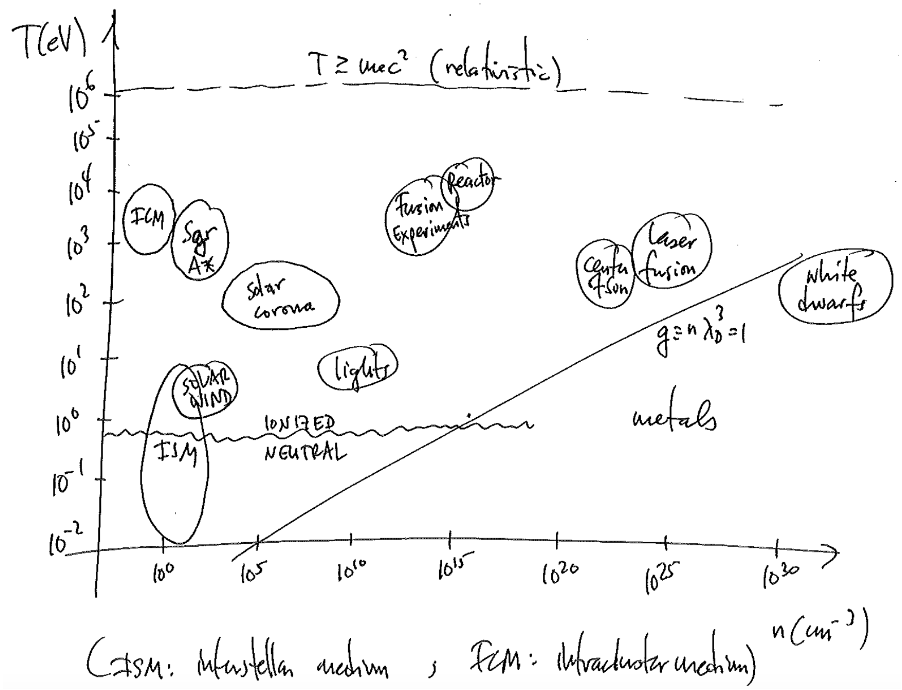
where $\overline{n}_\alpha$ is the mean number density of species α, q_α is its electric charge, and $\phi(\boldsymbol{r})$ is the Coulomb potential. In the limit $\Lambda \to \infty$, the distribution of charges becomes uniform, i.e., the plasma is said to be *quasi-neutral*, with equal amounts of positive and negative charge within a Debye sphere.

Debye shielding is fundamentally due to the polarization of the plasma and the associated redistribution of space charge, and is an example of how a plasma behaves as a dielectric medium. The hotter plasma, the more kinetic energy, the less bound individual electrons are to the protons. When $\Lambda \gg 1$, collective electrostatic interactions are much more important than binary particle–particle collisions, and the plasma is said to be

[1] In this course, sometimes temperature will be measured in Kelvin, and sometimes temperature will be measured in energy units (eV) after a hidden multiplication by Boltzmann's constant k_{B}. An energy of 1 eV corresponds to a temperature of $\sim 10^4$ K (more precisely, $\simeq 1.16 \times 10^4$ K).

weakly coupled. These are the types of plasmas that we will focus on in this course (e.g., the intracluster medium of galaxy clusters has $\Lambda \sim 10^{15}$).

Shown below is a rogue's gallery of astrophysical and space plasmas in the T–n plane, with the $\Lambda = 1$ line indicating a divide between quasi-neutral plasmas (to the left) and metals (to the right):



Clearly, there is a lot of parameter space here and so, to classify these plasmas further, we require additional dimensionless parameters.

I.2. Fundamental length and time scales

Another useful dividing line between different types of astrophysical and space plasmas is whether they are *collisional* or *collisionless*. In other words, is the mean free path between particle–particle collisions, $\lambda_{\rm mfp}$, larger or smaller than the macroscopic length scales of interest, L. If $\lambda_{\rm mfp} \ll L$, then the plasma is said to behave as a *fluid*, and various hydrodynamic and magnetohydrodynamic (MHD) equations can be used to describe its evolution. If, on the other hand, the mean free path is comparable to (or perhaps even larger than) the macroscopic length scales of interest, the plasma cannot be considered to be in local thermodynamic equilibrium, and the full six-dimensional phase space (3 spatial coordinates, 3 velocity coordinates) through which the constituent particles move must be retained in the description. Written in terms of the thermal speed of species α,

$$v_{{\rm th}\alpha} \doteq \left(\frac{2T_\alpha}{m_\alpha} \right)^{1/2}, \tag{I.5}$$

and the collision timescale τ_α, the *collisional mean free path* is

$$\lambda_{{\rm mfp},\alpha} \doteq v_{{\rm th}\alpha} \tau_\alpha. \tag{I.6}$$

For electron–ion collisions,

$$\tau_{\mathrm{ei}} = \frac{3\sqrt{m_{\mathrm{e}}}T_{\mathrm{e}}^{3/2}}{4\sqrt{2\pi}n_{\mathrm{e}}\lambda_{\mathrm{e}}Z^2e^4} \simeq 3.4\times 10^5 \left(\frac{T_{\mathrm{eV}}^{3/2}}{n_{\mathrm{cm}^{-3}}\lambda_{\mathrm{e}}Z^2}\right)\ \mathrm{s}, \tag{I.7}$$

where Ze is the ion charge and λ_{e} is the electron Coulomb logarithm; for ion–ion collisions,

$$\tau_{\mathrm{ii}} = \frac{3\sqrt{m_{\mathrm{i}}}T_{\mathrm{i}}^{3/2}}{4\sqrt{\pi}n_{\mathrm{i}}\lambda_{\mathrm{i}}Z^4e^4} \simeq 2.1\times 10^7 \left(\frac{T_{\mathrm{eV}}^{3/2}}{n_{\mathrm{cm}^{-3}}\lambda_{\mathrm{i}}Z^4}\right)\ \mathrm{s}, \tag{I.8}$$

where λ_{i} is the ion Coulomb logarithm. Note that the resulting $\lambda_{\mathrm{mfp,e}}$ and $\lambda_{\mathrm{mfp,i}}$ differ only by a factor of order unity:

$$\lambda_{\mathrm{mfp,e}} = \frac{3}{4\sqrt{\pi}}\frac{T_{\mathrm{e}}^2}{n_{\mathrm{e}}\lambda_{\mathrm{e}}Z^2e^4}, \quad \lambda_{\mathrm{mfp,i}} = \frac{3\sqrt{2}}{4\sqrt{\pi}}\frac{T_{\mathrm{i}}^2}{n_{\mathrm{i}}\lambda_{\mathrm{i}}Z^4e^4},$$

and so one often drops the species subscript on λ_{mfp}. With these definitions, it becomes clear that the plasma parameter (I.1) also reflects the ratio of the mean free path to the Debye length:

$$\Lambda \doteq \frac{n_{\mathrm{e}}\lambda_{\mathrm{D}}^4}{\lambda_{\mathrm{D}}} \sim \frac{T_{\mathrm{e}}^2/n_{\mathrm{e}}/e^4}{\lambda_{\mathrm{D}}} \sim \frac{\lambda_{\mathrm{mfp}}}{\lambda_{\mathrm{D}}}; \tag{I.9}$$

again, a measure of the relative importance of collective effects (λ_{D}) and binary collisions (λ_{mfp}).

Independent of whether a given astrophysical plasma is collisional or collisionless, nearly all such plasmas host magnetic fields, either inherited from the cosmic background in which they reside or produced *in situ* by a dynamo mechanism. There are two ways in which the strength of the magnetic field is quantified. First, the *plasma beta parameter*:

$$\beta_\alpha \doteq \frac{8\pi n_\alpha T_\alpha}{B^2}, \tag{I.10}$$

which reflects the relative energy densities of the thermal motions of the plasma particles and of the magnetic field. Note that

$$\beta_\alpha = \frac{2T_\alpha}{m_\alpha}\times\frac{4\pi m_\alpha n_\alpha}{B^2} = \frac{v_{\mathrm{th}\alpha}^2}{v_{\mathrm{A}\alpha}^2}, \tag{I.11}$$

where

$$v_{\mathrm{A}\alpha} \doteq \frac{B}{\sqrt{4\pi m_\alpha n_\alpha}} \tag{I.12}$$

is the *Alfvén speed* for species α.[2] Second, the *plasma magnetization*, ρ_α/L, where

$$\rho_\alpha \doteq \frac{v_{\mathrm{th}\alpha}}{\Omega_\alpha} \tag{I.13}$$

is the Larmor radius of species α and

$$\Omega_\alpha \equiv \frac{q_\alpha B}{m_\alpha c} \tag{I.14}$$

is the gyro- (or cyclotron, or Larmor) frequency. What distinguishes many astrophysical plasmas from their terrestrial laboratory counterparts is that the former can have $\beta \gg 1$ even though $\rho/L \lll 1$.[3] In other words, a magnetized astrophysical plasma need not have

[2] Usually, a single Alfvén speed, $v_{\mathrm{A}} \doteq B/\sqrt{4\pi\varrho}$, is given for a plasma with mass density ϱ.

[3] The $\sim$5 keV intracluster medium of galaxy clusters can be magnetized by a magnetic field as weak as $\sim 10^{-18}$ G.

an energetically important magnetic field, and $\beta \gg 1$ does not preclude the magnetic field from having dynamical consequences. You've been warned.

There are two more kinetic scales worth mentioning at this point, which we will come to later in this course: the *plasma frequency*,

$$\omega_{\mathrm{p}\alpha} = \left(\frac{4\pi n_\alpha e^2}{m_\alpha} \right)^{1/2}, \tag{I.15}$$

and the *skin depth* (or *inertial length*),

$$d_\alpha \doteq \frac{c}{\omega_{\mathrm{p}\alpha}} = \left(\frac{m_\alpha c^2}{4\pi n_\alpha e^2} \right)^{1/2}. \tag{I.16}$$

The former is the characteristic frequency at which a plasma oscillates when one sign of charge carriers is displaced from the other sign by a small amount (see § III.2). Indeed, the factor $(4\pi n_\alpha e^2)$ should look familiar from the definition of the Debye length (see (I.2)). The latter is the characteristic scale below which the inertia of species α precludes the propagation of (certain) electromagnetic waves. For example, the ion skin depth is the scale at which the ions decouple from the electrons and any fluctuations in which the electrons are taking part (e.g., whistler waves). The following relationship between the skin depth and the Larmor radius may one day come in handy:

$$d_\alpha = \frac{v_{\mathrm{A},\alpha}}{\Omega_\alpha} = \frac{\rho_\alpha}{\beta_\alpha^{1/2}}. \tag{I.17}$$

I.3. Examples of astrophysical and space plasmas

This part is given as a keynote presentation. Here I simply provide a chart of useful numbers on the next page (ICM = intracluster medium; JET = Joint European Torus, a nuclear fusion experiment; ISM = interstellar medium). For quick reference, the Earth has a ~0.5 G magnetic field, 1 eV ~ 10^4 K, 1 au $\approx 1.5 \times 10^{13}$ cm, 1 pc $\approx 3 \times 10^{18}$ cm, 1 pc Myr$^{-1} \simeq 1$ km s^{-1}.

	solar wind @ 1 au (earth location)	ICM @ ~100 kpc ("cooling radius")	galactic center @ 0.1 pc ("Bondi radius")	JET device (~1 meter)	ISM ("warm")
T	10 eV	8×10^3 eV	2×10^3 eV	10^4 eV	1 eV
n	10 cm^{-3}	5×10^{-3} cm^{-3}	100 cm^{-3}	10^{14} cm^{-3}	1 cm^{-3}
B	100 μG	1 μG	10^3 μG	3×10^4 G	5 μG

$v_{\mathrm{th}i}$	40 km/s	1000 km/s	600 km/s	600 km/s	10 km/s
$v_{\mathrm{A}i}$	70 km/s	30 km/s	200 km/s	4000 km/s	10 km/s
$\beta_i \equiv \dfrac{v_{\mathrm{th}i}^2}{v_{\mathrm{A}i}^2}$	$\sim 0.3-1$	$\sim 10^3$	~ 10	~ 0.02	~ 1
L	$\lesssim 1$ au	~ 10 kpc $-$ 100s kpc	$\lesssim 0.1$ pc	~ 1 m	~ 1 pc $-$ 10s pc
λ_{mfp}	$\sim 0.1-1$ au	$\sim 0.1-10$ kpc	~ 0.01 pc	~ 10 km	$\sim 10^{-7}$ pc
ρ_i	$\sim 10^{-7}$ au	~ 1 npc	~ 1 ppc	~ 0.2 cm	$\sim 10^{-11}$ pc
Ω_i	~ 1 Hz	~ 0.01 Hz	~ 10 Hz	~ 300 MHz	~ 0.05 Hz

PART II

Fundamentals of hydrodynamics

Unfortunately, fluid dynamics has all but disappeared from the US undergraduate curriculum, as physics departments have made way for quantum mechanics and condensed matter.[4] This is a shame – yes, it's classical physics and thus draws less 'oohs' and 'aahs' from the student (and professorial, for that matter) crowd. But there are many good reasons to study it. First, it forms the bedrock of fascinating and modern topics like non-equilibrium statistical mechanics, including the kinetic theory of gases and particles. Second, it is mathematically rich without being physically opaque. The more you really understand the mathematics, the more you really understand physically what is going on; the same cannot be said for many branches of modern physics. Third, nonlinear dynamics and chaos, burgeoning fields in their own right, are central to arguably the most important unsolved problem in classical physics: fluid turbulence. Solve that, and your solution would have immediate impact and practical benefits to society. Finally, follow in the footsteps of greatness: on Feynman's chalkboard at the time of his death was the remit 'to learn . . . nonlinear classical hydro'. With that, let's begin.

[4] An excellent textbook from which to learn elementary fluid dynamics is Acheson's *Elementary Fluid Dynamics*. It provides an engaging mix of history, physical insight, and transparent mathematics. I recommend it.

II.1. The equations of ideal hydrodynamics

The equations of hydrodynamics and MHD may be obtained rigorously by taking velocity-space moments of the Boltzmann and Vlasov–Landau kinetic equations. *What?* Okay, we'll get to that soon enough. For now, let's begin with things that you already know: mass is conserved, Newon's second law (force equals mass times acceleration), and the first law of thermodynamics (energy is conserved).

II.1.1. *Mass is conserved: The continuity equation*

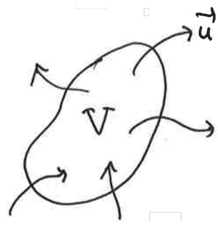
We describe our gaseous fluid by a mass density ρ, which in general is a function of time t and position $\boldsymbol{r}$.[5] Imagine an arbitrary volume $\mathcal{V}$ enclosing some of that fluid. The mass inside of the volume is simply

$$M = \int_{\mathcal{V}} \mathrm{d}V\, \rho. \tag{II.1}$$

Now let's mathematize our intuition: within this fixed volume, the only way the enclosed mass can change is by material flowing in or out of its surface $\mathcal{S}$:

$$\frac{\mathrm{d}M}{\mathrm{d}t} \doteq \int_{\mathcal{V}} \mathrm{d}V\, \frac{\partial \rho}{\partial t} = -\int_{\mathcal{S}} \mathrm{d}\boldsymbol{S} \boldsymbol{\cdot} \rho\boldsymbol{u}, \tag{II.2}$$

where $\boldsymbol{u}$ is the flow velocity.



Gauss' theorem may be applied to rewrite the right-hand side of this equation as follows:

$$\int_{\mathcal{S}} \mathrm{d}\boldsymbol{S} \boldsymbol{\cdot} \rho\boldsymbol{u} = \int_{\mathcal{V}} \mathrm{d}V\, \boldsymbol{\nabla} \boldsymbol{\cdot} (\rho\boldsymbol{u}). \tag{II.3}$$

Because the volume under consideration is arbitrary, the integrands of the volume integrals in (II.2) and (II.3) must be the same. Therefore,

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \boldsymbol{\cdot} (\rho\boldsymbol{u}) = 0} \tag{II.4}$$

This is the *continuity equation*; it's the differential form of mass conservation.

Exercise. Go to the bathroom and turn on the sink slowly to get a nice, laminar stream flowing down from the faucet. Go on, I'll wait. If you followed instructions, then you'll see that the stream becomes more narrow as it descends. Knowing that the density of water is very nearly constant, use the continuity equation to show that the cross-sectional area of the stream $A(z)$ as a function of distance from the faucet z is

$$A(z) = \frac{A_0}{\sqrt{1 + 2gz/v_0^2}},$$

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[5] I sometimes denote the mass density by ϱ to avoid confusion with the Larmor radius ρ. But, given that ρ is standard notation in hydrodynamics for the mass density, and ρ is standard notation in plasma physics for the Larmor radius, you should learn to tell the difference based on the context.

where A_0 is the cross-sectional area of the stream upon exiting the faucet with velocity v_0 and g is the gravitational acceleration. If you turn the faucet to make the water flow faster, what happens to the tapering of the stream?

II.1.2. *Newton's second law: The momentum equation*

So far we have an equation for the evolution of the mass density ρ expressed in terms of the fluid velocity $\boldsymbol{u}$. How does the latter evolve? Newton's second law provides the answer: simply add up the accelerations, divide by the mass (density), and you've got the time rate of change of the velocity. But there is a subtlety here: there is a difference between the time rate of change of the velocity in the lab frame and the time rate of change of the velocity in the fluid frame. So which time derivative of $\boldsymbol{u}$ do we take? The key is in how the accelerations are expressed. Are these accelerations acting on a fixed point in space, or are they acting on an element of our fluid? It is much easier (and more physical) to think of these accelerations in the latter sense: given a deformable patch of the fluid – large enough in extent to contain a very large number of atoms but small enough that all the macroscopic variables such as density, velocity, and pressure have a unique value over the dimensions of the patch – what forces are acting on that patch? These are relatively simple to catalog, and we will do so in short order. But first, let's answer our original question: which time derivative of $\boldsymbol{u}$ do we take? Since we have committed to expressing the forces in the frame of the fluid element, the acceleration must likewise be expressed in this frame. The acceleration is *not*

$$\frac{\partial \boldsymbol{u}}{\partial t}. \tag{II.5}$$

Remember what a partial derivative means: something is being fixed! Here, it is the instantaneous position $\boldsymbol{r}$ of the fluid element. Equation (II.5) is the answer to the question, 'how does the fluid velocity evolve at a fixed point in space?' Instead, we wish to fix our sights on the fluid element itself, which is moving. The acceleration we calculate must account for this frame transformation:

$$\boldsymbol{a} = \frac{\partial \boldsymbol{u}}{\partial t} + \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \boldsymbol{\cdot} \boldsymbol{\nabla} \boldsymbol{u}, \tag{II.6}$$

where $\mathrm{d}\boldsymbol{r}/\mathrm{d}t$ is the rate of change of the position of the fluid element, i.e., the velocity $\boldsymbol{u}(t, \boldsymbol{r})$. This combination of derivatives is so important that it has its own notation:

$$\frac{\mathrm{D}}{\mathrm{D}t} \doteq \frac{\partial}{\partial t} + \boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{\nabla}. \tag{II.7}$$

It is variously referred to as the *Lagrangian derivative*, or comoving derivative, or convective derivative. By contrast, the expression given by (II.5) is the *Eulerian deriative.* Note that the continuity equation (II.4) may be expressed using the Lagrangian derivative as

$$\frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{u}, \tag{II.8}$$

which states that incompressible flow corresponds to $\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{u} = 0$.

So, given some force $\boldsymbol{F}$ per unit volume that is acting on our fluid element, we now know how the fluid velocity evolves: force (per unit volume) equals mass (per unit volume) times acceleration (in the frame of the fluid element):

$$\boldsymbol{F} = \rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}. \tag{II.9}$$

Now we need only catalog the relevant forces. This could be, say, gravity: $\rho \boldsymbol{g} = -\rho \boldsymbol{\nabla} \Phi$.

Or, if the fluid element is conducting, electromagnetic forces (which we'll get to later in the course). But the most deserving of discussion at this point is the pressure force due to the internal thermal motions of the particles comprising the gas. For an ideal gas, the equation of state is

$$P = \frac{\rho k_{\mathrm{B}} T}{m} \doteq \rho C^2, \tag{II.10}$$

where T is the temperature in Kelvin, k_{B} is the Boltzmann constant, m is the mass per particle, and C is the speed of sound in an isothermal gas. Plasma physicists often drop Boltzmann's constant and register temperature in energy units (e.g., eV), and I will henceforth do the same in these notes. How does gas pressure due to microscopic particle motions exert a macroscopic force on a fluid element? First, the pressure must be spatially non-uniform: there must be more or less energetic content in the thermal motions of the particles in one region versus another, whether it be because the gas temperature varies in space or because there are more particles in one location as opposed to another. For example, the pressure force in the x direction in a slab of thickness $\mathrm{d}x$ and cross-sectional area $\mathrm{d}y\,\mathrm{d}z$ is

$$\bigl[P(t, x - \mathrm{d}x/2, y, z) - P(t, x + \mathrm{d}x/2, y, z)\bigr]\mathrm{d}y\,\mathrm{d}z = -\frac{\partial P}{\partial x}\,\mathrm{d}V. \tag{II.11}$$

Unless the thermal motions of the particles are not sufficiently randomized to be isotropic (e.g., if the collisional mean free path of the plasma is so long that inter-particle collisions cannot drive the system quickly enough towards local thermodynamic equilibrium), there is nothing particularly special about the x direction, and so the pressure force force acting on some differential volume $\mathrm{d}V$ is just $-\boldsymbol{\nabla}P\,\mathrm{d}V$.

Assembling the lessons we've learned here, we have the following force equation for our fluid:

$$\boxed{\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \doteq \rho\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\mathit{\Phi}} \tag{II.12}$$

This equation is colloquially known as the *momentum equation*, even though it evolves the fluid velocity rather than its momentum density. To obtain an equation for the latter, the continuity equation (II.4) may be used to move the mass density into the time and space derivatives:

$$\begin{aligned}
\frac{\partial(\rho\boldsymbol{u})}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\boldsymbol{u}) &= \frac{\partial\rho}{\partial t}\boldsymbol{u} + \rho\frac{\partial\boldsymbol{u}}{\partial t} + \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u})\boldsymbol{u} \\
&= \left[\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u})\right]\boldsymbol{u} + \rho\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u} \\
&= \Bigl[\qquad 0 \qquad\Bigr]\boldsymbol{u} + \rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{F}.
\end{aligned} \tag{II.13}$$

Thus, an equation for the momentum density:

$$\boxed{\frac{\partial(\rho\boldsymbol{u})}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\boldsymbol{u}) = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\mathit{\Phi}} \tag{II.14}$$

This form is particularly useful for deriving an evolution equation for the kinetic energy density. Dotting (II.14) with $\boldsymbol{u}$ and grouping terms,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho u^2\boldsymbol{u}\right) = -\boldsymbol{u}\cdot\boldsymbol{\nabla}P - \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\mathit{\Phi}, \tag{II.15}$$

which is a statement that the kinetic energy of a fluid element changes as work is done by the forces.

Now, how to we know the pressure P? There's an equation for that...

II.1.3. *First law of thermodynamics: The internal energy equation*

There are several ways to go about obtaining an evolution equation for the pressure. One way is to introduce the *internal energy*,

$$e \doteq \frac{P}{\gamma - 1} \tag{II.16}$$

and use the first law of thermodynamics to argue that e is conserved but for $P\,\mathrm{d}V$ work:

$$\boxed{\frac{\partial e}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(e\boldsymbol{u}) = -P\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}} \tag{II.17}$$

This is the *internal energy* equation.

Equation (II.17) may be used to derive a total (kinetic + internal + potential) energy equation for the fluid as follows. Do (II.15) + (II.17):

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + e\right) + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[\left(\frac{1}{2}\rho u^2 + e\right)\boldsymbol{u}\right] &= -\boldsymbol{\nabla}\boldsymbol{\cdot}(P\boldsymbol{u}) - \rho\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\Phi, \\ &= -(\gamma-1)\boldsymbol{\nabla}\boldsymbol{\cdot}(e\boldsymbol{u}) - \rho\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\Phi\end{aligned}$$

$$\Longrightarrow \quad \frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + e\right) + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[\left(\frac{1}{2}\rho u^2 + \gamma e\right)\boldsymbol{u}\right] = -\rho\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\Phi. \tag{II.18}$$

Now use the continuity equation (II.4) to write

$$\frac{\partial(\rho\Phi)}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\Phi\boldsymbol{u}) = \rho\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\Phi + \rho\frac{\partial\Phi}{\partial t}. \tag{II.19}$$

Adding this equation to (II.18) yields the desired result:

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + e + \rho\Phi\right) + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[\left(\frac{1}{2}\rho u^2 + \gamma e + \rho\Phi\right)\boldsymbol{u}\right] = \rho\frac{\partial\Phi}{\partial t}} \tag{II.20}$$

The first term in parentheses under the time derivative is sometimes denoted by $\mathcal{E}$.

Yet another way of expressing the internal energy equation (II.17) is to write $e = \rho T/m(\gamma-1)$ and use the continuity equation (II.4) to eliminate the derivatives of the mass density. The result is

$$\frac{\mathrm{D}\ln T}{\mathrm{D}t} = -(\gamma-1)\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}, \tag{II.21}$$

which states that the temperature of a fluid element is constant in an incompressible fluid (*viz.*, one with $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u} = 0$). If this seems intuitively unfamiliar to you, consider this: the hydrodynamic entropy of a fluid element is given by

$$s \doteq \frac{1}{\gamma-1}\ln P\rho^{-\gamma} = \frac{1}{\gamma-1}\ln T\rho^{1-\gamma}. \tag{II.22}$$

Taking the Lagrangian time derivative of the entropy along the path of a fluid element yields

$$\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{\mathrm{D}\ln T}{\mathrm{D}t} - (\gamma-1)\frac{\mathrm{D}\ln\rho}{\mathrm{D}t}. \tag{II.23}$$

It is then just a short trip back to (II.8) to see that (II.21) is, in fact, the second law of thermodynamics – entropy is conserved in the absence of sources or dissipative sinks:

$$\boxed{\frac{\mathrm{D}s}{\mathrm{D}t} = 0} \tag{II.24}$$

II.2. Summary: Adiabatic equations of hydrodynamics

The adiabatic equations of hydrodynamics, written in conservative form, are:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho \boldsymbol{u}) = 0, \tag{II.25a}$$

$$\frac{\partial (\rho \boldsymbol{u})}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho \boldsymbol{u}\boldsymbol{u}) = -\boldsymbol{\nabla} P - \rho\boldsymbol{\nabla}\varPhi, \tag{II.25b}$$

$$\frac{\partial e}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(e\boldsymbol{u}) = -P\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}. \tag{II.25c}$$

The left-hand sides of these equations express advection of, respectively, the mass density, the momentum density, and the internal energy density by the fluid velocity; the right-hand sides represents sources and sinks. If the gravitational potential is due to self-gravity, then one must additionally solve the Poisson equation,

$$\nabla^2 \varPhi = 4\pi G\rho. \tag{II.26}$$

where G is Newton's gravitational constant.

If we instead write these equations in terms of the density, fluid velocity, and entropy and make use of the Lagrangian derivative (II.7), we have

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}, \tag{II.27a}$$

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\varPhi, \tag{II.27b}$$

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0, \tag{II.27c}$$

where $s \doteq (\gamma-1)^{-1}\ln P\rho^{-\gamma}$. The limit $\gamma\rightarrow\infty$, often of utility for describing liquids, corresponds to $\mathrm{D}\rho/\mathrm{D}t = 0$, i.e., incompressibility.

Exercise. Show that the gravitational force on a self-gravitating fluid element may be written as

$$-\rho\boldsymbol{\nabla}\varPhi = -\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\frac{\boldsymbol{g}\boldsymbol{g}}{4\pi G} - \frac{g^2}{8\pi G}\boldsymbol{I}\right), \tag{II.28}$$

where $\boldsymbol{g} = -\boldsymbol{\nabla}\varPhi$, $g^2 = \boldsymbol{g}\boldsymbol{\cdot}\boldsymbol{g}$, and $\boldsymbol{I}$ is the unit dyadic. The quantity inside the divergence operator is known as the gravitational stress tensor. Because it's written in the form of a divergence, it represents the flux of total momentum through a surface due to gravitational forces.

II.3. Mathematical matters

II.3.1. *Vector identities*

As a start to this section, let me advise you to brush up on your vector calculus...

$$\begin{aligned}
\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) &= \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B}),\\
\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) &= \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B}),\\
\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) &= \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}),\\
\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) &= (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} - \boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) + \boldsymbol{A}(\boldsymbol{\nabla}\cdot\boldsymbol{B}),\\
\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) &= \boldsymbol{\nabla}(\boldsymbol{A}\cdot\boldsymbol{B}) - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} - (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A},\\
&\dots
\end{aligned}$$

Fluid dynamics is full of these things, and you should either (i) commit them to memory, (ii) carry your NRL formulary with you everywhere, or (iii) know how to quickly derive them using things like

$$\epsilon_{kij}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell},$$

where δ_{ij} is the Kronecker delta and ϵ_{ijk} is the Levi–Civita symbol.

II.3.2. *Leibniz's rule and the Lagrangian derivative of integrals*

In the proofs of many conservation laws, a Lagrangian time derivative is taken of a surface or volume integral whose integration limits are time-dependent. In this case, $\mathrm{D}/\mathrm{D}t$ does *not* commute with the integral sign. The trick to dealing with these situations is related to Leibniz's rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)}\mathrm{d}x\, f(t,x) = \int_{a(t)}^{b(t)}\mathrm{d}x\,\frac{\partial}{\partial t}f(t,x) + f(t,b(t))\frac{\mathrm{d}b}{\mathrm{d}t} - f(t,a(t))\frac{\mathrm{d}a}{\mathrm{d}t}. \tag{II.29}$$

In three dimensions, if we're taking the time derivative of a volume integral whose integration limits $\mathcal{V}(t)$ are time-dependent, the generalization of the above is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)}\mathrm{d}\mathcal{V}\, f(t,\boldsymbol{r}) = \int_{\mathcal{V}(t)}\mathrm{d}\mathcal{V}\,\frac{\partial}{\partial t}f(t,\boldsymbol{r}) + \oint_{\partial\mathcal{V}(t)}\mathrm{d}\boldsymbol{\mathcal{S}}\cdot\big[f(t,\boldsymbol{r})\boldsymbol{u}_{\mathrm{b}}(t,\boldsymbol{r})\big], \tag{II.30}$$

where $\boldsymbol{u}_{\mathrm{b}}$ is the velocity of the bounding surface $\partial\mathcal{V}(t)$. This is known as the Reynolds transport theorem. In words, the time rate-of-change of a quantity positioned within a moving volume is a combination of the lab-frame rate-of-change of that quantity (i.e., the time derivative at fixed position $\boldsymbol{r}$ – note the partial derivative) and how much of that quantity flowed through the surface. When the velocity of the bounding surface equals the fluid velocity, $\boldsymbol{u}_{\mathrm{b}} = \boldsymbol{u}(t,\boldsymbol{r})$, so that each moving volume corresponds to that of a fluid element, we may replace $\mathrm{d}/\mathrm{d}t$ in (II.30) with the Lagrangian derivative $\mathrm{D}/\mathrm{D}t$:

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}(t)}\mathrm{d}\mathcal{V}\, f(t,\boldsymbol{r}) = \int_{\mathcal{V}(t)}\mathrm{d}\mathcal{V}\,\frac{\partial}{\partial t}f(t,\boldsymbol{r}) + \oint_{\partial\mathcal{V}(t)}\mathrm{d}\boldsymbol{\mathcal{S}}\cdot\big[f(t,\boldsymbol{r})\boldsymbol{u}(t,\boldsymbol{r})\big]} \tag{II.31}$$

You've already encountered an example of this – mass conservation, in which the volume was a "material volume" moving with the fluid element itself:

$$0 = \frac{\mathrm{D}M}{\mathrm{D}t} \doteq \frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}(t)}\mathrm{d}\mathcal{V}\,\rho = \int_{\mathcal{V}(t)}\mathrm{d}\mathcal{V}\,\frac{\partial\rho}{\partial t} + \oint_{\partial\mathcal{V}(t)}\mathrm{d}\boldsymbol{\mathcal{S}}\cdot(\rho\boldsymbol{u}).$$

Using the divergence theorem on the final (surface-integral) term gives

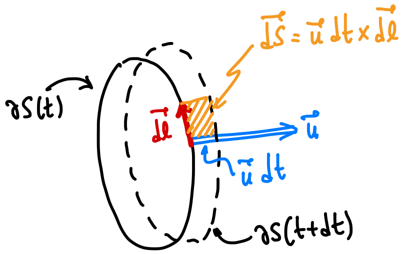
$$0 = \int_{\mathcal{V}(t)} \mathrm{d}\mathcal{V} \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) \right],$$

which provides us with our continuity equation.

A similar rule to (II.31) is needed for time derivatives of surface integrals whose integration limits $\mathcal{S}(t)$ are time-dependent. For a vector field $\boldsymbol{F} = \boldsymbol{F}(t, \boldsymbol{r})$ and a bounding surface $\mathcal{S}(t)$ whose contour $\partial\mathcal{S}(t)$ moves with the fluid velocity $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{r})$, this is given by

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t} \int_{\mathcal{S}(t)} \mathrm{d}\boldsymbol{\mathcal{S}} \cdot \boldsymbol{F} = \int_{\mathcal{S}(t)} \mathrm{d}\boldsymbol{\mathcal{S}} \cdot \left[\frac{\partial \boldsymbol{F}}{\partial t} + (\boldsymbol{\nabla} \cdot \boldsymbol{F}) \boldsymbol{u} \right] - \oint_{\partial\mathcal{S}(t)} \mathrm{d}\boldsymbol{\ell} \cdot (\boldsymbol{u} \times \boldsymbol{F})} \tag{II.32}$$

(By convention, the contour is taken in the counter-clockwise direction.) Note that $-\mathrm{d}\boldsymbol{\ell} \cdot (\boldsymbol{u} \times \boldsymbol{F}) = \boldsymbol{F} \cdot (\boldsymbol{u} \times \mathrm{d}\boldsymbol{\ell})$. In words, the comoving change of the differential surface element $\mathrm{d}\boldsymbol{\mathcal{S}}$ equals the amount of area swept out in a time $\mathrm{d}t$ via the advection of a differential line element $\mathrm{d}\boldsymbol{\ell}$ on $\partial\mathcal{S}$ by a distance $\boldsymbol{u}\,\mathrm{d}t$:



Equation (II.32) can be used to prove conservation of magnetic flux (§IV.1.1) and conservation of fluid vorticity (§II.4).

II.3.3. $\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}$ *and curvilinear coordinates*

Finally, the nonlinear combination $\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}$ that features prominently in the Lagrangian time derivative can be complicated, particularly in curvilinear coordinates where the gradient operator within it acts on the unit vectors within $\boldsymbol{u}$. For example, in cylindrical coordinates (R, φ, z),

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} &= \boldsymbol{u} \cdot \boldsymbol{\nabla} \big(u_R \hat{\boldsymbol{R}} + u_\varphi \hat{\boldsymbol{\varphi}} + u_z \hat{\boldsymbol{z}} \big) \\ &= (\boldsymbol{u} \cdot \boldsymbol{\nabla} u_R) \hat{\boldsymbol{R}} + (\boldsymbol{u} \cdot \boldsymbol{\nabla} u_\varphi) \hat{\boldsymbol{\varphi}} + (\boldsymbol{u} \cdot \boldsymbol{\nabla} u_z) \hat{\boldsymbol{z}} + \frac{u_\varphi^2}{R} \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \varphi} + \frac{u_R u_\varphi}{R} \frac{\partial \hat{\boldsymbol{R}}}{\partial \varphi} \\ &= (\boldsymbol{u} \cdot \boldsymbol{\nabla} u_i) \hat{\boldsymbol{e}}_i - \frac{u_\varphi^2}{R} \hat{\boldsymbol{R}} + \frac{u_R u_\varphi}{R} \hat{\boldsymbol{\varphi}}, \end{aligned} \tag{II.33}$$

where, to obtain the final equality, we have used $\partial \hat{\boldsymbol{\varphi}} / \partial \varphi = -\hat{\boldsymbol{R}}$ and $\partial \hat{\boldsymbol{R}} / \partial \varphi = \hat{\boldsymbol{\varphi}}$; summation over the repeated index i is implied in the first term in the final line.

Exercise. Follow a similar procedure to show that, in spherical coordinates (r, θ, φ),

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} &= \left(u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + \frac{u_\varphi}{r \sin\theta} \frac{\partial}{\partial \varphi} \right) \left(u_r \hat{\boldsymbol{r}} + u_\theta \hat{\boldsymbol{\theta}} + u_\varphi \hat{\boldsymbol{\varphi}} \right) \\ &= (\boldsymbol{u} \cdot \boldsymbol{\nabla} u_i) \hat{\boldsymbol{e}}_i - \frac{u_\theta^2 + u_\varphi^2}{r} \hat{\boldsymbol{r}} + \left(\frac{u_r u_\theta}{r} - \frac{u_\varphi^2 \cot\theta}{r} \right) \hat{\boldsymbol{\theta}} + \left(\frac{u_\theta u_\varphi \cot\theta}{r} + \frac{u_r u_\varphi}{r} \right) \hat{\boldsymbol{\varphi}}. \end{aligned}$$

The last two terms in the cylindrical $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}$, equation (II.33), might look familiar to you from working in rotating frames. Indeed, let us write $\boldsymbol{u} = \boldsymbol{v} + R\Omega(R,z)\hat{\boldsymbol{\varphi}}$, where Ω is an angular velocity, and substitute this decomposition into (II.33):

$$\begin{aligned}\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} &= \big[(\boldsymbol{v}+R\Omega\hat{\boldsymbol{\varphi}})\cdot\boldsymbol{\nabla}v_i\big]\hat{\boldsymbol{e}}_i + \big[(\boldsymbol{v}+R\Omega\hat{\boldsymbol{\varphi}})\cdot\boldsymbol{\nabla}(R\Omega)\big]\hat{\boldsymbol{\varphi}} \\ &\quad - \frac{(v_\varphi+R\Omega)^2}{R}\hat{\boldsymbol{R}} + \frac{v_R(v_\varphi+R\Omega)}{R}\hat{\boldsymbol{\varphi}} \\ &= \left[\left(\boldsymbol{v}\cdot\boldsymbol{\nabla}+\Omega\frac{\partial}{\partial\varphi}\right)v_i\right]\hat{\boldsymbol{e}}_i + \left[2\Omega\hat{\boldsymbol{z}}\times\boldsymbol{v} - R\Omega^2\hat{\boldsymbol{R}} + R\hat{\boldsymbol{\varphi}}(\boldsymbol{v}\cdot\boldsymbol{\nabla})\Omega\right] \\ &\quad + \left[\frac{v_R v_\varphi}{R}\hat{\boldsymbol{\varphi}} - \frac{v_\varphi^2}{R}\hat{\boldsymbol{R}}\right]. \end{aligned} \tag{II.34}$$

Each of these terms has a straightforward physical interpretation. The first term in brackets represents advection by the flow and the rotation. The second term in brackets contains the Coriolis force, the centrifugal force, and ‘tidal’ terms due to the differential rotation, in that order. (The ‘tidal’ terms can be thought of the fictitious acceleration required for a fluid element to maintain its presence in the local rotating frame as it is displaced radially or vertically. They come from Taylor expanding the angular velocity about a point in the disk.) The third and final term in brackets captures curvature effects due to the cylindrical geometry.

Exercise. Show that the $R\varphi$-component in cylindrical coordinates of the rate-of-strain tensor

$$W_{ij} \doteq \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

is given by

$$W_{R\varphi} = \frac{1}{R}\frac{\partial u_R}{\partial\varphi} + R\frac{\partial}{\partial R}\frac{u_\varphi}{R}.$$

Hint: $\partial u_i/\partial x_j = [(\hat{\boldsymbol{e}}_j\cdot\boldsymbol{\nabla})\boldsymbol{u}]\cdot\hat{\boldsymbol{e}}_i$ is coordinate invariant.

II.4. Vorticity and Kelvin's circulation theorem

With some vector identities in hand, let's take the curl of the force equation (II.27*b*):

$$\boldsymbol{\nabla}\times\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi\right).$$

The potential term vanishes, since the curl of a gradient is zero. Likewise, the pressure term becomes

$$-\boldsymbol{\nabla}\frac{1}{\rho}\times\boldsymbol{\nabla}P = \frac{1}{\rho^2}\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}P.$$

As for the left-hand side, the gradient operator commutes with $\partial/\partial t$, but not with $\boldsymbol{u}\cdot\boldsymbol{\nabla}$. Instead,

$$\boldsymbol{\nabla}\times\big[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\big] = \boldsymbol{\nabla}\times\left[\frac{1}{2}\boldsymbol{\nabla}u^2 - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})\right] = -\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{\omega}),$$

where

$$\boldsymbol{\omega} \doteq \boldsymbol{\nabla}\times\boldsymbol{u} \tag{II.35}$$

is the *fluid vorticity*. The vorticity measures how much rotation a velocity field has (and its direction). Note that it is divergence free, which means that vortex lines cannot end within the fluid – they must either close on themselves (like a smoke ring) or intersect a boundary (like a tornado). Any fresh vortex lines that are made must be created as continuous curves that grow out of points or lines where the vorticity vanishes.

Assembling the above gives the *vorticity equation*,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \frac{1}{\rho^2} \boldsymbol{\nabla} \rho \times \boldsymbol{\nabla} P. \tag{II.36}$$

Note that the right-hand side of this equation vanishes if the pressure is *barotropic*, i.e., if $P = P(\rho)$, so that surfaces of constant density and constant pressure coincide. If these surfaces do not coincide, then the fluid is said to have "baroclinicity" or to be "baroclinic". I'll demonstrate below using mathematics what (II.36) means physically, but you already know what the right-hand side means if you pay attention to the weather: areas of high atmospheric baroclinicity have frequent hurricanes and cyclones. In the parlance of fluid dynamics, this is called "baroclinic forcing". Now back to the math...

Dot (II.36) into a differential surface element $\mathrm{d}\boldsymbol{\mathcal{S}}$ normal to the surface $\mathcal{S}$ of a fluid element, integrate over that surface, and use Stokes' theorem to replace the surface integral of a curl with a line integral over the surface boundary $\partial\mathcal{S}$:

$$\int_{\mathcal{S}} \frac{\partial \boldsymbol{\omega}}{\partial t} \boldsymbol{\cdot} \mathrm{d}\boldsymbol{\mathcal{S}} - \oint_{\partial\mathcal{S}} (\boldsymbol{u} \times \boldsymbol{\omega}) \boldsymbol{\cdot} \mathrm{d}\boldsymbol{\ell} = \oint_{\partial\mathcal{S}} \left(-\frac{1}{\rho} \boldsymbol{\nabla} P \right) \boldsymbol{\cdot} \mathrm{d}\boldsymbol{\ell} = - \oint_{\partial\mathcal{S}} \frac{\mathrm{d}P}{\rho}.$$

Using (II.32) to replace the left-hand side by the Lagrangian time derivative of $\boldsymbol{\omega} \boldsymbol{\cdot} \mathrm{d}\boldsymbol{\mathcal{S}}$ yields

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_{\mathcal{S}} \boldsymbol{\omega} \boldsymbol{\cdot} \mathrm{d}\boldsymbol{\mathcal{S}} = - \oint_{\partial\mathcal{S}} \frac{\mathrm{d}P}{\rho}. \tag{II.37}$$

The surface integral on the left-hand side of this equation may be expressed using Stokes' theorem as the *circulation $\mathit{\Gamma}$*:

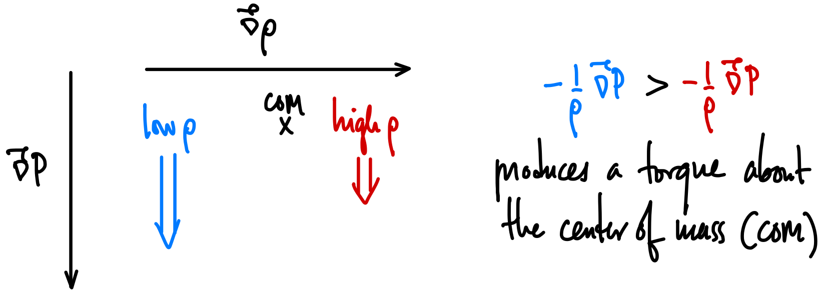
$$\int_{\mathcal{S}} \boldsymbol{\omega} \boldsymbol{\cdot} \mathrm{d}\boldsymbol{\mathcal{S}} = \oint_{\partial\mathcal{S}} \boldsymbol{u} \boldsymbol{\cdot} \mathrm{d}\boldsymbol{\ell} \doteq \mathit{\Gamma}. \tag{II.38}$$

The circulation around the boundary $\partial\mathcal{S}$ can be thought of as the number of vortex lines that thread the enclosed area S. Equation (II.37) then states that the circulation is conserved if the fluid is barotropic – Kelvin's circulation theorem:[6]

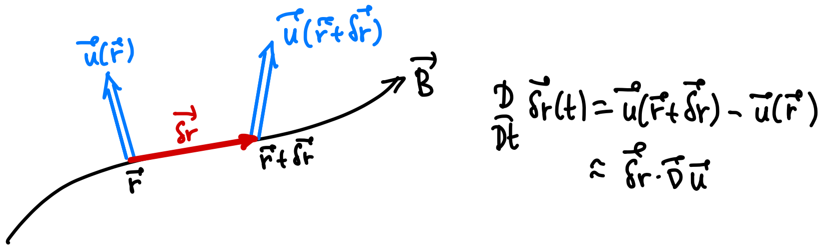
$$\boxed{\frac{\mathrm{D}\mathit{\Gamma}}{\mathrm{D}t} = - \oint_{\partial\mathcal{S}} \frac{\mathrm{d}P}{\rho} = 0 \quad \text{if } P = P(\rho)} \tag{II.39}$$

The figure below illustrates how baroclinic forcing generates vorticity.

[6]The above manipulations require that the surface is simply connected – that is, the region must be such that we can shrink the contour to a point without leaving the region. A region with a hole (like a bathtub drain) is not simply connected.



Another approach to proving (II.39) is to work with $\Gamma = \oint_{\partial S} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{\ell}$ rather than $\int_S \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{\mathcal{S}}$ and use the following for how an advected line element of ∂S changes in time:



Exercise. The helicity of a region of fluid is defined to be $\mathcal{H} \doteq \int \boldsymbol{\omega} \cdot \boldsymbol{u} \, \mathrm{d}\mathcal{V}$, where the integral is taken over the volume of that region. Assume that $\Gamma = \text{const}$ and that $\boldsymbol{\omega} \cdot \hat{\boldsymbol{n}}$ vanishes when integrated over the surface bounding $\mathcal{V}$, where $\hat{\boldsymbol{n}}$ is the unit normal to that surface. Prove that the helicity $\mathcal{H}$ is conserved in a frame moving with the fluid, *viz.* $\mathrm{D}\mathcal{H}/\mathrm{D}t = 0$. Note that the fluid need not be incompressible for this property to hold.

The calculation leading to (II.39) can be repeated in a reference frame rotating at a constant angular velocity $\boldsymbol{\Omega}$, in which the fluid velocity is measured to be $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{\Omega} \times \boldsymbol{r}$ (here, $\boldsymbol{u}$ is the fluid velocity in the inertial frame; see §II.3). The associated vorticity in this rotating frame is

$$\boldsymbol{\omega}_{\mathrm{rot}} = \boldsymbol{\omega} - \boldsymbol{\nabla} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{\omega} - \boldsymbol{\Omega}(\boldsymbol{\nabla} \cdot \boldsymbol{r}) + (\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\boldsymbol{r} = \boldsymbol{\omega} - 3\boldsymbol{\Omega} + \boldsymbol{\Omega} = \boldsymbol{\omega} - 2\boldsymbol{\Omega}, \quad \text{(II.40)}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$. The circulation in the rotating reference frame is then given by

$$\begin{aligned} \Gamma_{\mathrm{rot}} &= \int_S \boldsymbol{\omega}_{\mathrm{rot}} \cdot \mathrm{d}\boldsymbol{\mathcal{S}} = \int_S (\boldsymbol{\omega} - 2\boldsymbol{\Omega}) \cdot \mathrm{d}\boldsymbol{\mathcal{S}} \\ &= \oint_{\partial S} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{\ell} - \int_S 2\boldsymbol{\Omega} \cdot \mathrm{d}\boldsymbol{\mathcal{S}} \\ &= \Gamma - \int_S 2\boldsymbol{\Omega} \cdot \mathrm{d}\boldsymbol{\mathcal{S}}. \end{aligned} \quad \text{(II.41)}$$

Kelvin's circulation theorem in this rotating frame is therefore

$$\frac{\mathrm{D}\Gamma_{\mathrm{rot}}}{\mathrm{D}t} = -\oint_{\partial S} \frac{\mathrm{d}P}{\rho} - 2\Omega \frac{\mathrm{D}\mathcal{S}_{\mathrm{n}}}{\mathrm{D}t}, \quad \text{(II.42)}$$

where $\mathcal{S}_{\mathrm{n}}$ is component of the surface area oriented normally to $\boldsymbol{\Omega}$. In words, if the projected area of the vortex tube in the plane perpendicular to the rotation vector changes, then the circulation in the rotating frame must change to compensate. This is the origin of Rossby waves, something that will be discussed further in §II.5.2.

II.5. Rotating reference frames

The final calculation in the preceding section provides a natural segue into a discussion of fluid dynamics in rotating reference frames. To begin this discussion, let us first recall equation (II.34), in which the nonlinearity $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}$ was written out in cylindrical coordinates for a fluid velocity $\boldsymbol{u}$ consisting of a cylindrical rotation $R\Omega\hat{\boldsymbol{\varphi}}$ with angular velocity $\Omega = \Omega(R,z)$ and a residual velocity $\boldsymbol{v} \doteq \boldsymbol{u} - R\Omega\hat{\boldsymbol{\varphi}}$:

$$\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = \left[\left(\boldsymbol{v}\cdot\boldsymbol{\nabla} + \Omega\frac{\partial}{\partial\varphi}\right)v_i\right]\hat{\boldsymbol{e}}_i + \left[2\Omega\hat{\boldsymbol{z}}\times\boldsymbol{v} - R\Omega^2\hat{\boldsymbol{R}} + R\hat{\boldsymbol{\varphi}}(\boldsymbol{v}\cdot\boldsymbol{\nabla})\Omega\right] + \left[\frac{v_R v_\varphi}{R}\hat{\boldsymbol{\varphi}} - \frac{v_\varphi^2}{R}\hat{\boldsymbol{R}}\right].$$

When this expansion was introduced in §II.3, each of its components were described physically: 'The first term in brackets represents advection by the flow and the rotation. The second term in brackets contains the Coriolis force, the centrifugal force, and "tidal" terms due to the differential rotation... The third and final term in brackets captures curvature effects due to the cylindrical geometry.' Let's see these terms in action.

Using (II.34), we may express the equations of hydrodynamics (II.27) in cylindrical coordinates in a frame co-moving with the differential rotation. With

$$\frac{\mathrm{D}}{\mathrm{D}t} \rightarrow \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} + \Omega\frac{\partial}{\partial\varphi} \tag{II.43}$$

to include advection by the rotation, we have

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}, \tag{II.44a}$$

$$\frac{\mathrm{D}v_R}{\mathrm{D}t} = f_R + 2\Omega v_\varphi + R\Omega^2 + \frac{v_\varphi^2}{R}, \tag{II.44b}$$

$$\frac{\mathrm{D}v_\varphi}{\mathrm{D}t} = f_\varphi - \frac{\kappa^2}{2\Omega}v_R - R\frac{\partial\Omega}{\partial Z}v_z - \frac{v_R v_\varphi}{R}, \tag{II.44c}$$

$$\frac{\mathrm{D}v_z}{\mathrm{D}t} = f_z, \tag{II.44d}$$

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0, \tag{II.44e}$$

where

$$\boldsymbol{f} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi \tag{II.45}$$

and the combination

$$\boxed{\kappa^2 \doteq 4\Omega^2 + \frac{\partial\Omega^2}{\partial\ln R} = \frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial R}} \tag{II.46}$$

is known as the (square of the) epicyclic frequency. Note that $R^4\Omega^2 = \ell^2$, the square of the specific angular momentum ℓ, and so κ^2 measures how much the specific angular momentum associated with the rotation increases or decreases outwards. For Keplerian rotation, $\kappa^2 = \Omega^2$.

In §VI.9, these equations will be modified for the presence and evolution of magnetic fields and used to look at linear waves and instabilities that rely on differential rotation. In the meantime, I'll close this portion of the notes by remarking on two useful applications of what you've learned here: the thermal wind equation (§II.5.1) and Rossby waves (§II.5.2).

II.5.1. *Thermal wind equation*

In steady state with $\boldsymbol{v}=0$, equations (II.44*b*) and (II.44*d*) become

$$0=-\frac{1}{\rho}\frac{\partial P}{\partial R}-\frac{\partial \Phi}{\partial R}+R\Omega^2 \quad \text{and} \quad 0=-\frac{1}{\rho}\frac{\partial P}{\partial z}-\frac{\partial \Phi}{\partial z}. \tag{II.47}$$

Taking $\partial/\partial z$ of the first equation, using the second equation, and rearranging yields

$$R\frac{\partial \Omega^2}{\partial z}=\frac{\hat{\boldsymbol{\varphi}}}{\rho^2}\boldsymbol{\cdot}(\boldsymbol{\nabla}P\times\boldsymbol{\nabla}\rho). \tag{II.48}$$

This is the $\hat{\boldsymbol{\varphi}}$ component of the vorticity equation. Note that, if ρ is constant or if $P=P(\rho)$, then the angular velocity Ω must be constant on cylinders (this is related to *von Zeippel's theorem*). Now, let us recall the definition of the hydrodynamic entropy, $s=(\gamma-1)^{-1}\ln P\rho^{-\gamma}$ and use it to replace $\boldsymbol{\nabla}\ln\rho$. The result is

$$R\frac{\partial \Omega^2}{\partial z}=\frac{\gamma-1}{\gamma}\,\hat{\boldsymbol{\varphi}}\boldsymbol{\cdot}\left(\boldsymbol{\nabla}s\times\frac{1}{\rho}\boldsymbol{\nabla}P\right)=\hat{\boldsymbol{\varphi}}\boldsymbol{\cdot}\left(\frac{1}{\rho}\boldsymbol{\nabla}P\times\boldsymbol{\nabla}\ln T\right). \tag{II.49}$$

In the Sun, $\boldsymbol{g}=(1/\rho)\boldsymbol{\nabla}P$ is an excellent approximation, with only a tiny angular component due to centrifugal effects. Adopting this simplification and working in spherical coordinates (r,θ,φ), equation (II.49) becomes

$$\boxed{R\frac{\partial \Omega^2}{\partial z}=\frac{\gamma-1}{\gamma}\frac{g}{r}\frac{\partial s}{\partial \theta}} \tag{II.50}$$

where $g=GM/r^2$. [The right-hand side of (II.50) can also be written as $-(g/r)\partial\ln T/\partial\theta$.] Equation (II.50) is known as the *thermal wind equation*. It is used often in geophysical applications (e.g., longitudinal entropy gradients driven by temperature differences cause wind shear) and to understand the rotation profile in the convection zone of the Sun.

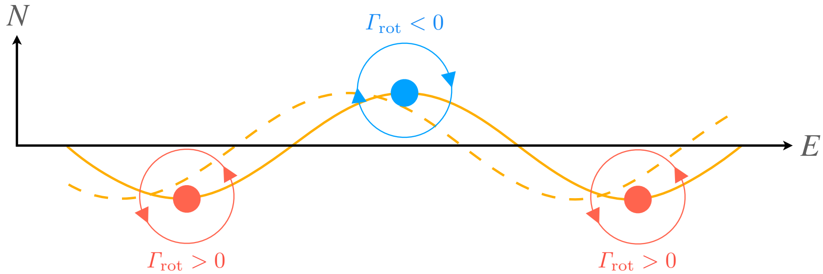
II.5.2. *Rossby waves*

Consider a two-dimensional, incompressible fluid on the surface of uniformly rotating sphere (e.g., a planetary atmosphere). For a constant density or a barotropic equation of state, equation (II.42) becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}\bigl(\Gamma_{\rm rot}+2\Omega\mathcal{S}\cos\theta\bigr)=0, \tag{II.51}$$

where θ is the angle between the rotation vector and the surface oriented normal to the fluid element. (Note that incompressibility assures $\mathcal{S}=\text{const.}$) This equation states that, as a fluid element makes its way from the equator northwards (*viz.*, from $\theta=\pi/2$ towards $\theta=0$), its circulation as measured in the rotating frame must decrease. This means that the element must then rotate in the clockwise direction. Likewise, a fluid element that starts at the north pole and moves southwards towards the equator (*viz.*, from $\theta=0$ towards $\theta=\pi/2$) increases its relative vorticity and thus rotates in the counterclockwise direction.

With this behavior in mind, let's now imagine a small-amplitude, wave-like disturbance at constant latitude (see diagram below). Northward displacements in this wave acquire negative relative vorticity and rotate clockwise; southward displacements acquire positive relative vorticity and rotate counterclockwise. These changes in the velocity of the disturbance actually feed back on the wave itself to make it travel westward; in effect, the wave is advecting itself to the west.



The relationship between the frequency ω and wavevector $\boldsymbol{k}$ for this wave – the *dispersion relation* – is given by

$$\omega = -\frac{k_y}{k_x^2 + k_y^2} \frac{2\Omega \sin\theta}{r}, \tag{II.52}$$

where x denotes the local poloidal direction (pointing southward), y denotes the local azimuthal direction (pointing eastward), and r the spherical radial distance. With $\Omega > 0$ and $k_y > 0$, the phase velocity of the wave $\omega/k_y < 0$, i.e., the wave travels westward. Note that the group velocity, $\partial\omega/\partial k_y$, can be either positive or negative; in general, shorter wavelengths (higher k) have an eastward group velocity and longer wavelengths (smaller k) have a westward group velocity.

These waves are named after the meteorologist Carl Rossby, who derived the mathematics governing this phenomenon in 1939 while at MIT (after which he became assistant director of research at the U.S. Weather Bureau and then moved to University of Chicago as Chair of the Department of Meteorology).[7]

PART III
Fundamentals of plasmas

Now that we have the fluid equations under our belts, let us discuss why we might expect them to apply to a plasma (instead of the more familiar fluid). There are three concepts to cover in this regard: Debye shielding and quasi-neutrality, plasma oscillations, and collisional relaxation of the plasma to take on a Maxwell–Boltzmann distribution of particle velocities.

III.1. Debye shielding and quasi-neutrality

In § I.1, we mentioned the concept of the *Debye length* and explained its importance in the definition of a plasma. Here we actually derive it from first principles. This derivation starts by recalling that a large plasma parameter $\Lambda \gg 1$ implies that the kinetic energy of the plasma particles is much greater than the potential energy due to Coulomb interactions amongst binary pairs of particles. In this case, the plasma temperature T is much bigger than the Coulomb energy $e\phi \sim e^2/\Delta r \sim e^2 n^{1/3}$, where ϕ is the electrostatic potential, $\Delta r \sim n^{-1/3}$ is the typical interparticle distance, and n is the number density of the particles. Assuming a plasma in local thermodynamic equilibrium, the number density of species α' with charge $q_{\alpha'}$ sitting in the potential ϕ_α of one 'central' particle

[7] See https://elischolar.library.yale.edu/journal_of_marine_research/516.

of species α ought to satisfy the Boltzmann relation

$$n_{\alpha'}(\boldsymbol{r}) = \overline{n}_{\alpha'} \exp\left(-\frac{q_\alpha \phi_\alpha(\boldsymbol{r})}{T}\right) \approx \overline{n}_{\alpha'}\left(1 - \frac{q_{\alpha'}\phi_\alpha(\boldsymbol{r})}{T}\right), \tag{III.1}$$

where the potential $\phi_\alpha(\boldsymbol{r})$ depends on the distance $\boldsymbol{r}$ from the 'central' particle. To obtain the approximate equality, we have used the assumption $T \gg e\phi_\alpha$ to Taylor expand the Boltzmann factor in its small argument. Inserting (III.1) into the Gauss–Poisson law for the electric field $\boldsymbol{E} = -\boldsymbol{\nabla}\phi_\alpha$, we have

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E} = -\nabla^2\phi_\alpha &= 4\pi q_\alpha \delta(\boldsymbol{r}) + 4\pi \sum_{\alpha'} q_{\alpha'} n_{\alpha'} \\ &\approx 4\pi q_\alpha \delta(\boldsymbol{r}) + 4\pi \sum_{\alpha'} q_{\alpha'}\overline{n}_{\alpha'} - \underbrace{\left(\sum_{\alpha'} \frac{4\pi \overline{n}_{\alpha'} q_{\alpha'}^2}{T}\right)}_{\doteq\, \lambda_{\mathrm{D}}^{-2}} \phi_\alpha. \end{aligned} \tag{III.2}$$

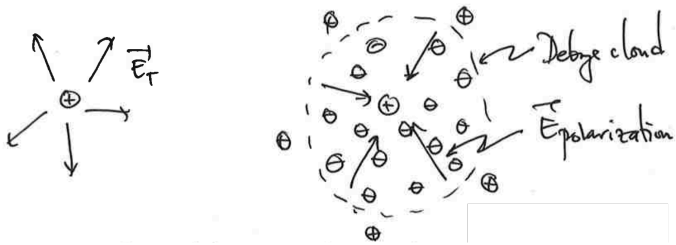
The first term in (III.2) is the point-like charge of the 'central' particle located at $\boldsymbol{r} = \boldsymbol{0}$. The second term is the sum over all charges in the plasma, and equals zero if the plasma is overall charge-neutral (as it should be). The final term introduces the Debye length (see (I.2)), which is the only characteristic scale in (III.2). Note further that this equation has no preferred direction, and so we may exploit its spherical symmetry to recast it as follows:

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \phi_\alpha}{\partial r} - \frac{1}{\lambda_{\mathrm{D}}^2}\phi_\alpha = 4\pi q_\alpha \delta(\boldsymbol{r}). \tag{III.3}$$

The solution to this equation that asymptotes to the Coulomb potential $\phi_\alpha \to q_\alpha/r$ as $r \to 0$ and to zero as $r \to \infty$ is

$$\boxed{\phi_\alpha = \frac{q_\alpha}{r} \exp\left(-\frac{r}{\lambda_{\mathrm{D}}}\right)} \tag{III.4}$$

This equation states that the bare potential of the 'central' charge is exponentially attenuated ('shielded') on typical distances $\sim\lambda_{\mathrm{D}}$. This is *Debye shielding*, and the sphere of neutralizing charge accompanying the 'central' charge is referred to as the *Debye sphere* (or cloud). Debye shielding of an ion by preferential accumulation of electrons in its vicinity is sketched below:



Note that the electric field due to the polarization of the plasma in response to the ion's bare Coulomb potential acts in the opposite direction to the unshielded electric field.

Now, there was nothing particularly special about the charge that we singled out as our 'central' charge. Indeed, we could have performed the above integration for any charge in the plasma. This leads us to the fundamental tenet in the statistical mechanics of a weakly coupled plasma with $\Lambda \gg 1$: every charge simultaneously hosts its own Debye sphere while being a member of another charge's Debye sphere. The key points are that, by involving a huge number of particles in the small-scale electrostatics of the plasma, these Coulomb-mediated relations (i) make the plasma 'quasi-neutral' on scales $\gg\lambda_{\mathrm{D}}$ and (ii) make collective effects in the plasma much more important than individual binary effects due to particle-particle pairings. The latter is what makes a plasma very different from a neutral gas, in which particle-particle interactions occur through hard-body collisions on scales comparable to the mean particle size.

One consequence of Debye shielding is that the electric fields that act on large scales due to the self-consistent collective interactions between $\sim\Lambda$ Debye clouds are smoothly varying in space and time. As a result, when we write down Maxwell's equations for our quasi-neutral plasma, the fields that appear are these smooth, coarse-grained fields whose spatial structure resides far above the Debye length. Mathematically, we average the Maxwell equations over the microscopic (i.e., Debye) scales, and what remains are the collective macroscopic fields that ultimately make their way into the magnetohydrodynamics of the plasma 'fluid'.

III.2. Plasma oscillations

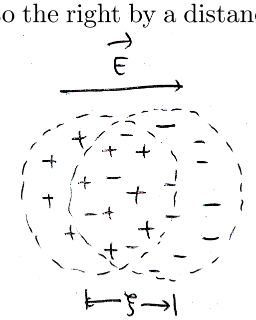
In the previous section, we spoke of a characteristic length scale below which particle-particle interactions are important and above which they are supplanted by collective effects between a large number of quasi-neutral Debye spheres. Is there a corresponding characteristic time scale? The answer is yes, and it may be obtained simply by dimensional analysis: take our Debye length and divide by a velocity to get time. The only velocity in our plasma thus far is the thermal speed, $v_{\mathrm{th}\alpha} = \sqrt{2T/m_\alpha}$, and so that must be it... we have obtained the *plasma frequency* of species α,

$$\omega_{\mathrm{p}\alpha} \doteq \sqrt{\frac{4\pi q_\alpha^2 n_\alpha}{m_\alpha}} \sim \frac{\lambda_{\mathrm{D}}}{v_{\mathrm{th}\alpha}}. \tag{III.5}$$

Of particular importance, given the smallness of the electron mass, is the electron plasma frequency ω_{pe}, which is $\sim\sqrt{m_{\mathrm{i}}/m_{\mathrm{e}}}$ larger than the ion plasma frequency and is generally the largest frequency in a weakly coupled plasma.

Fine. Dimensional analysis works. But what does this frequency actually mean? Go back to our picture of Debye shielding. That was a static picture, in that we waited long enough for the plasma to settle down into charge distributions governed by Boltzmann relations. What if we didn't wait? Surely there was some transient process whereby the particles moved around to configure themselves into these nice equilibrated Debye clouds. There was, and this transient process is referred to as a *plasma oscillation*, and it has a characteristic frequency of (you guessed it) ω_{pe}. Let's show this.

Imagine a spatially uniform, quasi-neutral plasma with well-equilibrated Debye clouds. Shift all of the electrons slightly to the right by a distance ξ, as shown in the figure below:



The offset between the electrons and the ions will cause an electric field pointing from the ions to the displaced electrons, given by $E = 4\pi e n_{\mathrm{e}}\xi$. The equation of motion for the electrons is then

$$m_{\mathrm{e}}\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = -eE = -4\pi e^2 n_{\mathrm{e}}\xi = -m_{\mathrm{e}}\omega_{\mathrm{pe}}^2\xi \quad\Longrightarrow\quad \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = -\omega_{\mathrm{pe}}^2\xi. \tag{III.6}$$

This is just the equation for a simple harmonic oscillator with frequency ω_{pe}. So, small displacements between oppositely charged species result in *plasma oscillations* (or 'Langmuir oscillations'), a collective process that occurs as the plasma attempts to restore quasi-neutrality in response to some disturbance. Retaining the effects of electron pressure makes these oscillations propagate dispersively with a non-zero group velocity; these *Langmuir waves* have the dispersion relation $\omega^2 \approx \omega_{\mathrm{pe}}^2(1+3k^2\lambda_{\mathrm{De}}^2)$, where k is the wavenumber of the perturbation. More on that later.

III.3. Collisional relaxation and the Maxwell–Boltzmann distribution

In order for the plasma particles to move freely as plasma oscillations attempt to set up equilibrated Debye clouds, the mean free path between particle–particle collisions must be larger than the Debye length. We may estimate the former in term of the collision cross-section σ,

$$\lambda_{\mathrm{mfp}} \sim \frac{1}{n\sigma} \sim \frac{T^2}{ne^4},$$

where the cross-section $\sigma = \pi b^2$ is given by a balance between the Coulomb potential energy, $\sim e^2/b$, across some typical impact parameter b and the kinetic energy of the particles, $\sim T$. Comparing this mean free path to the Debye length (I.2), we find

$$\frac{\lambda_{\mathrm{mfp}}}{\lambda_{\mathrm{D}}} \sim \frac{T^2}{ne^4}\left(\frac{ne^2}{T}\right)^{1/2} \sim n\lambda_{\mathrm{D}}^3 \doteq \mathit{\Lambda} \gg 1.$$

Thus, a particle can travel a long distance and experience the macroscopic fields exerted by the collective electrodynamics of the plasma before being deflected by much the shorter-range, microscopic electric fields generated by another individual particle (recall (I.9)).

The scale separation between the collisional mean free path and the Debye length due to the enormity of the plasma parameter in a weakly coupled plasma says something very important about the statistical mechanics of the plasma. Because $\lambda_{\mathrm{mfp}}/\lambda_{\mathrm{D}} \sim \omega_{\mathrm{pe}}\tau_{\mathrm{ei}} \gg 1$, the particle motions are randomized and the velocity distribution of the plasma particles relaxes to a local Maxwell–Boltzmann distribution on (collisional) timescales that are much longer than the timescale on which particle correlations are established and Coulomb potentials are shielded. As a result, collisions in the plasma occur between partially equilibrated Debye clouds instead of between individual particles, the mathematical result being that the ratio $\lambda_{\mathrm{mfp}}/\lambda_{\mathrm{D}}$ is attenuated by a factor $\sim \ln\mathit{\Lambda} \approx$ 10–40. Thus, the logarithmic factors in the collision times (I.7) and (I.8).

Now, about this collisional relaxation. This school isn't the place to go through all the details of how collision operators are derived, but we need to establish a few facts. First, because of Debye shielding, the vast majority of scatterings that a particle experiences as it moves through a plasma are *small-angle scatterings*, with each event changing the trajectory of a particle by a small amount. These accumulate like a random walk in angle away from the original trajectory of the particle, with an average deflection angle $\langle\theta\rangle = 0$ but with a mean-square deflection angle $\langle\theta^2\rangle$ proportional to the number of scattering events. For a typical electron scattering off a sea of Debye-shielded ions of charge Ze and density n, this angle satisfies

$$\langle\theta^2\rangle \approx \frac{8\pi n L Z^2 e^4}{m_{\rm e}^2 v_{\rm the}^4} \ln\Lambda \tag{III.7}$$

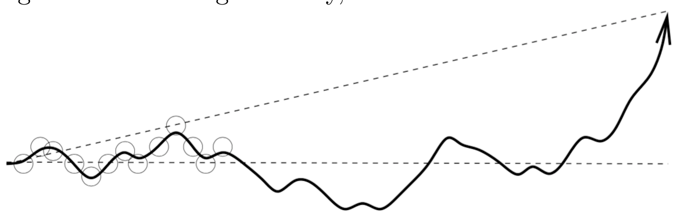
after the electron has traversed a distance L. A large deflection angle, i.e. $\langle\theta^2\rangle \sim 1$, is reached once this distance

$$L \sim \frac{m_{\rm e}^2 v_{\rm the}^4}{8\pi n Z^2 e^4} \frac{1}{\ln\Lambda} \sim v_{\rm the}\tau_{\rm ei} \doteq \lambda_{\rm mfp,e}, \tag{III.8}$$

the collisional mean free path (recall the definition of the electron–ion collision time, equation (I.7)). Noting that the impact parameter for a single 90-degree scattering is $\sim Ze^2/T$, we find the ratio of the cross-section for many small-angle scatterings to accumulate a 90-degree deflection, $\sigma_{\rm multi,90^\circ} \sim 1/nL$ using (III.8), to the cross-section for a single 90-degree scattering, $\sigma_{\rm single,90^\circ} = \pi b^2$ with $b \sim Ze^2/T$, is

$$\frac{\sigma_{\rm multi,90^\circ}}{\sigma_{\rm single,90^\circ}} \sim \ln\Lambda \gg 1. \tag{III.9}$$

Thus, in a weakly coupled plasma, multiple small-angle scatterings are more important than a single large-scale scattering. Visually,



This is the physical origin of the $\ln\Lambda$ reduction in collision time mentioned in the prior paragraph.

So what do these collisions mean for treating our plasma as a fluid? If $\lambda_{\rm mfp}$ is much less than any other macroscopic scale of dynamical interest (i.e., scales on which hydrodynamics occurs), then the *velocity distribution function* $f(\boldsymbol{v})$ of the plasma – that is, the differential number of particles with velocities between $\boldsymbol{v}$ and $\boldsymbol{v}+{\rm d}\boldsymbol{v}$ – is well described by a Maxwell–Boltzmann distribution (often simply called a 'Maxwellian'):

$$f_{\rm M}(v) \doteq \frac{n}{\pi^{3/2} v_{\rm th}^3} \exp\left(-\frac{v^2}{v_{\rm th}^2}\right). \tag{III.10}$$

The factor of $\pi^{3/2} v_{\rm th}^3$ is there for normalization purposes:

$$\int {\rm d}^3\boldsymbol{v}\, f_{\rm M}(\boldsymbol{v}) = 4\pi \int {\rm d}v\, v^2 f_{\rm M}(v) = n \tag{III.11}$$

is the number of particles per unit volume. (Any particle distribution function should satisfy this constraint.) Note that the Maxwellian is isotropic in velocity space, depending only on the speed of the particles (rather than their vector velocity). If these particles are all co-moving with some bulk velocity $\boldsymbol{u}$, then this 'fluid' velocity is subtracted off to ensure an isotropic distribution function in that 'fluid' frame:

$$f_{\rm M}(\boldsymbol{v}) \doteq \frac{n}{\pi^{3/2} v_{\rm th}^3} \exp\left(-\frac{|\boldsymbol{v}-\boldsymbol{u}|^2}{v_{\rm th}^2}\right). \tag{III.12}$$

Note that the first moment of this distribution

$$\int {\rm d}^3\boldsymbol{v}\, \boldsymbol{v} f_{\rm M}(\boldsymbol{v}) = n\boldsymbol{u}; \tag{III.13}$$

and that the (mass-weighted) second moment of this distribution

$$\int \mathrm{d}^3\boldsymbol{v}\, m|\boldsymbol{v}-\boldsymbol{u}|^2 f_{\mathrm{M}}(\boldsymbol{v}) = 3P. \tag{III.14}$$

(Again, any velocity distribution function should satisfy these constraints.)

Different species collisionally relax to a Maxwellian at different rates (e.g., $\tau_{\mathrm{ee}} \sim \tau_{\mathrm{ei}} \sim \sqrt{m_{\mathrm{i}}/m_{\mathrm{e}}}\,\tau_{\mathrm{ii}} \sim (m_{\mathrm{i}}/m_{\mathrm{e}})\tau_{ie}$), and so each species may be described by their own Maxwellians:

$$f_{\mathrm{M},\alpha}(\boldsymbol{v}) \doteq \frac{n_\alpha}{\pi^{3/2} v_{\mathrm{th}\alpha}^3} \exp\left(-\frac{|\boldsymbol{v}-\boldsymbol{u}_\alpha|^2}{v_{\mathrm{th}\alpha}^2}\right). \tag{III.15}$$

But, in the long-time limit, unless some process actively dis-equilibrates the species on a timescale comparable to or smaller than these collision times, all species will take on the *same* $\boldsymbol{u}$ and the *same* T. Their densities are, of course, the same as well, as guaranteed by quasi-neutrality (*viz.*, $\omega_{\mathrm{pe}}\tau \gg 1$ for all collision times τ).

Note then, that when we wrote down our hydrodynamic equations for a scalar pressure (see (II.14) and (II.17)) and didn't affix any species labels to any quantities, we were implicitly assuming that our hydrodynamics occurs on time scales much longer than the collisional equilibration times, so that all species can be well described by local Maxwellians with the same density, fluid velocity, and temperature. Not all astrophysical systems are so cooperative, and anisotropic pressures, velocity drifts between species, and dis-equilibration of species temperatures can often be the norm. Yes, hydrodynamics and MHD are fairly simple, but do not let their simplicity lure you into using them when it's not appropriate to do so – a hard-earned lesson for many astrophysicists.

PART IV

Fundamentals of magnetohydrodynamics

IV.1. The equations of ideal magnetohydrodynamics

Ideal magnetohydrodynamics (MHD) describes the hydrodynamics of a perfectly conducting fluid in the presence of electromagnetic fields. Mass is still conserved, so we still have the continuity equation:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla\cdot}(\rho\boldsymbol{u}) = 0. \tag{IV.1}$$

The first law of thermodynamics still holds, so we still have the internal energy equation:

$$\frac{\partial e}{\partial t} + \boldsymbol{\nabla\cdot}(e\boldsymbol{u}) = -P\boldsymbol{\nabla\cdot u}. \tag{IV.2}$$

And Newton's second law still governs the dynamics, so we still have the momentum equation:

$$\frac{\partial(\rho\boldsymbol{u})}{\partial t} + \boldsymbol{\nabla\cdot}(\rho\boldsymbol{u}\boldsymbol{u}) = \boldsymbol{f}. \tag{IV.3}$$

But now we must supplement the force $\boldsymbol{f}$, which was equal to $-\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\varPhi$ in §II, with the force due to the electromagnetic fields on the conducting fluid elements. To do so, let us view our conducting fluid elements as a coherent collection of ions (with charge $q_{\mathrm{i}} = Ze > 0$) and electrons (with charge $q_{\mathrm{e}} = -e < 0$), and ask how electric and magnetic fields influence each of these species.

The electromagnetic force per unit volume on a collection of charges of species α is given by

$$\boldsymbol{f}_{\mathrm{EM}} = q_\alpha n_\alpha \left(\boldsymbol{E} + \frac{\boldsymbol{u}_\alpha}{c} \times \boldsymbol{B} \right), \tag{IV.4}$$

where n_α is the number density of the species and $\boldsymbol{u}_\alpha$ is that species' bulk velocity. You can think of this simply as the Lorentz force $q_\alpha(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}/c)$ integrated over the ensemble of α charges in each fluid element and divided by the volume of said fluid element. Separating (IV.3) into its charged constituent parts, we then have the momentum equation for species α,

$$\frac{\partial(\rho_\alpha \boldsymbol{u}_\alpha)}{\partial t} + \boldsymbol{\nabla \cdot} (\rho_\alpha \boldsymbol{u}_\alpha \boldsymbol{u}_\alpha) = -\boldsymbol{\nabla} P_\alpha - \rho_\alpha \boldsymbol{\nabla} \Phi + q_\alpha n_\alpha \left(\boldsymbol{E} + \frac{\boldsymbol{u}_\alpha}{c} \times \boldsymbol{B} \right). \tag{IV.5}$$

At the moment, the trouble is that our continuity equation (IV.1) and internal energy equation (IV.2) make reference to the *total* mass density ρ, the *total* fluid velocity $\boldsymbol{u}$, the *total* pressure P, and the *total* internal energy e. The obvious thing to do, then, is to sum (IV.5) over both species,

$$\sum_\alpha \left[\frac{\partial(\rho_\alpha \boldsymbol{u}_\alpha)}{\partial t} + \boldsymbol{\nabla \cdot} (\rho_\alpha \boldsymbol{u}_\alpha \boldsymbol{u}_\alpha) = -\boldsymbol{\nabla} P_\alpha - \rho_\alpha \boldsymbol{\nabla} \Phi + q_\alpha n_\alpha \left(\boldsymbol{E} + \frac{\boldsymbol{u}_\alpha}{c} \times \boldsymbol{B} \right) \right], \tag{IV.6}$$

and simplify each of the sums one by one. The first term in (IV.6) becomes familiar after introducing the center-of-mass fluid velocity,

$$\boldsymbol{u} \doteq \frac{1}{\rho} \sum_\alpha \rho_\alpha \boldsymbol{u}_\alpha, \quad \text{where} \quad \rho \doteq \sum_\alpha \rho_\alpha. \tag{IV.7}$$

The second term in (IV.6) requires a bit more work. Write $\boldsymbol{u}_\alpha = \boldsymbol{u} + \Delta \boldsymbol{u}_\alpha$, so that $\Delta \boldsymbol{u}_\alpha$ measures the difference between the bulk flow of species α and the center-of-mass velocity $\boldsymbol{u}$. Then

$$\sum_\alpha \rho_\alpha \boldsymbol{u}_\alpha \boldsymbol{u}_\alpha = \rho \boldsymbol{u} \boldsymbol{u} + \boldsymbol{u} \left(\cancelto{0}{\sum_\alpha \rho_\alpha \Delta \boldsymbol{u}_\alpha} \right) + \left(\cancelto{0}{\sum_\alpha \rho_\alpha \Delta \boldsymbol{u}_\alpha} \right) \boldsymbol{u} + \sum_\alpha \rho_\alpha \Delta \boldsymbol{u}_\alpha \Delta \boldsymbol{u}_\alpha.$$

The first term here ($\rho \boldsymbol{u} \boldsymbol{u}$) should look familiar: it's the flux of momentum density associated with the total fluid, the same as was seen in §II. Moving the final term of the above expression to the right-hand side of (IV.6) and writing $\sum_\alpha P_\alpha \doteq P$, we have a momentum equation that is starting to look more like (IV.3):

$$\frac{\partial(\rho \boldsymbol{u})}{\partial t} + \boldsymbol{\nabla \cdot} (\rho \boldsymbol{u} \boldsymbol{u}) = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi - \sum_\alpha \rho_\alpha \Delta \boldsymbol{u}_\alpha \Delta \boldsymbol{u}_\alpha + \sum_\alpha q_\alpha n_\alpha \left(\boldsymbol{E} + \frac{\boldsymbol{u}_\alpha}{c} \times \boldsymbol{B} \right). \tag{IV.8}$$

Now, this term involving $\Delta \boldsymbol{u}_\alpha$ has nothing really to do with MHD, and was in fact implicitly discarded in §II.1.2, the reason being either that our fluid element is composed of a single species, or that collisions between different species keep their bulk flows very close to the center-of-mass velocity, or that the total mass density and total momentum density are completely dominated by a single species (e.g., the ions). In any of these cases, we may safely drop this term.

Almost there. All that remains to consider is

$$\sum_\alpha q_\alpha n_\alpha \left(\boldsymbol{E} + \frac{\boldsymbol{u}_\alpha}{c} \times \boldsymbol{B} \right).$$

In §III.1, we showed that the densities of the positive and negative charge carriers surrounding a point charge Q in a weakly coupled plasma satisfies

$$\sum_\alpha q_\alpha n_\alpha = \cancelto{0}{\sum_\alpha q_\alpha \bar{n}_\alpha} - \frac{Q}{4\pi\lambda_{\rm D}^3}\,\frac{\exp(-r/\lambda_{\rm D})}{r/\lambda_{\rm D}},$$

and therefore is extremely close to zero well outside of that charge's Debye sphere, i.e., the plasma is *quasi-neutral* on scales $r \gg \lambda_{\rm D}$. MHD concerns itself with just such scales, and so the total electric force on a fluid element in MHD vanishes under quasi-neutrality. This leaves the magnetic term, $(\sum_\alpha q_\alpha n_\alpha \boldsymbol{u}_\alpha) \times \boldsymbol{B}/c$. The sum in parentheses is equivalent to the *current density* of the plasma, $\boldsymbol{j}$, the amount of electric current flowing per unit cross-sectional area. We these principles implemented, our MHD momentum equation is finally here:

$$\boxed{\frac{\partial(\rho\boldsymbol{u})}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\boldsymbol{u}\boldsymbol{u}) = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\varPhi + \frac{\boldsymbol{j}}{c}\times\boldsymbol{B}} \tag{IV.9}$$

Another way to this of this additional term is by analogy with circuits: when a current $\boldsymbol{I}$ flows through a wire of length ℓ in the presence of a magnetic field $\boldsymbol{B}$, there is a force on the wire given by $\boldsymbol{I}\ell \times \boldsymbol{B}/c$. In the fluid context, the 'wire' is the conducting fluid element through which electrons and ions move differentially.

We now have our continuity equation, internal energy equation, and MHD momentum equation. However, in deriving the latter, we have introduced two new variables, $\boldsymbol{j}$ and $\boldsymbol{B}$. The remaining tasks are then to express the current density $\boldsymbol{j}$ in terms of the magnetic field $\boldsymbol{B}$ (since by summing over the momentum equations of each species, we've lost information about each species' bulk flow), and to provide an equation for how the magnetic field evolves. Both of these tasks are solved by Maxwell's equations:

$$\frac{\partial\boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla}\times\boldsymbol{E}, \quad \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{B} = 0, \quad \frac{\partial\boldsymbol{E}}{\partial t} = c\boldsymbol{\nabla}\times\boldsymbol{B} - 4\pi\boldsymbol{j}, \quad \color{red}{\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{E} = 4\pi\sum_\alpha q_\alpha n_\alpha},$$

with the important caveat that the final equation in red (Gauss' law) is rendered completely useless by the quasi-neutrality assumption, $\sum_\alpha q_\alpha n_\alpha \approx 0$. The other equations are (from left to right) Faraday's law of induction, Gauss' law for magnetism (no magnetic monopoles), and Maxwell's version of Ampère's law. No offense to Maxwell, but it turns out that the original Ampère's law,

$$\boldsymbol{j} = \frac{c}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{B}, \tag{IV.10}$$

is just fine our purposes. The displacement current, $(4\pi)^{-1}\partial\boldsymbol{E}/\partial t$, which mathematically and physically connects electromagnetism with the propagation of light, may be rigorously dropped if the fluid velocity satisfies $u^2 \ll c^2$. Why, you ask? Well, this brings us back to the first sentence of this section: we are interested in *perfect conductors*.

A perfect conductor is one that has exactly zero electrical resistance, and so by Ohm's law must have zero electrostatic field. But this doesn't necessarily mean that $\boldsymbol{E} = 0$, because an electric field can be induced by the motion of a conductor through a magnetic field (sometimes called the 'motional emf'). What we mean by a perfect conductor is then that the electric field vanishes *in the frame of the conductor*, or

$$\boldsymbol{E} + \frac{\boldsymbol{u}}{c}\times\boldsymbol{B} = 0. \tag{IV.11}$$

Inserting this equation into the Maxwell–Ampère law and ordering $\partial/\partial t \sim u/\ell$ for some characteristic bulk flow velocity u and gradient lengthscale ℓ, we find that the displacement current

$$\frac{\partial \boldsymbol{E}}{\partial t} \sim \frac{u^2}{c^2}\frac{cB}{\ell} \ll \frac{cB}{\ell} \sim c\boldsymbol{\nabla}\times\boldsymbol{B}$$

if the flow is non-relativistic. As claimed, the original Ampère's law is just fine.

Altogether then, we may close our MHD momentum equation with the following subset of Maxwell's equations:

$$\boxed{\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}), \quad \boldsymbol{\nabla\cdot B} = 0, \quad \boldsymbol{j} = \frac{c}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{B}} \tag{IV.12}$$

These equations for the electromagnetic fields $\boldsymbol{B}$ and $\boldsymbol{j}$ – taken alongside (IV.1), (IV.2), and (IV.8) specifying the evolution of the hydrodynamics variables $(\rho,\rho\boldsymbol{u},e)$ – constitute the equations of ideal MHD.

IV.1.1. *Flux freezing: Alfvén's theorem*

Arguably the most important prediction of the ideal MHD equations is that the magnetic flux Φ_B through the surface of any fluid element is exactly conserved as that element is advected and deformed by a flow $\boldsymbol{u} = \boldsymbol{u}(t,\boldsymbol{r})$. This is known as 'Alfvén's theorem' or, more colloquially, *flux freezing*. Given Leibniz's rule regarding the time derivatives of surface integrals whose integrations limits $\mathcal{S}(t)$ are time-dependent (eq. (II.32)), the proof itself is trivial:

$$\begin{aligned}
\frac{\mathrm{D}\Phi_B}{\mathrm{D}t} \doteq \frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{S}(t)} \mathrm{d}\boldsymbol{\mathcal{S}}\boldsymbol{\cdot}\boldsymbol{B} &= \int_{\mathcal{S}(t)} \mathrm{d}\boldsymbol{\mathcal{S}}\boldsymbol{\cdot}\left[\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{\nabla\cdot B})\boldsymbol{u}\right] - \oint_{\partial\mathcal{S}(t)} \mathrm{d}\boldsymbol{\ell}\boldsymbol{\cdot}(\boldsymbol{u}\times\boldsymbol{B}) \\
\text{(use equation (IV.12))} &= \int_{\mathcal{S}(t)} \mathrm{d}\boldsymbol{\mathcal{S}}\boldsymbol{\cdot}\Big[\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})\Big] - \oint_{\partial\mathcal{S}(t)} \mathrm{d}\boldsymbol{\ell}\boldsymbol{\cdot}(\boldsymbol{u}\times\boldsymbol{B}) \\
\text{(use Stokes' theorem)} &= \oint_{\partial\mathcal{S}(t)} \mathrm{d}\boldsymbol{\ell}\boldsymbol{\cdot}(\boldsymbol{u}\times\boldsymbol{B}) - \oint_{\partial\mathcal{S}(t)} \mathrm{d}\boldsymbol{\ell}\boldsymbol{\cdot}(\boldsymbol{u}\times\boldsymbol{B}) \\
&= 0.
\end{aligned} \tag{IV.13}$$

In words, the magnetic flux is conserved in a frame comoving with a fluid element. (This is analogous to Kelvin's circulation theorem governing the circulation; cf. (II.39).)

An alternative description of flux freezing can be stated in terms of line tying: fluid elements that lie on a field line initially will remain on that field line (Lundquist 1951). See Problem 9 in Problem Set 1.

IV.1.2. *Ideal MHD induction equation*

Using a particular vector identity (see §II.3.1), the ideal MHD induction equation may be written in the following form:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) = -\boldsymbol{u\cdot\nabla B} + \boldsymbol{B\cdot\nabla u} - \boldsymbol{B\nabla\cdot u}. \tag{IV.14}$$

Each of the terms on the right-hand side has a physical meaning. The first indicates that the magnetic field is advected (carried around by) the fluid flow; when placed on the left-hand side, we obtain the Lagrangian derivative of the magnetic field, $\mathrm{D}\boldsymbol{B}/\mathrm{D}t$. In this Lagrangian frame, the magnetic field can evolve because of two effects. The second term on the right-hand side, $\boldsymbol{B\cdot\nabla u}$, represents stretching of the magnetic field: if the fluid velocity has a gradient along the direction of the magnetic field, different parts of the field line will be carried along at different velocities, causing the field line to stretch. The final term, $-\boldsymbol{B\nabla\cdot u}$, corresponds to compression or rarefaction of the magnetic field.

Indeed, with the continuity equation giving $-\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u} = \mathrm{D}\ln\rho/\mathrm{D}t$, we see that co-moving increases (decreases) in the fluid density go hand-in-hand with increases (decreases) in the magnetic-field strength.

A rarely publicized but useful form of the induction equation (IV.14) is obtained by defining the magnetic-field unit vector $\hat{\boldsymbol{b}} \doteq \boldsymbol{B}/B$ and writing separate equations for it and the magnetic-field strength B:

$$\frac{\mathrm{D}\ln B}{\mathrm{D}t} = \big(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \mathsf{I}\big) \boldsymbol{:} \boldsymbol{\nabla}\boldsymbol{u} \quad \text{and} \quad \frac{\mathrm{D}\hat{\boldsymbol{b}}}{\mathrm{D}t} = \big(\mathsf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big) \boldsymbol{\cdot} \big(\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}\big). \tag{IV.15}$$

These may come in handy one day...

IV.1.3. *Lorentz force: Magnetic pressure and tension*

We now know that perfectly conducting fluids advect, stretch, and compress magnetic fields while conserving magnetic flux. What is the effect of that flux on the dynamics of the fluid element itself? For that, we revisit the Lorentz force in the MHD momentum equation (IV.9), and use Ampère's law to cast the current density in terms of the magnetic field:

$$\boldsymbol{f}_{\mathrm{M}} = \frac{\boldsymbol{j}}{c}\times\boldsymbol{B} = \frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi} = -\boldsymbol{\nabla}\frac{B^2}{8\pi} + \frac{\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{B}}{4\pi}, \tag{IV.16}$$

where to obtain the final equality we have used a well-known vector identity (see §II.3.1). Because $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{B} = 0$, this can also be written as

$$\boldsymbol{f}_{\mathrm{M}} = -\boldsymbol{\nabla}\boldsymbol{\cdot}\left[\frac{B^2}{8\pi}\mathsf{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right] = -\boldsymbol{\nabla}\boldsymbol{\cdot}\mathsf{M}, \tag{IV.17}$$

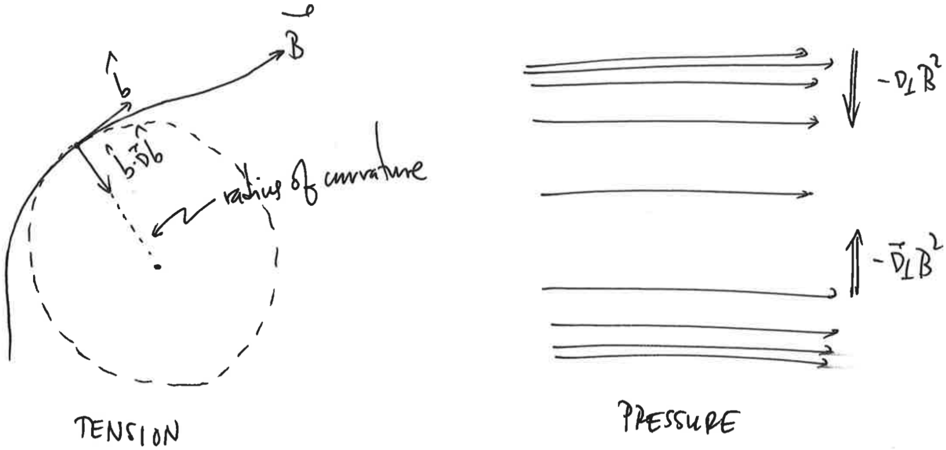
which implicitly defines the 'Maxwell stress', M. This form of the magnetic force suggests a kind of elasticity. To further see this, use the definition of the magnetic unit vector $\hat{\boldsymbol{b}} \doteq \boldsymbol{B}/B$ to write

$$\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{B} = B\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}(B\hat{\boldsymbol{b}}) = B^2(\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\hat{\boldsymbol{b}}) + \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\frac{B^2}{2}.$$

Using this in (IV.16) and collecting terms yields

$$\boldsymbol{f}_{\mathrm{M}} = \frac{B^2}{4\pi}(\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\hat{\boldsymbol{b}}) - \big(\mathsf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\big)\boldsymbol{\cdot}\boldsymbol{\nabla}\frac{B^2}{8\pi}. \tag{IV.18}$$

The first term here corresponds to a curvature force, with $\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}\hat{\boldsymbol{b}} \doteq \boldsymbol{\kappa}$ being the curvature of the field lines (see the diagram below). Note that $1/|\boldsymbol{\kappa}|$ is the radius of curvature. When a field line is bent, there is a force pointing towards the local center of curvature that is trying to un-bend the field line and push the plasma towards a lower-energy state in which the magnetic field is straight. The second term in (IV.18) corresponds to a magnetic pressure force acting perpendicular to the field (thus the projection of the gradient onto $\mathsf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}$). This term causes the magnetic-field strength to evolve towards being uniform across itself, again seeking a lower-energy state. Magnetic fields like to be straight and evenly spaced, and they will coerce the fluid to adopt motions that drive them towards being straight and evenly spaced.



IV.1.4. *MHD energy equation*

In §II.1.3, we derived an evolution equation for the total energy of a neutral fluid (eq. (II.20)). Here we augment that equation for a perfectly conducting fluid to include the energy of the magnetic field, $B^2/8\pi$. Take the ideal MHD induction equation (IV.14) and dot it with $\boldsymbol{B}/4\pi$:

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{B^2}{8\pi} = \frac{\boldsymbol{B}}{4\pi}\boldsymbol{\cdot}\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) &= \frac{B_i}{4\pi}\,\epsilon_{ijk}\frac{\partial}{\partial x_j}(\boldsymbol{u}\times\boldsymbol{B})_k \\
&= \epsilon_{ijk}\frac{\partial}{\partial x_j}\left[\frac{B_i}{4\pi}(\boldsymbol{u}\times\boldsymbol{B})_k\right] - \epsilon_{ijk}(\boldsymbol{u}\times\boldsymbol{B})_k\frac{\partial}{\partial x_j}\frac{B_i}{4\pi} \\
&= \epsilon_{ijk}\frac{\partial}{\partial x_j}\left[\frac{B_i}{4\pi}(\boldsymbol{u}\times\boldsymbol{B})_k\right] - \epsilon_{ijk}\epsilon_{k\ell m}u_\ell B_m\frac{\partial}{\partial x_j}\frac{B_i}{4\pi} \\
&= \epsilon_{ijk}\frac{\partial}{\partial x_j}\left[\frac{B_i}{4\pi}(\boldsymbol{u}\times\boldsymbol{B})_k\right] - (\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell})u_\ell B_m\frac{\partial}{\partial x_j}\frac{B_i}{4\pi} \\
&= -\boldsymbol{\nabla}\boldsymbol{\cdot}\left[\frac{\boldsymbol{B}\times(\boldsymbol{u}\times\boldsymbol{B})}{4\pi}\right] - \frac{\boldsymbol{u}\boldsymbol{B}\,\boldsymbol{:}\,\boldsymbol{\nabla}\boldsymbol{B}}{4\pi} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\frac{B^2}{8\pi}
\end{aligned}
$$

$$
\Longrightarrow\quad \frac{\partial}{\partial t}\frac{B^2}{8\pi} + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[\frac{\boldsymbol{B}\times(\boldsymbol{u}\times\boldsymbol{B})}{4\pi}\right] = -\frac{\boldsymbol{u}\boldsymbol{B}\,\boldsymbol{:}\,\boldsymbol{\nabla}\boldsymbol{B}}{4\pi} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\frac{B^2}{8\pi}.
$$

Note that the quantity inside the divergence on the left-hand side of this equation equals $(c/4\pi)\boldsymbol{E}\times\boldsymbol{B} \doteq \boldsymbol{\mathcal{S}}$... the Poynting flux! In words, magnetic energy (as measured in the lab frame; note the partial time derivative) is transported by the Poynting flux. Those two terms on the right-hand side corresponding to will be cancelled by two equal-and-opposite terms found in the equation for the kinetic energy, obtained by dotting the momentum equation (IV.9) with $\boldsymbol{u}$ and focusing on the Lorentz force:

$$
\boldsymbol{u}\boldsymbol{\cdot}\left(-\boldsymbol{\nabla}\frac{B^2}{8\pi} + \frac{\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{B}}{4\pi}\right).
$$

Yep, they cancel. So, adding the total hydrodynamic energy equation including these Lorentz-force contributions to the magnetic energy equation leads to

$$
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + e + \rho\Phi + \frac{B^2}{8\pi}\right) + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[\left(\frac{1}{2}\rho u^2 + \gamma e + \rho\Phi\right)\boldsymbol{u} + \boldsymbol{\mathcal{S}}\right] = \rho\frac{\partial\Phi}{\partial t}.
$$

But for the impact of a time-varying gravitational potential, the total MHD energy $\mathcal{E} \doteq (1/2)\rho u^2 + e + \rho\Phi + B^2/8\pi$ is conserved.

IV.1.5. *Rotating reference frames*

In §II.3.3, we examined the nonlinear combination $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}$ in curvilinear coordinates, finding additional terms that stemmed from differentiating unit vectors and which included Coriolis, centrifugal, and tidal accelerations. Here we take a similar look at the combination $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{B} - \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u}$ that features in the induction equation (IV.14).

First, use $\partial\hat{\boldsymbol{\varphi}}/\partial\varphi = -\hat{\boldsymbol{R}}$ and $\partial\hat{\boldsymbol{R}}/\partial\varphi = \hat{\boldsymbol{\varphi}}$ in (IV.14) to obtain

$$\frac{\partial\boldsymbol{B}}{\partial t} + \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{u} = (-\boldsymbol{u}\cdot\boldsymbol{\nabla}B_i)\hat{\boldsymbol{e}}_i + (\boldsymbol{B}\cdot\boldsymbol{\nabla}u_i)\hat{\boldsymbol{e}}_i + \frac{B_\varphi u_R - B_R u_\varphi}{R}\hat{\boldsymbol{\varphi}}.$$

As in §II.5, if we then decompose the fluid velocity as $\boldsymbol{u} = \boldsymbol{v} + R\Omega(R,z)\hat{\boldsymbol{\varphi}}$, where Ω is an angular velocity, substitute this decomposition into the above equation, and re-group terms, we have

$$\frac{\mathrm{D}B_R}{\mathrm{D}t} = \boldsymbol{B}\cdot\boldsymbol{\nabla}v_R - B_R\boldsymbol{\nabla}\cdot\boldsymbol{v}, \tag{IV.19a}$$

$$\frac{\mathrm{D}B_\varphi}{\mathrm{D}t} = \boldsymbol{B}\cdot\boldsymbol{\nabla}v_\varphi - B_\varphi\boldsymbol{\nabla}\cdot\boldsymbol{v} + B_R\frac{\partial\Omega}{\partial\ln R} + B_z R\frac{\partial\Omega}{\partial z} + \frac{B_\varphi v_R - B_R v_\varphi}{R}, \tag{IV.19b}$$

$$\frac{\mathrm{D}B_z}{\mathrm{D}t} = \boldsymbol{B}\cdot\boldsymbol{\nabla}v_z - B_z\boldsymbol{\nabla}\cdot\boldsymbol{v}, \tag{IV.19c}$$

with $\mathrm{D}/\mathrm{D}t \doteq \partial/\partial t + \boldsymbol{v}\cdot\boldsymbol{\nabla} + \Omega\,\partial/\partial\varphi$. Note that poloidal magnetic fields are sheared into the azimuthal direction by differential rotation.

IV.2. Summary: Adiabatic equations of ideal MHD

The adiabatic equations of MHD, written in conservative form, are:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \tag{IV.20a}$$

$$\frac{\partial(\rho\boldsymbol{u})}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\boldsymbol{u}) = -\boldsymbol{\nabla}\cdot\left[\left(P + \frac{B^2}{8\pi}\right)\boldsymbol{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right] - \rho\boldsymbol{\nabla}\Phi, \tag{IV.20b}$$

$$\frac{\partial e}{\partial t} + \boldsymbol{\nabla}\cdot(e\boldsymbol{u}) = -P\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{IV.20c}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) = 0. \tag{IV.20d}$$

The left-hand sides of these equations express advection of, respectively, the mass density, the momentum density, the internal energy density, and the magnetic flux by the fluid velocity; the right-hand sides represents sources and sinks.

If we instead write these equations in terms of the density, fluid velocity, and entropy and make use of the Lagrangian derivative (II.7), we have

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}, \tag{IV.21a}$$

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{B}}{4\pi\rho} - \boldsymbol{\nabla}\mathit{\Phi}, \tag{IV.21b}$$

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0, \tag{IV.21c}$$

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}, \tag{IV.21d}$$

where $s \doteq (\gamma - 1)^{-1} \ln P\rho^{-\gamma}$.

PART V
Linear theory of MHD waves

MHD waves and linear theory

Let us summarize our dissipationless "ideal MHD" eqns:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

$$\rho\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\vec{u} = -\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) - \rho\vec{\nabla}\Phi + \frac{\vec{B}\cdot\vec{\nabla}\vec{B}}{4\pi}$$

$$\frac{P}{\gamma-1}\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\ln P\rho^{-\gamma} = 0$$

$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$$

Now, let us consider a uniform, stationary MHD fluid, threaded by a uniform magnetic field. To orient our coordinate system, we will use $\vec{B} = B_0\hat{z}$, with the directions $\perp$ to the field being x and y. We perturb the fluid with small displacements, which we take (freely) to be sinusoidal:

$$\rho = \rho_0 + \delta\rho\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$$
$$\vec{B} = B_0\hat{z} + \vec{\delta B}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$$
$$\vec{u} = \phi + \vec{\delta u}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$$
$$P = P_0 + \delta P\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

Small? What's "small"? By "small", I mean that all nonlinearities ($\propto O(\delta^2)$) will be dropped. The result is <u>linear theory</u>.

Before we do this, note that, when computing actual observed quantities, we should take the real part (e.g. $e^{i\theta} \to \cos\theta$, $ie^{i\theta} \to -\sin\theta$, etc.)

First, let's do the simplest thing: $\vec{k} = k\hat{z}$. My notation is usually "$k_{\parallel}$" in this case, to remind me that k is parallel to the guide field. This notation is used in a lot of plasma physics, but less so in astronomy. Our linearized MHD eqns. are then

$$-i\omega \frac{\delta\rho}{\rho_0} + ik_{\parallel}\delta u_{\parallel} = 0$$

$$-i\omega \vec{\delta u} = -\frac{ik_{\parallel}\hat{z}}{\rho_0}\left(\delta p + \frac{B_0 \delta B_{\parallel}}{4\pi}\right) + \frac{ik_{\parallel}B_0}{4\pi\rho_0}\vec{\delta B}$$

$$-i\omega \frac{\vec{\delta B}}{B_0} = ik_{\parallel}\vec{\delta u} - \hat{z}\, ik_{\parallel}\delta u_{\parallel} \longrightarrow \delta B_{\parallel} = 0 \quad \text{(as is required by } \vec{k}\cdot\vec{\delta B} = 0\text{)}$$

Note that we don't need to know $\delta\rho$ or δp to solve for the perpendicular ($\perp$) dynamics:

$$\left.\begin{aligned} -i\omega \vec{\delta u_{\perp}} &= \frac{ik_{\parallel}B_0}{4\pi\rho_0}\vec{\delta B_{\perp}} \\ -i\omega \frac{\vec{\delta B_{\perp}}}{B_0} &= ik_{\parallel}\vec{\delta u_{\perp}} \end{aligned}\right\} \left(\omega^2 - k_{\parallel}^2 v_A^2\right)\frac{\vec{\delta B_{\perp}}}{B_0} = 0$$

$$\downarrow$$

$$\boxed{\begin{aligned} &\omega = \pm k_{\parallel} v_A \\ &\text{with } v_A \equiv \frac{B_0}{\sqrt{4\pi\rho_0}} \end{aligned}}$$

These are "Alfvén waves", which are polarized across the guide field and which propagate at speed v_A, the "Alfvén speed". These waves are not associated with any motion along the field nor any changes in density.

Using $\frac{\delta p}{p_0} = \gamma\frac{\delta\rho}{\rho_0}$, the other modes are sound waves: $\boxed{\omega = \pm k_{\parallel} c_s}$, with $c_s \equiv \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}$ being the "sound speed".

The fifth mode is $\omega=0$, and corresponds to a relabeling of fluid elements. It's called the "entropy mode"

Now, let's let $\vec{k} = k_\parallel \hat{z} + \vec{k}_\perp$ — a more general wavevector. Then our linearized equations are

$$\text{(a)} \quad -i\omega \frac{\delta\rho}{\rho_0} + ik_\parallel \delta u_\parallel + i\vec{k}_\perp \cdot \delta\vec{u}_\perp = 0$$

$$-i\omega \delta\vec{u} = -\frac{i\vec{k}}{\rho_0}\left(\delta p + \frac{B_0 \delta B_\parallel}{4\pi}\right) + \frac{ik_\parallel B_0}{4\pi\rho_0}\delta\vec{B}$$

$$-i\omega \frac{\delta\vec{B}}{B_0} = ik_\parallel \delta\vec{u} - \hat{z}\left(ik_\parallel \delta u_\parallel + i\vec{k}_\perp \cdot \delta\vec{u}_\perp\right)$$

$$\text{(b)} \quad -i\omega \frac{\delta\vec{B}_\perp}{B_0} = ik_\parallel \delta\vec{u}_\perp \quad \text{and} \quad \text{(c)} \quad -i\omega\frac{\delta B_\parallel}{B_0} = -i\vec{k}_\perp\cdot\delta\vec{u}_\perp$$

$$\text{(d)} \quad -i\omega\delta\vec{u}_\perp = -\frac{i\vec{k}_\perp}{\rho_0}\left(\delta p + \frac{B_0\delta B_\parallel}{4\pi}\right) + \frac{ik_\parallel B_0}{4\pi\rho_0}\delta\vec{B}_\perp \quad \text{and}$$

$$\text{(e)} \quad -i\omega\delta u_\parallel = -\frac{ik_\parallel}{\rho_0}\left(\delta p + \frac{B_0\delta B_\parallel}{4\pi}\right) + \frac{ik_\parallel B_0}{4\pi\rho_0}\delta B_\parallel .$$

$$\vec{k}_\perp\cdot(d) + k_\parallel(e) \Rightarrow -i\omega\left(\vec{k}_\perp\cdot\delta\vec{u}_\perp + k_\parallel\delta u_\parallel\right) = -\frac{ik^2}{\rho_0}\left(\delta p + \frac{B_0\delta B_\parallel}{4\pi}\right) + \cancel{0}$$

use (a) : $\cancel{+}\frac{\cancel{i}\omega}{\cancel{i}}\left(\frac{\cancel{i}\omega\delta\rho}{\rho_0}\right) = \cancel{+}\frac{\cancel{i}k^2}{\rho_0}\left(\delta p + \frac{B_0\delta B_\parallel}{4\pi}\right)$

* here we've lost the entropy mode *

use $\frac{\delta p}{p_0} = \gamma\frac{\delta\rho}{\rho_0}$: $(\omega^2 - k^2 c_s^2)\frac{\delta\rho}{\rho_0} = k^2 V_A^2 \frac{\delta B_\parallel}{B_0}$

Now, (d) with (b) gives: $(\omega^2 - k_\parallel^2 V_A^2)\frac{\delta\vec{B}_\perp}{B_0} = -k_\parallel\vec{k}_\perp\left(c_s^2\frac{\delta\rho}{\rho_0} + V_A^2\frac{\delta B_\parallel}{B_0}\right)$

$$(\omega^2 - k_\parallel^2 V_A^2)\frac{\delta\vec{B}_\perp}{B_0} = -k_\parallel\vec{k}_\perp\frac{\delta B_\parallel}{B_0}\left[c_s^2\frac{k^2 V_A^2}{\omega^2 - k^2 c_s^2} + V_A^2\right]$$

Note that the parallel and perpendicular components are now coupled!

$$\left(\omega^2 - k_\parallel^2 v_A^2\right)\frac{\delta\vec{B}_\perp}{B_0} = -k_\parallel \vec{k}_\perp v_A^2\left[\frac{\omega^2}{\omega^2 - k^2 c_s^2}\right]\frac{\delta B_\parallel}{B_0}$$

Before we go any further, note that, if $c_s^2/v_A^2 \gg 1$, then we have $\omega^2 - k_\parallel^2 v_A^2 \simeq 0$, so we get back something like an Alfvén wave in this limit. Proceeding by using $\frac{\delta B_\parallel}{B_0} = -\frac{\vec{k}_\perp \cdot \delta\vec{B}_\perp}{k_\parallel B_0}$, we have

$$\left[\overleftrightarrow{I}\left(\omega^2 - k_\parallel^2 v_A^2\right) - \vec{k}_\perp \vec{k}_\perp v_A^2 \frac{\omega^2}{\omega^2 - k^2 c_s^2}\right]\cdot \frac{\delta\vec{B}_\perp}{B_0} = 0.$$

Taking the determinant and setting it to zero gives the dispersion relation

$$\left(\omega^2 - k_\parallel^2 v_A^2\right)\left[\omega^2 - k_\parallel^2 v_A^2 - k_\perp^2 v_A^2 \frac{\omega^2}{\omega^2 - k^2 c_s^2}\right] = 0.$$

you'll often see this written as

$$\left[\omega^4 - \omega^2 k^2\left(c_s^2 + v_A^2\right) + k_\parallel^2 v_A^2 k^2 c_s^2\right]$$

But I like it like this because you can take β limits easier.

Note that we recover the Alfvén wave solution $\omega = \pm k_\parallel v_A$. Now we also have $\omega^2 = \frac{k^2(c_s^2 + v_A^2)}{2} \pm \sqrt{\frac{k^4(c_s^2+v_A^2)^2}{4} - k_\parallel^2 v_A^2 k^2 c_s^2}$.

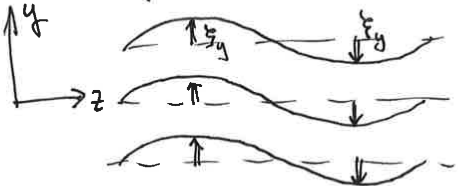
These are the "magnetosonic" modes — the $\oplus$ solution being the "fast wave" and the $\ominus$ solution being the "slow wave". Note that, in the high-β limit, we have

$$\omega_+^2 \simeq \underbrace{k^2 c_s^2}_{\text{sound!}} \quad \text{and} \quad \omega_-^2 \simeq \underbrace{k_\parallel^2 v_A^2}_{\text{Alfvén!}}$$

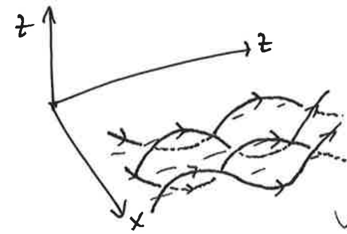
The difference between the slow mode here and an actual shear Alfvén wave is the latter involves no compressive fluctuations, being polarized with $\delta B_\parallel$ exactly $= 0$. This is sometimes called a "pseudo-Alfvén" wave.

Here are some pictures of these waves: ($\vec{\xi}$ is displacement)

Alfvén:

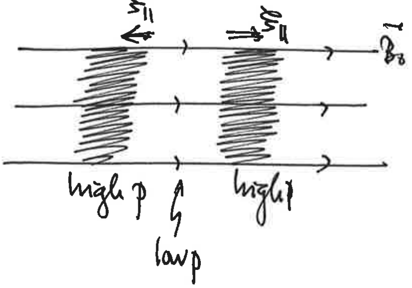


(NB: can have $k_\perp \neq 0$

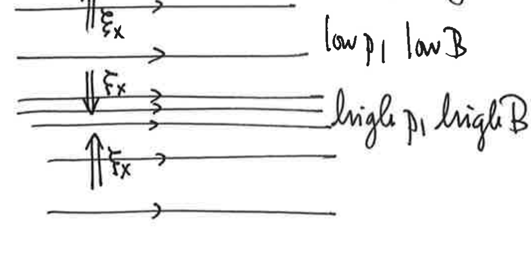


)

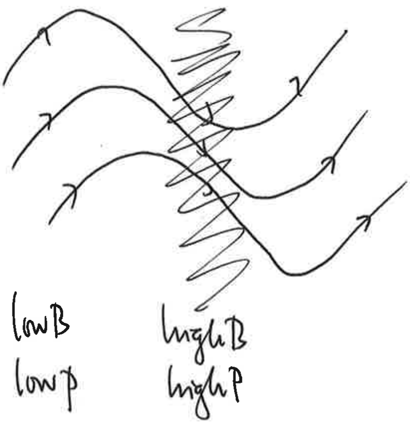
Sound:



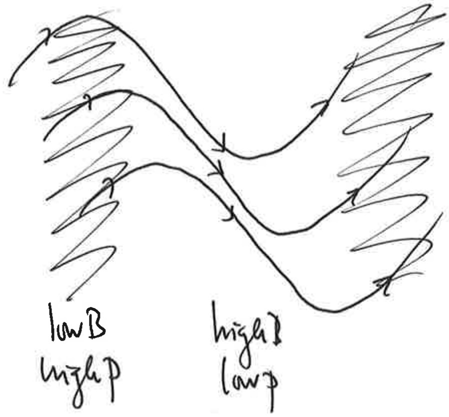
fast w/ $k_\parallel = 0$:



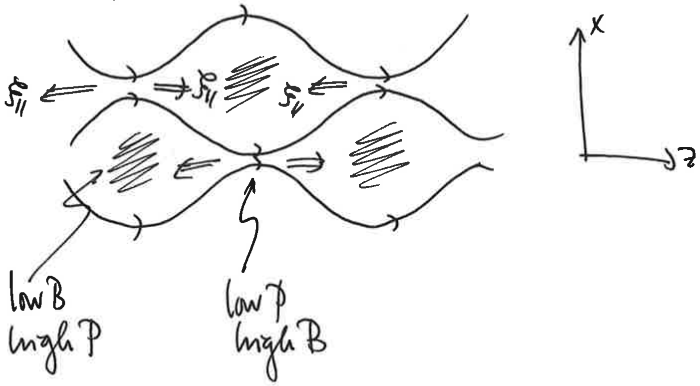
Fast:



Slow:



Slow with $k_\parallel/k_\perp \ll 1$:



Now, this last limit, $k_\parallel/k_\perp \ll 1$, is quite useful for studies of Alfvénic turbulence. What are all waves in this limit? Alfvén is the same: $\pm k_\parallel v_A$. Magnetosonic waves become

$$\omega^2 \approx \frac{k_\perp^2 (c_s^2 + v_A^2)}{2} \left[1 \pm \left(1 - \frac{2 k_\parallel^2 v_A^2 k_\perp^2 c_s^2}{k_\perp^4 (c_s^2 + v_A^2)^2} \right) \right]$$

$\oplus$ FAST $\rightarrow k_\perp^2 (c_s^2 + v_A^2)$

$\ominus$ SLOW $\rightarrow k_\parallel^2 v_A^2 \left(\dfrac{c_s^2}{c_s^2 + v_A^2} \right)$

· Let's look at the slow mode in this limit. Recall from our linear calculation that

$$\delta p = \delta\rho\, c_s^2 = \rho_0 c_s^2 \left(\frac{k^2 v_A^2}{\omega^2 - k^2 c_s^2}\right)\frac{\delta B_\parallel}{B_0} \Rightarrow \frac{\delta p}{p_0} + \left(\frac{k^2 v_A^2}{k^2 c_s^2 - \omega^2}\right)\frac{\delta B_\parallel}{B_0} = 0.$$

With $k_\perp \gg k_\parallel$ and $\omega^2 \approx k_\parallel^2 v_A^2 \left(\frac{c_s^2}{c_s^2 + v_A^2}\right)$, this becomes

$$\frac{\delta p}{p} + \frac{k_\perp^2 v_A^2 (\delta B_\parallel / B_0)}{k_\perp^2 c_s^2 - \frac{k_\parallel^2 v_A^2 c_s^2}{c_s^2 + v_A^2}} \approx \underbrace{\frac{\delta p}{p} + \frac{v_A^2}{c_s^2}\frac{\delta B_\parallel}{B_0} \approx 0}_{\text{pressure balance!}}$$

PART VI

Linear theory of MHD instabilities

Now let's do some MHD linear instabilities. The program is to set up some equilibria and then subject them to small-amplitude perturbations in the fluid and magnetic field. There are a few different ways of doing this and assessing whether the system is stable or unstable to these perturbations. There's something called the MHD energy principle, which will tell you whether a given set of perturbations about some equilibrium state will bring the system profitably to a lower energy state. There's something called Eulerian perturbation theory, where you subject the equilibrium state to small-amplitude perturbations, formulate those perturbations in the lab frame, and ask whether the perturbations oscillate, grow, or decay. And there's something called Lagrangian perturbation theory, which is same as Eulerian perturbation theory but is formulated in the frame of fluid. Each of these has its advantages depending on the equilibrium state, boundary conditions, and questions being asked. Eulerian perturbation theory is the most straightforward procedure, so we'll start there.

VI.1. A primer on instability

Before attacking the MHD equations, though, let's do something simpler to establish notation and learn the procedure. Consider the following ordinary differential equation:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\nu\frac{\mathrm{d}x}{\mathrm{d}t} + \Omega^2(x - x_0) = 0, \tag{VI.1}$$

where ν and $\Omega > 0$ are constants. You may recognize this as the equation for a damped simple harmonic oscillator of natural frequency Ω whose velocity along the x axis is damped at a rate $\nu > 0$. But let's not yet commit to any particular sign of ν. First, the equilibrium state. This is easy: the oscillator is at rest at $x = x_0$. We now displace the oscillator by a small amount ξ, so that $x(t) = x_0 + \xi(t)$. The equation governing this displacement is

$$\frac{\mathrm{d}^2 \xi}{\mathrm{d}t^2} + 2\nu\frac{\mathrm{d}\xi}{\mathrm{d}t} + \Omega^2\xi = 0. \tag{VI.2}$$

This equation admits solutions $\xi \sim \exp(-\mathrm{i}\omega t)$, where ω is a complex frequency that satisfies the *dispersion relation*

$$\omega^2 + 2\mathrm{i}\omega\nu - \Omega^2 = 0 \quad\Longrightarrow\quad \omega = -\mathrm{i}\nu \pm \sqrt{\Omega^2 - \nu^2}. \tag{VI.3}$$

How do we assess stability? If the imaginary part of ω is positive, then $-\mathrm{i}\omega$ has a positive real part, and the displacements will grow exponentially in time. If the imaginary part of ω is negative, then $-\mathrm{i}\omega$ has a negative real part, and this corresponds to exponential decay of the perturbation. If ω additionally has a real part, then this represents a growing or decaying oscillator. It's clear from a cursory glance at the dispersion relation (VI.3) that the perturbations oscillate and decay exponentially if $\Omega > \nu > 0$. If $\nu > \Omega > 0$, then the perturbations decay without oscillating. But if $\nu < 0$, then there is always an exponentially growing solution. Thus, $\nu > 0$ is the *stability criterion* for this system.

Now, suppose the equation of interest were instead

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\nu\frac{\mathrm{d}x}{\mathrm{d}t} + \Omega^2\sin(x - x_0) = 0. \tag{VI.4}$$

The equilibrium is still the same, but if we want simple harmonic oscillator solutions, we're only go to get them if the displacement is small, i.e., $|\xi| \ll x_0$. In that case, we can Taylor expand $\sin(x - x_0) \approx \xi - \xi^3/6 + \dots$. To leading order in ξ, we're back to where we started with (VI.2). This is *linear theory*: identify an equilibrium, perturb the system about that equilibrium, and drop all terms nonlinear in the perturbation amplitude.

Note that we are not solving an initial value problems. We are agnostic about the initial conditions and only ask whether some disturbance will ultimately grow or decay. In some situations (most notably, Landau damping), solving the initial value problem is absolutely essential to obtain the full solution and all the physics involved. But if you just want to calculate the wave-like response of a system to infinitesimally small perturbations and learn whether such a response grows or decays, you need only adopt solutions $\sim \exp(-\mathrm{i}\omega t)$, find the dispersion relation for ω vs $\boldsymbol{k}$, and examine the sign of its imaginary part. (The difference is related to a Laplace vs a Fourier transform in time.)

VI.2. Linearized MHD equations

Take (IV.21) and write

$$\rho = \rho_0(\boldsymbol{r}) + \delta\rho(t,\boldsymbol{r}), \quad \boldsymbol{u} = \delta\boldsymbol{u}(t,\boldsymbol{r}), \quad P = P_0(\boldsymbol{r}) + \delta P(t,\boldsymbol{r}), \quad \boldsymbol{B} = \boldsymbol{B}_0(\boldsymbol{r}) + \delta\boldsymbol{B}(t,\boldsymbol{r});$$

i.e., consider a stratified, stationary equilibrium state threaded by a magnetic field and subject it to perturbations. Never mind how the equilibrium is set up – it is what it is, and we'll perturb it. Neglecting all terms quadratic in δ, equations (IV.21) become

$$\frac{\partial\delta\rho}{\partial t} = -(\delta\boldsymbol{u}\cdot\boldsymbol{\nabla})\rho_0 - \rho_0(\boldsymbol{\nabla}\cdot\delta\boldsymbol{u}), \tag{VI.5}$$

$$\begin{aligned}\frac{\partial\delta\boldsymbol{u}}{\partial t} &= -\frac{1}{\rho_0}\boldsymbol{\nabla}\left(\delta P + \frac{\boldsymbol{B}_0\cdot\delta\boldsymbol{B}}{4\pi}\right) + \frac{\delta\rho}{\rho_0^2}\boldsymbol{\nabla}\left(P_0 + \frac{B_0^2}{8\pi}\right) \\ &\quad + \frac{(\boldsymbol{B}_0\cdot\boldsymbol{\nabla})\delta\boldsymbol{B}}{4\pi\rho_0} + \frac{(\delta\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}_0}{4\pi\rho_0} - \boldsymbol{\nabla}\delta\Phi,\end{aligned} \tag{VI.6}$$

$$\frac{\partial\delta\boldsymbol{B}}{\partial t} = -(\delta\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{B}_0 + (\boldsymbol{B}_0\cdot\boldsymbol{\nabla})\delta\boldsymbol{u} - \boldsymbol{B}_0(\boldsymbol{\nabla}\cdot\delta\boldsymbol{u}), \tag{VI.7}$$

$$\frac{\partial}{\partial t}\left(\frac{\delta P}{P_0} - \gamma\frac{\delta\rho}{\rho_0}\right) = -\delta\boldsymbol{u}\cdot\boldsymbol{\nabla}\ln\frac{P_0}{\rho_0^\gamma}. \tag{VI.8}$$

(A quick way of getting these is to think of δ as a differential operator that commutes with partial differentiation.) Pretty much every gradient of an equilibrium quantity here will give an instability! (Otherwise, you just get back simple linear waves on a homogeneous background.) So let's not analyze this all at once. But I write this system of equations here for two important reasons: (i) it makes clear that we can adopt solutions $\delta \sim \exp(-\mathrm{i}\omega t)$ for the perturbations, since the equations are linear in the fluctuation amplitudes; (ii) we can only adopt full plane-wave solutions $\delta \sim \exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$ if the fluctuations vary on length scales much smaller than that over which the background varies (the so-called WKB approximation). Otherwise, we have to worry about the exact structure of the background gradients and their boundary conditions.

So these are the themes of most linear stability analyses: a WKB approximation whereby plane-wave solutions are assumed on top of a background state that is slowly varying, and a focus only on whether fluctuations grow or decay rather than their specific spatio-temporal evolution from a set of initial conditions.

VI.3. Lagrangian versus Eulerian perturbations

There is one last thing worth discussing before proceeding with a linear stability analysis of the MHD equations. Just as there is an Eulerian time derivative and a Lagrangian time derivative, there is Eulerian perturbation theory and Lagrangian perturbation theory. The former, in which perturbations are denoted by a 'δ', measures the change in a quantity at a particular point in space. For example, if the equilibrium density at $\boldsymbol{r}$, $\rho(\boldsymbol{r})$, is changed at time t by some disturbance to become $\rho'(t,\boldsymbol{r})$, then we denote the Eulerian perturbation of the density by

$$\rho'(t,\boldsymbol{r}) - \rho(\boldsymbol{r}) \doteq \delta\rho \ll \rho(\boldsymbol{r}). \tag{VI.9}$$

Again, these perturbations are taken *at fixed position*. The latter – Lagrangian perturbation theory – concerns the evolution of small perturbations about a background state *within a particular fluid element* as it undergoes a displacement $\boldsymbol{\xi}$. For example, if a particularly fluid element is displaced from its equilibrium position $\boldsymbol{r}$ to position $\boldsymbol{r}+\boldsymbol{\xi}$, then the density of that fluid element changes by an amount

$$\rho'(t,\boldsymbol{r}+\boldsymbol{\xi}) - \rho(\boldsymbol{r}) \doteq \Delta\rho. \tag{VI.10}$$

This is a Lagrangian perturbation. To linear order, δ and Δ are related by

$$\Delta\rho \simeq \rho'(t,\boldsymbol{r}) + \boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}\rho(\boldsymbol{r}) - \rho(\boldsymbol{r}) = \delta\rho + \boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}\rho. \tag{VI.11}$$

There are many situations in which a Lagrangian approach is easier to use than an Eulerian approach; there are also some situations in which doing so is absolutely necessary (e.g., see §IIIe of Balbus (1988) and §Ic of Balbus & Soker (1989) for discussions of the perils of using Eulerian perturbations in the context of local thermal instability).

Question: It is possible to have zero Eulerian perturbation and yet have finite Lagrangian perturbation. What does this mean physically? Is there a physical change in the system?

The Lagrangian velocity perturbation $\Delta\boldsymbol{u}$ is given by

$$\Delta\boldsymbol{u} \doteq \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\right)\boldsymbol{\xi}, \tag{VI.12}$$

where $\boldsymbol{u}$ is the background velocity. It is the instantaneous time rate of rate of the displacement of a fluid element, taken relative to the unperturbed flow. Because $\Delta\boldsymbol{u} = \delta\boldsymbol{u} + \boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}$, we have

$$\delta\boldsymbol{u} = \frac{\partial\boldsymbol{\xi}}{\partial t} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\xi} - \boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}. \tag{VI.13}$$

Note the additional $\boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}$ term, representing a measurement of the background fluid gradients by the fluid displacement.

Exercise. Let $\boldsymbol{u} = R\Omega(R)\hat{\boldsymbol{\varphi}}$, as in a differentially rotating disk in cylindrical coordinates. Consider a displacement $\boldsymbol{\xi}$ with radial and azimuthal components ξ_R and ξ_φ, each depending upon R and φ. Show that

$$\frac{\mathrm{D}\xi_R}{\mathrm{D}t} = \delta u_R \quad \text{and} \quad \frac{\mathrm{D}\xi_\varphi}{\mathrm{D}t} = \delta u_\varphi + \xi_R\frac{\mathrm{d}\Omega}{\mathrm{d}\ln R}. \tag{VI.14}$$

The second term in the latter equation accounts for the stretching of radial displacements into the azimuthal direction by the differential rotation.

You can think of δ and $\mathit{\Delta}$ as difference operators, since we're only working to linear order in the perturbation amplitude: e.g.,

$$\delta\left(\frac{1}{\rho}\right) = \frac{1}{\rho+\delta\rho} - \frac{1}{\rho} \simeq -\frac{\delta\rho}{\rho^2}.$$

But you must be very careful when mixing Eulerian and Lagrangian points of view. Prove the following commutation relations:

$$\begin{aligned}
&(i) && \left[\delta,\, \frac{\partial}{\partial t}\right] = 0;\\
&(ii) && \left[\delta,\, \frac{\partial}{\partial x_i}\right] = 0;\\
&(iii) && \left[\mathit{\Delta},\, \frac{\partial}{\partial t}\right] = -\frac{\partial \xi_j}{\partial t}\frac{\partial}{\partial \xi_j};\\
&(iv) && \left[\mathit{\Delta},\, \frac{\partial}{\partial x_i}\right] = -\frac{\partial \xi_j}{\partial x_i}\frac{\partial}{\partial \xi_j};\\
&(v) && \left[\mathit{\Delta},\, \frac{\mathrm{D}}{\mathrm{D}t}\right] = 0;\\
&(vi) && \left[\mathit{\Delta},\, \frac{\mathrm{D}}{\mathrm{D}x_i}\right] = -\xi_j\frac{\partial}{\partial x_j}\frac{\mathrm{D}}{\mathrm{D}t};\\
&(vii) && \left[\frac{\partial}{\partial x_i},\, \frac{\mathrm{D}}{\mathrm{D}t}\right] = \frac{\partial u_j}{\partial x_i}\frac{\partial}{\partial x_j}.
\end{aligned}$$

You can use these to show that the linearized continuity equation, induction equation, and internal energy equation are

$$\frac{\mathit{\Delta}\rho}{\rho} = -\boldsymbol{\nabla\cdot\xi}, \tag{VI.15}$$

$$\mathit{\Delta}\boldsymbol{B} = \boldsymbol{B\cdot\nabla\xi} - \boldsymbol{B\nabla\cdot\xi}, \tag{VI.16}$$

$$\frac{\mathit{\Delta}T}{T} = -(\gamma-1)\boldsymbol{\nabla\cdot\xi}, \tag{VI.17}$$

respectively. These forms are particularly useful for linear analyses.

Now to calculate something... I'll start with two simple instabilities, the first of which (Jeans instability) will be analyzed using Eulerian perturbation theory, and the second of which (Kelvin–Helmholtz instability) will be analyzed using Lagrangian perturbation theory. Hopefully you'll see why one approach is sometimes easier than the other.

VI.4. Self-gravity: Jeans instability

One of the simplest hydrodynamical waves is a small-amplitude sound wave propagating on an infinite, homogeneous background. Take (IV.21), set $\boldsymbol{B}_0 = 0$, and assume ρ_0 and P_0 to be constant. The resulting linearized equations are

$$\frac{\partial}{\partial t}\frac{\delta\rho}{\rho_0} = -\boldsymbol{\nabla\cdot}\delta\boldsymbol{u}, \quad \frac{\partial\delta\boldsymbol{u}}{\partial t} = -\frac{1}{\rho_0}\boldsymbol{\nabla}\delta P - \boldsymbol{\nabla}\delta\mathit{\Phi}, \quad \frac{\partial}{\partial t}\left(\frac{\delta P}{P_0} - \gamma\frac{\delta\rho}{\rho_0}\right) = 0. \tag{VI.18a}$$

I've retained the perturbed gravitational potential $\delta\mathit{\Phi}$ in the second equation, because we're going to assume that the fluid is self-gravitating with a potential that obey's

Poisson's equation:[8]

$$\nabla^2 \delta\Phi = 4\pi G \delta\rho. \tag{VI.18b}$$

These equations are linear in δ, and so we may adopt plane-wave solutions, $\delta \sim \exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$. Substituting this form into (VI.18) gives

$$-\mathrm{i}\omega\frac{\delta\rho}{\rho_0} = -\mathrm{i}\boldsymbol{k}\cdot\delta\boldsymbol{u}, \quad -\mathrm{i}\omega\delta\boldsymbol{u} = -\mathrm{i}\boldsymbol{k}\frac{\delta P}{\rho_0} - \mathrm{i}\boldsymbol{k}\delta\Phi, \quad -\mathrm{i}\omega\left(\frac{\delta P}{P_0} - \gamma\frac{\delta\rho}{\rho_0}\right) = 0, \tag{VI.19a}$$

$$-k^2\delta\Phi = 4\pi G\delta\rho. \tag{VI.19b}$$

Taking $\boldsymbol{k}\cdot$ the second equation and using the other three yields the dispersion relation

$$\omega\bigl(\omega^2 - k^2a^2 + 4\pi G\rho_0\bigr) = 0, \tag{VI.20}$$

where $a^2 \doteq \gamma P_0/\rho_0$. The $\omega = 0$ root comes from the perturbed entropy equation, and corresponds to a isentropic relabelling of the fluid elements; its name is the 'entropy mode'. The other two roots correspond to forward- and backward-propagating sound waves under the influence of their own self-gravity:

$$\omega = \pm ka\sqrt{1 - \frac{4\pi G\rho_0}{k^2a^2}} \tag{VI.21}$$

Self-gravity reduces the speed of the wave for wavenumbers satisfying $ka > (4\pi G\rho_0)^{1/2}$, for which the (expansive) pressure force is greater than the (attractive) gravitational force. At $ka = (4\pi G\rho_0)^{1/2}$, these two forces balance exactly, and the mode is neutrally stable. But for $ka < (4\pi G\rho_0)^{1/2}$, the wavelength is long enough to include a sufficiently large amount of mass in the perturbation to overwhelm the pressure force. Instability ensues, and the mode grows without propagating. This is the *Jeans instability*, named after Sir James Jeans (although Sir Isaac Newton understood the concept over 200 years before the calculation).

The critical wavelength

$$\lambda_{\rm J} = a\sqrt{\frac{\pi}{G\rho_0}} \tag{VI.22}$$

is referred to as the *Jeans length*. For an isothermal ($\gamma = 1$) molecular cloud of temperature 10 K, number density 200 cm^{-3}, and mean mass per particle $2.33m_{\rm p}$, the Jeans length is $\simeq$1.5 pc. The corresponding *Jeans mass* enclosed within a spherical volume with $\lambda_{\rm J}$ as its diameter is

$$M_{\rm J} = \frac{\pi}{6}\rho_0\lambda_{\rm J}^3 = 20.3\left(\frac{T_0}{10~\mathrm{K}}\right)^{3/2}\left(\frac{n}{200~\mathrm{cm}^{-3}}\right)^{-1/2}~\mathrm{M}_\odot. \tag{VI.23}$$

Giant molecular clouds with these parameters have typical masses $\gtrsim$$10^4$ M$_\odot$, indicating that more must be going on than just thermal pressure support against self-gravity (see: magnetic fields and turbulence). Note that $M_{\rm J} = M_\odot$ at a density $n \simeq 8.2\times10^4$ cm^{-3}.

VI.5. Shear: Kelvin–Helmholtz instability

Consider two uniform fluids separated by a discontinuous interface at $z = 0$, as in the figure below:

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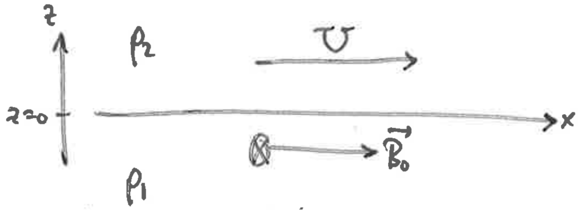
[8]Wouldn't an infinite, homogeneous, self-gravitating fluid collapse under its own weight? Indeed it would. Ignoring this inconvenience is known as the *Jeans swindle*. Following Binney & Tremaine (1987): 'it is a swindle because in general there is no formal justification for discarding the unperturbed gravitational field'.



The fluid above the interface ($z > 0$) has density ρ_2 and equilibrium velocity $\boldsymbol{u}_0 = U\hat{\boldsymbol{x}}$. The fluid below the interface ($z < 0$) has density ρ_1 and is stationary. (We can always transform to a frame in which this fluid is stationary, so why not take advantage of that?) There is a uniform magnetic field $\boldsymbol{B}_0 = B_{0x}\hat{\boldsymbol{x}} + B_{0y}\hat{\boldsymbol{y}}$ oriented parallel to the interface that permeates all of the fluid, which we take to be perfectly conducting. For simplicity, take the fluid to be incompressible, *viz.* $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u} = 0$.

We seek the dispersion relation governing small-amplitude perturbations. It turns out that this problem is most easily analyzed using Lagrangian perturbations rather than Eulerian perturbations – the reason being that the interface and the interfacial pressure between the two fluids must remain continuous as the fluid is perturbed, and it's easier to measure this interface in the frame of the fluid element than in the lab frame.

Take the momentum equation in each of the fluids, above and below, and apply the difference operator $\varDelta \doteq \delta + \boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}$ while recalling that $[\varDelta, \mathrm{D}/\mathrm{D}t] = 0$ and $\varDelta\boldsymbol{u} = \mathrm{D}\boldsymbol{\xi}/\mathrm{D}t$:

$$\varDelta\left[\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{B}}{4\pi}\right]$$
$$\Longrightarrow \quad \rho\frac{\mathrm{D}^2\boldsymbol{\xi}}{\mathrm{D}t^2} = -\boldsymbol{\nabla}\delta\left(P + \frac{B^2}{8\pi}\right) + \frac{\boldsymbol{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\delta\boldsymbol{B}}{4\pi}, \tag{VI.24}$$

the form of the right-hand side following because $\boldsymbol{\nabla}B_0 = \boldsymbol{\nabla}P_0 = 0$. Use the linearized induction equation (VI.16) with $\boldsymbol{\nabla}B_0 = 0$, which reads $\delta\boldsymbol{B} = (\boldsymbol{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{\xi}$, and rearrange to obtain

$$\left[\frac{\mathrm{D}^2}{\mathrm{D}t^2} - \frac{(\boldsymbol{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla})^2}{4\pi\rho}\right]\boldsymbol{\xi} = -\frac{1}{\rho}\boldsymbol{\nabla}\delta\left(P + \frac{B^2}{8\pi}\right) \doteq -\frac{1}{\rho}\boldsymbol{\nabla}\delta\varPi. \tag{VI.25}$$

Note that taking the divergence of this equation and using $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\xi} = 0$ (incompressibility) implies that the total perturbed pressure $\varPi$ satisfies

$$\nabla^2\delta\varPi = 0. \tag{VI.26}$$

With the x and y directions being infinite in extent and the background state possessing no structure in those directions, we may write $\delta\varPi = \delta\varPi(z)\exp(\mathrm{i}k_x x + \mathrm{i}k_y y)$ to find

$$\left(-k^2 + \frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)\delta\varPi(z) = 0 \quad \Longrightarrow \quad \delta\varPi(z) \propto \exp(-|kz|), \quad k \equiv \sqrt{k_x^2 + k_y^2}. \tag{VI.27}$$

The absolute value in the argument of the exponential indicates that the perturbation must die off as $z \to \pm\infty$. We may now adopt solutions of the form $\exp(-\mathrm{i}\omega t)$ and evaluate the z component of (VI.25) above and below the interface:

$$\left[(-\mathrm{i}\omega + \mathrm{i}k_x U)^2 + \frac{(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{B}_0)^2}{4\pi\rho_2}\right]\xi_{z2} = +\frac{1}{\rho_2}|k|\delta\varPi_2, \tag{VI.28a}$$
$$\left[(-\mathrm{i}\omega \qquad\quad)^2 + \frac{(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{B}_0)^2}{4\pi\rho_1}\right]\xi_{z1} = -\frac{1}{\rho_1}|k|\delta\varPi_1, \tag{VI.28b}$$

respectively. At the interface, $\xi_{z1} = \xi_{z2}$ and $\varDelta\varPi_1 = \varDelta\varPi_2$, i.e., the two fluids must move together at the interface and their pressures must hold continuous as they are perturbed.

Because $\nabla B_0 = \nabla P_0 = 0$, the latter implies $\delta\Pi_1 = \delta\Pi_2$. Using this information to match (VI.28a) and (VI.28b) leads to

$$(\omega - k_x U)^2 \rho_2 + \omega^2 \rho_1 = \frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2}{2\pi} \tag{VI.29}$$

$$\Longrightarrow \quad \boxed{\omega = \frac{k_x U}{2}\frac{\overline{\rho}}{\rho_1}\left\{1 \pm \mathrm{i}\sqrt{\frac{\rho_1}{\rho_2}\left[1 - \frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2}{\pi\overline{\rho}k_x^2U^2}\right]}\right\}} \tag{VI.30}$$

where $\overline{\rho} \doteq 2\rho_1\rho_2/(\rho_1+\rho_2)$ is the reduced mass density. For

$$\frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2}{4\pi\overline{\rho}} < \left(\frac{k_x U}{2}\right)^2, \tag{VI.31}$$

the discriminant is positive and there is a growing (and propagating) mode whose growth rate is proportional to the wavenumber and the velocity shear across the interface. Note that, for $\rho_1 = \rho_2 = \rho$, we have $\overline{\rho} = \rho$, and then (VI.30) becomes

$$\omega = \frac{k_x U}{2}\left[1 \pm \mathrm{i}\sqrt{1 - \frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2}{\pi\rho k_x^2 U^2}}\right];$$

for $U = 0$, this returns a stably propagating shear Alfvén wave, $\omega = \mp(\boldsymbol{k}\cdot\boldsymbol{v}_\mathrm{A})$. This indicates that it is the tension in the magnetic-field lines that is responsible for stabilizing the instability. That being said, if the magnetic field is oriented such that $B_{0x} = 0$, then (VI.31) can always be satisfied for small enough $|k_y/k_x|$, no matter how strong is B_{0y}.

The physics is as follows. An upwardly displaced distortion of the interface into region 2 causes a constriction of the velocity there, and the fluid must move faster to conserve its mass. But when it moves faster, the pressure must drop (Bernoulli!). The opposite happens below the interface. Now there is a pressure gradient pushing upwards, reinforcing the displacement, and the process runs away (unless the magnetic tension can stabilize the displacements and propagate them away as Alfvén waves). That's why pressure perturbations were vital in (VI.25).

Question: Does this instability occur in a simple linear shear flow, e.g., $\boldsymbol{u}_0 = Sz\hat{\boldsymbol{x}}$? No! The proof goes as follows. Drop the magnetic field for simplicity. With $\boldsymbol{u}_0 = u_0(z)\hat{\boldsymbol{x}}$, one can show using $\nabla\cdot\boldsymbol{\xi} = 0$ and the momentum equation that

$$\frac{\mathrm{d}^2\xi_z}{\mathrm{d}z^2} - k_x^2\xi_z = \frac{k u_0''}{\omega - k_x u_0}\xi_z.$$

Multiply this by ξ_z^* (the '$*$' denotes the complex conjugate) and integrate between the upper and lower boundaries $z = \pm L$ to obtain

$$\int_{-L}^{L}\mathrm{d}z\,\left(\xi_z^*\xi_z'' - k_x^2|\xi_z|^2\right) = \int_{-L}^{L}\mathrm{d}z\,\frac{k_x u_0''}{\omega - k_x u_0}|\xi_z|^2.$$

The first term on the left-hand side may be simplified using integration by parts and assuming either periodicity or that ξ_z or ξ_z' vanish at the boundaries. Then

$$\int_{-L}^{L}\mathrm{d}z\,\left(-|\xi_z'|^2 - k_x^2|\xi_z|^2\right) = \int_{-L}^{L}\mathrm{d}z\,\frac{k_x u_0''}{\omega - k_x u_0}|\xi_z|^2,$$

If the system is unstable, then ω must have an imaginary part, ω_I. Writing $\omega = \omega_\mathrm{R} + \mathrm{i}\omega_\mathrm{I}$, the imaginary part of the above equation is simply

$$\omega_{\rm I} \int_{-L}^{L} {\rm d}z \, \frac{k_x u_0''}{|\omega - k_x u_0|^2} |\xi_z|^2 = 0.$$

This states that u_0'' must be positive over part of the integration range, and negative over the remainder, i.e., u_0'' must pass through zero. Thus, instability requires an *inflection point* (Rayleigh 1880). (Note that the converse is not true: a velocity profile *with* an inflection point is not necessarily unstable.)

VI.6. Buoyancy: Rayleigh–Taylor instability

Using Lagrangian perturbation theory, it is easy to generalize the calculation in the previous section (§VI.5) to include gravity. Again, let the fluid above the interface ($z > 0$) have uniform density ρ_2, and the fluid below the interface ($z < 0$) have uniform density ρ_1. Include the same uniform background magnetic field as before, $\boldsymbol{B}_0 = B_{0x}\hat{\boldsymbol{x}} + B_{0y}\hat{\boldsymbol{y}}$. But now place these fluids in a constant gravitational field $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$, with the gas pressure either side of the interface satisfying hydrostatic equilibrium in the vertical direction:

$$g = -\frac{1}{\rho_1}\frac{{\rm d}P_1}{{\rm d}z} = -\frac{1}{\rho_2}\frac{{\rm d}P_2}{{\rm d}z}.$$

The entire calculation goes through as before, but with the following additions and modifications. First, we must include the perturbed gravitational force in the momentum equation (VI.24), *viz.* $\varDelta(\rho\boldsymbol{g}) = -(\varDelta\rho)g\hat{\boldsymbol{z}}$. Secondly, because of the background pressure gradient in each of the fluids, we no longer have that $\varDelta(\boldsymbol{\nabla}P) = \boldsymbol{\nabla}\delta P$, but rather that $\varDelta(\boldsymbol{\nabla}P) = \boldsymbol{\nabla}\delta P + \boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}(\boldsymbol{\nabla}P)$. Using hydrostatic equilibrium, this may equivalently be written as $\varDelta(\boldsymbol{\nabla}P) = \boldsymbol{\nabla}\delta P - \boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}(\rho g\hat{\boldsymbol{z}})$. Making these two changes in (VI.24) leads to

$$\rho\frac{{\rm D}^2\boldsymbol{\xi}}{{\rm D}t^2} = -\boldsymbol{\nabla}\delta\left(P + \frac{B^2}{8\pi}\right) + \frac{\boldsymbol{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\delta\boldsymbol{B}}{4\pi} - (\varDelta\rho)g\hat{\boldsymbol{z}} + \boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}(\rho g). \tag{VI.32}$$

Despite this extra work, however, those two additional terms cancel one another if the fluid is incompressible, since then $\varDelta\rho - \boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}\rho \doteq \delta\rho = 0$. As a result, the *only* difference between this calculation and the Kelvin–Helmholtz calculation in §VI.5 is that the imposition of pressure continuity at the perturbed interface does not imply that $\delta\varPi_1 = \delta\varPi_2$, but rather

$$\varDelta\varPi_1 = \varDelta\varPi_2 \implies \delta\varPi_1 - \xi_{z1}\rho_1 g = \delta\varPi_2 - \xi_{z2}\rho_2 g \implies \delta\varPi_2 - \delta\varPi_1 = \xi_z(\rho_2 - \rho_1)g.$$

We may then use this in (VI.28) to jump straight to the dispersion relation (cf. (VI.29))

$$(\omega - k_x U)^2\rho_2 + \omega^2\rho_1 = \frac{(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{B}_0)^2}{2\pi} + |k|g(\rho_1 - \rho_2), \tag{VI.33}$$

whose solutions are (cf. (VI.30))

$$\omega = \frac{k_x U}{2}\frac{\overline{\rho}}{\rho_1}\left\{1 \pm {\rm i}\sqrt{\frac{\rho_1}{\rho_2}\left[1 + \frac{2|k|g}{k_x^2U^2}\frac{\rho_2 - \rho_1}{\overline{\rho}} - \frac{(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{B}_0)^2}{\pi\overline{\rho}k_x^2U^2}\right]}\right\}. \tag{VI.34}$$

By design, this has both Kelvin–Helmholtz and Rayleigh–Taylor in it; let's set $U = 0$ to eliminate the former, in which case

$$\boxed{\omega = \pm{\rm i}\sqrt{|k|g\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} - \frac{(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{B}_0)^2}{2\pi(\rho_1 + \rho_2)}}} \tag{VI.35}$$

This equation states that linear instability requires $\rho_2 > \rho_1$ (heavy on top, light on the bottom), with the difference between the densities being large enough for the destabilizing pressure gradient (Bernoulli!) to overcome the stabilizing magnetic tension. Note that, if $\boldsymbol{B}_0$ is not oriented along the interface, no amount of magnetic field can stabilize the system.

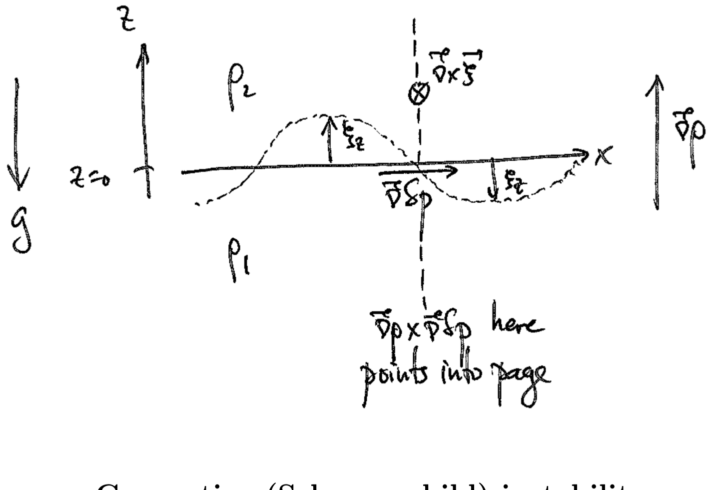
Usually a physical explanation of the Rayleigh–Taylor instability isn't provided in derivations of its linear theory; indeed, the fact that heavy stuff falls down if given the opportunity to do so is fairly obvious. But it's worth thinking about the physics a little harder, and connecting this intuition back to the math. Let's return to (VI.32) with $U = 0$ and $\boldsymbol{B} = 0$ in order to isolate the Rayleigh–Taylor physics, and take curl of this equation:

$$\boldsymbol{\nabla} \times \left(\rho \frac{\mathrm{D}^2 \boldsymbol{\xi}}{\mathrm{D}t^2} \right) = 0.$$

Chain-ruling the curl, dividing through by ρ, and using (VI.32) to eliminate $\mathrm{D}^2\boldsymbol{\xi}/\mathrm{D}t^2$ yields

$$\frac{\mathrm{D}^2}{\mathrm{D}t^2} \boldsymbol{\nabla} \times \boldsymbol{\xi} = \frac{1}{\rho^2} \boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}\delta P.$$

If you read §II.4, you'll recognize this as an equation for the vorticity, which is being baroclinically forced by the misalignment between the background density gradient and the *perturbed* pressure gradient. Because of the rippled interface, this perturbed pressure gradient has a component in the x direction, which points from regions where $\xi_z > 0$ towards regions where $\xi_z < 0$. With $\boldsymbol{\nabla}\rho$ pointing upwards (in the unstable situation), the baroclinic forcing is pointing in just the right direction to accentuate the vorticity in the original perturbation. See the figure below.



VI.7. Buoyancy: Convective (Schwarzschild) instability

Next up: stratification. Henceforth, ignore self-gravity. Suppose our plasma is immersed in a constant, externally imposed gravitational field $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$ and that its thermal-pressure gradient balances the gravitational acceleration to produce a stationary, equilibrium state. Ignoring for the moment magnetic fields, this hydrostatic equilibrium is described by the equation

$$\frac{1}{\rho_0} \frac{\mathrm{d}P_0}{\mathrm{d}z} = g = \text{const}, \tag{VI.36}$$

where $\rho_0 = \rho_0(z)$. The hydrodynamic equations linearized about this equilibrium are

$$\frac{\partial}{\partial t}\frac{\delta\rho}{\rho_0} + \boldsymbol{\nabla}\boldsymbol{\cdot}\delta\boldsymbol{u} + \delta u_z \frac{\mathrm{d}\ln\rho_0}{\mathrm{d}z} = 0, \tag{VI.37}$$

$$\frac{\partial\delta\boldsymbol{u}}{\partial t} = -\frac{1}{\rho_0}\boldsymbol{\nabla}\delta P - \frac{\delta\rho}{\rho_0} g\hat{\boldsymbol{z}}, \tag{VI.38}$$

$$\frac{\partial}{\partial t}\left(\frac{\delta P}{P_0} - \gamma\frac{\delta\rho}{\rho_0}\right) + \delta u_z \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{P_0}{\rho_0^\gamma} = 0. \tag{VI.39}$$

Solutions to this set of equations are $\propto \exp(-\mathrm{i}\omega t)$:

$$-\mathrm{i}\omega\frac{\delta\rho}{\rho_0} + \boldsymbol{\nabla}\boldsymbol{\cdot}\delta\boldsymbol{u} + \delta u_z \frac{\mathrm{d}\ln\rho_0}{\mathrm{d}z} = 0, \tag{VI.40}$$

$$-\mathrm{i}\omega\delta\boldsymbol{u} = -\frac{1}{\rho_0}\boldsymbol{\nabla}\delta P - \frac{\delta\rho}{\rho_0} g\hat{\boldsymbol{z}}, \tag{VI.41}$$

$$-\mathrm{i}\omega\left(\frac{\delta P}{P_0} - \gamma\frac{\delta\rho}{\rho_0}\right) + \delta u_z \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{P_0}{\rho_0^\gamma} = 0. \tag{VI.42}$$

Continued on hand-written notes…

In general, we cannot Fourier transform these eqns. in z, because the coefficients in front of the perturbed quantities are z-dependent. But we can do so in the horizontal (say, x) direction:

$$-i\omega \frac{\delta\rho}{\rho_0} + ik_x \delta u_x + \frac{d\,\delta u_z}{dz} + \delta u_z \frac{d\ln\rho_0}{dz} = 0,$$

$$-i\omega \delta u_x = -ik_x \frac{\delta p}{\rho_0},$$

$$-i\omega \delta u_z = -\frac{1}{\rho_0}\frac{d\,\delta p}{dz} - \frac{\delta\rho}{\rho_0} g,$$

$$-i\omega \left(\frac{\delta p}{p_0} - \gamma \frac{\delta\rho}{\rho_0} \right) + \delta u_z \frac{d\ln p_0\rho_0^{-\gamma}}{dz} = 0,$$

where now the fluctuations are z-dependent Fourier amplitudes. Denoting $\delta\vec{u} = -i\omega\vec{\xi}$, and dropping the equilibrium "0" subscripts for notational ease, we have

(A) $$\frac{\delta\rho}{\rho} + ik_x\xi_x + \xi_z' + \xi_z \frac{d\ln\rho}{dz} = 0,$$

(B) $$-\omega^2 \xi_x = -ik_x \frac{\delta p}{\rho},$$

(C) $$-\omega^2 \xi_z = -\frac{1}{\rho}\frac{d}{dz}\delta p - \frac{\delta\rho}{\rho} g,$$

(D) $$\frac{\delta p}{p} = \gamma \frac{\delta\rho}{\rho} - \xi_z \frac{d\ln p\rho^{-\gamma}}{dz}.$$

Ⓑ and Ⓓ $\Rightarrow$ $$-\omega^2 \xi_x = -ik_x \frac{P}{\rho}\left[\gamma \frac{\delta\rho}{\rho} - \xi_z \frac{d\ln P\rho^{-\gamma}}{dz}\right]$$

and Ⓐ $\Rightarrow$ $$-\omega^2 \xi_x = +ik_x \frac{P}{\rho}\gamma\left[+ik_x\xi_x + \xi_z' + \xi_z \frac{d\ln\rho}{dz}\right] + ik_x\frac{P}{\rho}\xi_z \frac{d\ln P\rho^{-\gamma}}{dz}$$

$$\Rightarrow (-\omega^2 + k_x^2 a^2)\xi_x = ik_x a^2 \xi_z' + \frac{ik_x a^2}{\gamma}\frac{d\ln P}{dz}\xi_z,$$

where $a^2 \equiv \gamma P/\rho$. Note: $g = -\frac{a^2}{\gamma}\frac{d\ln P}{dz}$, so this is

Ⓑ $$\boxed{(-\omega^2 + k_x^2 a^2)\xi_x = ik_x a^2 \xi_z' - ik_x \xi_z g}$$

Ⓐ $\Rightarrow$ $$\frac{\delta\rho}{\rho} = -\xi_z' - \xi_z\frac{d\ln\rho}{dz} - \frac{ik_x\left[ik_x a^2\xi_z' - ik_x\xi_z g\right]}{-\omega^2 + k_x^2 a^2}$$

$$\Rightarrow \boxed{\frac{\delta\rho}{\rho} = \frac{\omega^2\xi_z' + \left[(\omega^2 - k_x^2a^2)\frac{d\ln\rho}{dz} - k_x^2 g\right]\xi_z}{k_x^2a^2 - \omega^2}} \quad ⓐ$$

$$\Rightarrow \frac{\delta P}{P} = \frac{1}{k_x^2a^2 - \omega^2}\left[\gamma\omega^2\xi_z' + \gamma\left((\omega^2 - k_x^2a^2)\frac{d\ln\rho}{dz} - k_x^2 g\right)\xi_z + (\omega^2 - k_x^2 a^2)\frac{d\ln P\rho^{-\gamma}}{dz}\xi_z\right]$$

$$= \frac{1}{k_x^2a^2 - \omega^2}\left[\gamma\omega^2\xi_z' + \omega^2\frac{d\ln P}{dz}\xi_z\right]$$

$$= \frac{\omega^2}{k_x^2a^2 - \omega^2}\left[\gamma\xi_z' + \frac{d\ln P}{dz}\xi_z\right] \quad ✪$$

into ⓒ:

$$+\omega^2\xi_z = +g\left[\frac{\omega^2\xi_z' + (\omega^2 - k_x^2a^2)\frac{d\ln p}{dz}\xi_z - k_x^2 g\xi_z}{k_x^2a^2-\omega^2}\right]$$

$$+\frac{1}{\rho}\frac{d}{dz}\left[\frac{\omega^2 p}{k_x^2a^2-\omega^2}\left(\xi_z'\gamma + \xi_z\frac{d\ln p}{dz}\right)\right]$$

$$= \frac{\frac{\omega^2}{\rho}\frac{dp}{dz}\left(\xi_z'\gamma + \xi_z\frac{d\ln p}{dz}\right)}{k_x^2a^2-\omega^2} + \frac{p}{\rho}\frac{\omega^2}{k_x^2a^2-\omega^2}\left(\gamma\xi_z'' + \xi_z'\frac{d\ln p}{dz} + \xi_z\frac{d^2\ln p}{dz^2}\right)$$

$$-\frac{\omega^2 p}{\rho}\frac{k_x^2}{(k_x^2a^2-\omega^2)^2}\frac{da^2}{dz}\left(\xi_z'\gamma + \xi_z\frac{d\ln p}{dz}\right)$$

$$\Rightarrow \omega^2\xi_z = \frac{g}{k_x^2a^2-\omega^2}\left[\omega^2\xi_z' - k_x^2 g\xi_z + (\omega^2 - k_x^2a^2)\frac{d\ln p}{dz}\xi_z\right]$$

$$+\frac{\omega^2}{k_x^2a^2-\omega^2}(-g)\left[\xi_z'\gamma + \xi_z\frac{d\ln p}{dz}\right]$$

$$+\left(\frac{a^2}{\gamma}\right)\frac{\omega^2}{k_x^2a^2-\omega^2}\left[\xi_z''\gamma + \xi_z'\frac{d\ln p}{dz} + \xi_z\frac{g\gamma}{a^2}\frac{d\ln T}{dz}\right]$$

$$-\frac{\omega^2}{\gamma}\frac{a^2k_x^2a^2}{(k_x^2a^2-\omega^2)^2}\frac{d\ln T}{dz}\left[\xi_z'\gamma + \xi_z\frac{d\ln p}{dz}\right]$$

Multiply by $\frac{k_x^2a^2-\omega^2}{\omega^2a^2}$ and group:

ξ_z'': 1.

ξ_z': $\cancel{\frac{g}{a^2}} - \frac{g\gamma}{a^2} + \frac{1}{\cancel{\gamma}}\cancel{\frac{d\ln p}{dz}} - \frac{k_x^2 a^2}{\cancel{\gamma}}\frac{1}{(k_x^2 a^2-\omega^2)}\frac{d\ln T}{dz}\cancel{\gamma}$

$$= \frac{d\ln p}{dz} - \left(\frac{k_x^2 a^2}{k_x^2 a^2-\omega^2}\right)\frac{d\ln T}{dz} = \frac{d\ln p/dz}{k_x^2 a^2-\omega^2}\left[k_x^2 a^2-\omega^2 - k_x^2 a^2\frac{d\ln T}{d\ln p}\right]$$

$$= \frac{d\ln p/dz}{k_x^2 a^2-\omega^2}\left[-\omega^2 + k_x^2 a^2\frac{d\ln \rho}{d\ln p}\right] = \frac{\omega^2\frac{d\ln p}{dz} - k_x^2 a^2\frac{d\ln \rho}{dz}}{\omega^2-k_x^2 a^2}$$

ξ_z: $-\frac{(k_x^2 a^2-\omega^2)}{a^2} - \frac{k_x^2 g^2}{\omega^2 a^2} - g\frac{d\ln p}{dz}\frac{k_x^2 a^2 \cancel{-\omega^2}}{\omega^2 a^2}$

$$\cancel{-\frac{g}{a^2}\frac{d\ln p}{dz}} + \cancel{\frac{a^2}{\gamma}\frac{1}{a^2}\frac{g\gamma}{a^2}\frac{d\ln T}{dz}} - \frac{k_x^2 a^2}{\gamma(k_x^2 a^2-\omega^2)}\frac{d\ln T}{dz}\frac{d\ln p}{dz}$$

$$= \frac{-1}{k_x^2 a^2-\omega^2}\left[\frac{k_x^2 a^2}{\gamma}\frac{d\ln T}{dz}\frac{d\ln p}{dz} + \frac{g k_x^2}{\omega^2}\frac{d\ln p}{dz}(k_x^2 a^2-\omega^2) + \frac{k_x^2 g^2}{a^2\omega^2}(k_x^2 a^2-\omega^2) + \frac{(k_x^2 a^2-\omega^2)^2}{a^2}\right]$$

$$= \frac{1}{\omega^2-k_x^2 a^2}\left[\frac{\omega^4}{a^2} - 2\omega^2 k_x^2 + k_x^4 a^2 - \frac{k_x^2 g^2}{a^2} - \cancel{g k_x^2\frac{d\ln p}{dz}} + \frac{k_x^2 a^2}{\gamma}\frac{d\ln T}{dz}\frac{d\ln p}{dz} + \frac{g k_x^4 a^2}{\omega^2}\frac{d\ln p}{dz} + \frac{k_x^4 \cancel{a^2} g^2}{\cancel{a^2}\omega^2}\right]$$

$$-k_x^2 g\left(\frac{d\ln p}{dz} - \cancel{\frac{d\ln p}{dz}}\right)$$

So,

$$\boxed{\xi_z'' + \xi_z' \left[\frac{\omega^2 \frac{d\ln p}{dz} - k_x^2 a^2 \frac{d\ln \rho}{dz}}{\omega^2 - k_x^2 a^2} \right] + \xi_z \frac{1}{\omega^2 - k_x^2 a^2}\left[\frac{\omega^4}{a^2} - 2\omega^2 k_x^2 + k_x^4 a^2 - k_x^2 g \frac{d\ln p}{dz}\left(1-\frac{1}{\gamma}\right) + g\frac{k_x^4 a^2}{\omega^2}\frac{d\ln \rho}{dz} + \frac{k_x^4 g^2}{\omega^2}\right] = 0.}$$

This is UGLY!!! And we can't solve it analytically anyhow. It's just a stratified fluid — why is it so complicated?! The reason is twofold: (1) this equation mixes up buoyancy and sound waves — distinct physical effects; and (2) the sound and buoyancy frequencies are functions of height. Let's fix this by adopting an <u>ordering</u>: let

$$\frac{d\xi_z}{dz} \sim i k_z \xi_z = i k_z H \left(\frac{\xi_z}{H}\right) \gg \frac{\xi_z}{H},$$

where $H \equiv \left|\frac{dz}{d\ln p}\right| \sim \left|\frac{dz}{d\ln \rho}\right|$. In other words, we assume that ξ_z varies on a scale $\ll$ the scale of the background. This is a WKB approach. So...

Let $\epsilon \equiv \frac{1}{k_z H} \ll 1$. Also, $k_x \sim k_z$. Now, we must make a decision about the size of ω, by comparing it with

$\frac{a}{H} = \frac{\gamma g}{a} = \sqrt{\frac{\gamma g}{H}}$. There are two choices of interest:

(i) $\omega \sim a/H$

(ii) $\omega \sim ka \sim \frac{(a/H)}{\epsilon} \gg \frac{a}{H}$.

First, write $\frac{d\xi_z}{dz} = ik_z\xi_z$ with $k_zH \equiv \frac{1}{\epsilon}$;  becomes

$$-k_z^2\xi_z + ik_z\xi_z\left[\frac{\omega^2\frac{d\ln\rho}{dz} - k_x^2a^2\frac{d\ln\rho}{dz}}{\omega^2-k_x^2a^2}\right]$$

$$+\frac{\xi_z}{\omega^2-k_x^2a^2}\left[\frac{\omega^4}{a^2} - 2\omega^2k_x^2 + k_x^4a^2 - k_x^2g\frac{d\ln\rho}{dz}\left(1-\frac{1}{\gamma}\right)\right.$$

$$\left.+\frac{gk_x^4a^2}{\omega^2}\frac{d\ln\rho}{dz} + \frac{k_x^4g^2}{\omega^2}\right] = 0.$$

Now, (i) $\omega \sim a/H$ gives $k_x^2a^2 \gg \omega^2$ and so the dominant terms are

$$-k_z^2\xi_z + \xi_z(-k_x^2) - \xi_z\frac{gk_x^2}{\omega^2}\frac{d\ln\rho}{dz} - \xi_z\frac{k_x^2g^2}{a^2\omega^2} = 0$$

$$\Rightarrow \quad k^2 + \frac{gk_x^2}{\omega^2}\underbrace{\left[\frac{d\ln\rho}{dz} + \frac{g}{a^2}\right]}_{\frac{d\ln\rho}{dz} - \frac{1}{\gamma}\frac{d\ln p}{dz} = -\frac{1}{\gamma}\frac{d\ln p\rho^{-\gamma}}{dz}} = 0.$$

$$\Rightarrow \boxed{\begin{aligned} \omega^2 &= \frac{k_x^2}{k^2}\frac{g}{\gamma}\frac{d\ln p\rho^{-\gamma}}{dz} \\ &= -\frac{k_x^2}{k^2}\frac{1}{\gamma p}\frac{dp}{dz}\frac{d\ln p\rho^{-\gamma}}{dz} \equiv \frac{k_x^2}{k^2}N^2 \end{aligned}}$$

where N^2 is the square of the Brunt – Väisälä frequency.
If $N^2 > 0$, these are called <u>internal waves</u> or <u>g-modes</u>.

Note that different wavenumbers have different velocities (i.e., dispersion) and that ω depends on the direction of $\vec{k}$: $\frac{\partial \omega}{\partial \vec{k}} = \frac{\omega}{k^2}\frac{k_z}{k_x}\left(k_z\hat{x} - k_x\hat{z}\right)$, so that $\vec{k}\cdot\frac{\partial \omega}{\partial \vec{k}} = 0$.

We'll return to the physical cause of these waves later, after the "Boussinesq approximation" is introduced, but, for now, note that $N^2 < 0$ (i.e., upwardly decreasing entropy) gives instability. Go boil some water and think about it.

(ii) $\omega \sim ka \gg a/H$. This gives the following dominant terms:

$$-k_z^2 + \frac{1}{\omega^2 - k_x^2 a^2}\underbrace{\left[\frac{\omega^4}{a^2} - 2\omega^2 k_x^2 + k_x^4 a^2\right]}_{(\omega^2 - k_x^2 a^2)^2/a^2} = 0$$

$$\Rightarrow \boxed{\omega^2 = (k_x^2 + k_z^2)\,a^2}$$ <u>Sound waves!</u>

Now, sound waves are often a nuisance in many calculations. They mostly play a rather boring role, and often serve only to make the algebra more tedious. There is something called the Boussinesq approximation, which rigorously filters out sound waves. Let's see how this works in our convection problem...

Return to $\star$ with $\omega \sim \epsilon k a$:

$$\frac{\delta p}{p} \simeq \frac{\omega^2}{k_x^2 a^2}\, i k_z \xi_z \gamma \quad \text{if } k_z \neq 0;$$

$$\simeq \underbrace{\frac{\omega^2}{k_x^2 a^2}}_{\sim \epsilon^2 \ll 1} \frac{d\ln p}{dz}\, \xi_z \quad \text{otherwise.}$$

So, perhaps we should have dropped perturbations to the gas pressure at some point. Where? In the momentum equation? Hmm... careful. Consider B with $\frac{\delta p}{p} \simeq \frac{\omega^2}{k_x^2 a^2} i k_z \xi_z \gamma$

$\Rightarrow \xi_x = -\xi_z \frac{k_z}{k_x}$, or $\vec{k} \cdot \vec{\xi} = 0$. Looks like pressure fluctuations are enforcing (near) incompressibility. Best not to drop them! And C?

$$-\omega^2 \xi_z = -\frac{i k_z p}{\rho} \frac{\omega^2}{k_x^2 a^2}\, i k_z \xi_z \gamma - \frac{\delta\rho}{\rho} g$$

same order!

Okay. So, pressure fluctuations are small, but not so small that they can be dropped from the momentum eqn. What about the entropy eqn?

$$\text{(D)} \Longrightarrow \quad \frac{\delta P}{P} = \gamma \frac{\delta\rho}{\rho} - \xi_z \frac{d \ln P\rho^{-\gamma}}{dz}$$

$\frac{\delta P}{P} \sim \frac{\xi_z}{H} \frac{\omega^2}{k_x^2 a^2}$ or $\sim i k_z \xi_z \gamma \frac{\omega^2}{k_x^2 a^2}$, either way... it's small. So, drop δP from entropy equation!

$\xi_z \frac{d \ln P\rho^{-\gamma}}{dz} \sim \frac{\xi_z}{H}$, required for internal waves

What does that leave us with?

$$\gamma \frac{\delta\rho}{\rho} \simeq \xi_z \frac{d \ln P\rho^{-\gamma}}{dz}$$

$\Longrightarrow \frac{\delta\rho}{\rho} \sim \frac{\xi_z}{H}$. Ah! Look at (A):

$$\frac{\delta\rho}{\rho} + i k_z \xi_z + i k_x \xi_x + \xi_z \frac{d \ln \rho}{dz} = 0$$

$\frac{\delta\rho}{\rho} \sim \frac{\xi_z}{H}$, $\quad i k_z \xi_z \sim k \xi_z$, $\quad \xi_z \frac{d \ln \rho}{dz} \sim \frac{\xi_z}{H}$

So, to leading order, we have $\vec{k} \cdot \vec{\xi} = 0$ — incompressibility! Okay, things are consistent, and we have the Boussinesq approx:

$$\frac{\delta P}{P} \sim \frac{1}{kH}\frac{\delta p}{p} \ll \frac{\delta \rho}{\rho} \sim \frac{\delta u}{a} \ll \frac{k\,\delta u}{\omega} \sim k\xi \sim (kH)\frac{\delta u}{a}.$$

Or, defining the Mach number $\mathcal{M}$ and taking it to be small ($\sim \epsilon$),

$$\frac{\delta u}{a} \sim \frac{\delta \rho}{\rho} \sim \frac{\delta T}{T} \sim \frac{1}{\mathcal{M}}\frac{\delta p}{p} \sim \frac{1}{kH} \sim \epsilon \ll 1.$$

In practice, this means:

1) Assume (near) incompressibility ($\vec{\nabla}\cdot\vec{\delta u} = 0$)

2) Drop δp everywhere EXCEPT the momentum eqn. They are enforcing (near) incompressibility.

3) Keep $\delta \rho$ everywhere EXCEPT the continuity eqn. They interact with gravity to give buoyancy.

Watch how much simpler this is...

$$Ⓐ \longrightarrow \quad i k_x \xi_x + i k_z \xi_z = 0$$

$$Ⓑ \longrightarrow \quad -\omega^2 \xi_x = -i k_x \frac{\delta p}{\rho}$$

$$\longrightarrow \quad \frac{\delta p}{\rho} = \frac{i\omega^2 k_z}{k_x^2}\xi_z$$

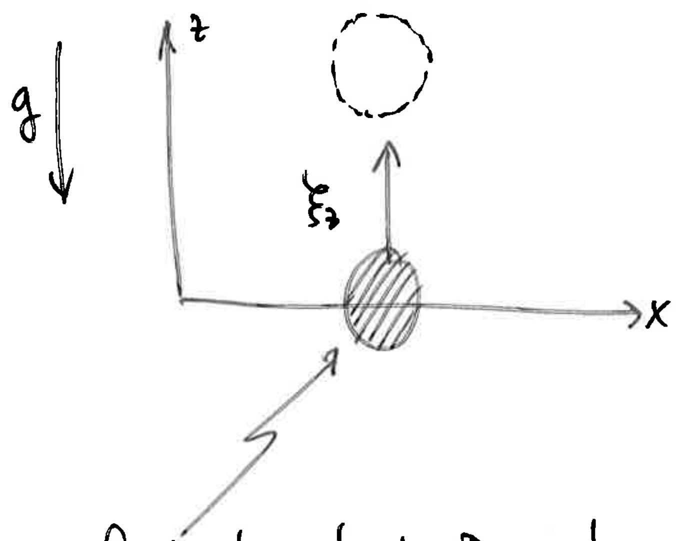
$$Ⓒ \longrightarrow \quad -\omega^2 \xi_z = -i k_z \frac{\delta p}{\rho} - \frac{\delta \rho}{\rho} g$$

$$\omega^2 \xi_z = \frac{k_x^2}{k^2} g \frac{\delta \rho}{\rho}$$

$$Ⓓ \longrightarrow \quad 0 = \gamma \frac{\delta \rho}{\rho} - \xi_z \frac{d \ln p\,\rho^{-\gamma}}{dz}$$

$$\omega^2 = \frac{k_x^2}{k^2} N^2. \quad \text{Done!}$$

What we've done here is eliminated the restoring pressure forces that drive sound waves, essentially by assuming that a^2 is so large that sound waves propagate instantaneously. When the restoring force is purely external (e.g., gravity), the flow behaves as though it were incompressible (nearly). Physically, a slow-moving fluid element remains in pressure balance with its surroundings. This readjustment is what makes buoyancy waves and convection possible. Let us see that explicitly.



$P_2 < P_1$ $\sigma_2 > \sigma_1$

P_1 σ_1

where $\sigma \equiv P\rho^{-\gamma}$ is the entropy variable.

fluid element at P_1 and σ_1

Displace fluid element upwards while conserving its entropy. Now it has less entropy than its surroundings. With pressure balance holding, this means that it is also denser than its surroundings. It must fall back to its equilibrium position. Overshooting, it will oscillate at frequency N. (Mathematically, $\Delta\sigma = 0 \Rightarrow \Delta p/p = \gamma\, \Delta\rho/\rho \Rightarrow \vec{\xi}\cdot\vec{\nabla}\ln p = \gamma \frac{\delta\rho}{\rho} + \gamma\, \vec{\xi}\cdot\vec{\nabla}\ln\rho \Rightarrow \frac{\delta\rho}{\rho} = \frac{N^2}{g}\xi_z$.)

Now, consider $\delta_2 < \delta_1$. Our upwardly displaced fluid element has <u>more</u> entropy than its surroundings, and it will continue to rise $\longrightarrow$ convective instability. The (Karl) <u>Schwarzschild criterion</u> for convective stability is $\boxed{N^2 > 0}$.

Bonus: Exact solution to  for an isothermal atmosphere.

Suppose $\frac{d\ln p}{dz} = \frac{d\ln \rho}{dz}$ (T = const). Then $\frac{d\ln p}{dz} = -\frac{\gamma g}{a^2}$ = const.

$\Rightarrow p = p_0 \exp(-z/H)$ with $H \equiv a^2/\gamma g$. Then we have

$$\xi_z'' - \frac{\xi_z'}{H} + \left[\frac{\omega^2 - k_x^2 a^2}{a^2} + \frac{k_x^2 g}{H}\left(1 - \frac{1}{\gamma}\right)\frac{1}{\omega^2}\right]\xi_z = 0.$$

Let $\xi_z = f(z)\exp\left(\frac{z}{2H}\right)$. Then

$$\xi_z' = f' e^{z/2H} + \frac{f}{2H} e^{z/2H} = f' e^{z/2H} + \xi_z/2H$$

$$\xi_z'' = f'' e^{z/2H} + \frac{f'}{H} e^{z/2H} + \frac{\xi_z}{(2H)^2}$$

$$f'' + \frac{f'}{H} + \frac{f}{(2H)^2} - \frac{f'}{H} - \frac{f}{2H^2} + [\dots]f = 0.$$

$$f'' + \underbrace{\left[-\frac{1}{4H^2} + \frac{\omega^2 - k_x^2 a^2}{a^2} + \frac{k_x^2 a^2}{\gamma H^2}\left(1 - \frac{1}{\gamma}\right)\frac{1}{\omega^2}\right]}_{= \text{const.}} f = 0.$$

bracket

$\Rightarrow f = \exp(\pm i k_z z)$ with $k_z^2 =$

$$\Rightarrow -k_z^2 - \frac{1}{4H^2} + \frac{\omega^2 - k_x^2 a^2}{a^2} + \frac{k_x^2 a^2}{\omega^2 H^2}\left(\frac{\gamma-1}{\gamma^2}\right) = 0.$$

Mult. by $\omega^2 a^2$ and regroup terms:

$$\omega^4 + \omega^2\left[-k_z^2 a^2 - \frac{a^2}{4H^2} - k_x^2 a^2\right] + k_x^2 a^2 \left(\frac{a^2}{H^2}\right)\left(\frac{\gamma-1}{\gamma^2}\right) = 0.$$

$$\Rightarrow \omega^2 = \frac{k^2 a^2 + \frac{a^2}{4H^2}}{2} \pm \frac{1}{2}\left[\left(k^2 a^2 + \frac{a^2}{4H^2}\right)^2 - 4k_x^2 a^2 \left(\frac{a^2}{H^2}\right)\left(\frac{\gamma-1}{\gamma^2}\right)\right]^{1/2}.$$

Note that $N^2 = \frac{g}{\gamma}\frac{d\ln P\rho^{-\gamma}}{dz} = \left(\frac{1-\gamma}{\gamma}\right) g \frac{d\ln \rho}{dz} = \frac{a^2}{H^2}\left(\frac{\gamma-1}{\gamma^2}\right).$

So,

$$\boxed{\omega^2 = \frac{k^2 a^2 + \frac{a^2}{4H^2}}{2} \pm \frac{1}{2}\left[\left(k^2 a^2 + \frac{a^2}{4H^2}\right)^2 - 4k_x^2 a^2 N^2\right]^{1/2}.}$$

If $(kH)^2 \gg 1$, this becomes $\omega^2 = k^2 a^2$ for the sound wave ($+$ sign) and $\omega^2 = \frac{k_x^2 N^2}{k_z^2 + k_x^2}$ for the g-mode ($-$ sign).

The final term in the square root captures the coupling between these modes.

VI.8. Buoyancy: Parker instability

Continued on hand-written notes...

- A related problem is the Parker Instability, or "magnetic Rayleigh-Taylor instability" (although it is different in detail from RTI and is closer to Schwarzschild Convection). Consider an atmosphere similar to that in our convective instability calculation, but with a magnetic field oriented perpendicularly to gravity with a z-dependence: $\vec{B}_0 = B_0(z)\hat{x}$. The force balance in the equilibrium state now includes a contribution from the magnetic pressure: $g = -\frac{1}{\rho}\left(\frac{dp}{dz} + \frac{dB_0^2}{dz}\right) = \text{const.}$, or

$$\frac{g}{a^2} = -\frac{1}{\gamma}\frac{d\ln P}{dz} - \frac{V_{A0}^2}{a^2}\frac{d\ln B_0}{dz}.$$

Our equations are almost the same:

$$\frac{\delta\rho}{\rho} + ik_x\xi_x + \xi_z' + \xi_z\frac{d\ln\rho}{dz} = 0,$$

$$-\omega^2\xi_x = -ik_x\left(\frac{\delta p}{\rho} + \frac{B_0\,\delta B_x}{4\pi\rho}\right) + \frac{ik_x B_0}{4\pi\rho}\delta B_x + \frac{\delta B_z}{4\pi\rho}\frac{dB_0}{dz},$$

$$-\omega^2\xi_z = -\frac{1}{\rho}\frac{d}{dz}\left(\delta p + \frac{B_0\,\delta B_x}{4\pi}\right) - \frac{\delta\rho}{\rho}g + \frac{ik_x B_0}{4\pi\rho}\delta B_z,$$

$$\frac{\delta p}{p} = \gamma\frac{\delta\rho}{\rho} - \xi_z\frac{d\ln p\rho^{-\gamma}}{dz},$$

but now with magnetic-field perturbations and gradients. The former are given by $\vec{\delta B} = \vec{\nabla}\times(\vec{\xi}\times\vec{B}_0) \Rightarrow \delta B_x = -\frac{d}{dz}(\xi_z B_0)$

$$\delta B_z = ik_x B_0\xi_z$$

First, note that $\dfrac{\delta p + B_0\,\delta B_x/4\pi}{\rho}$

$$= a^2\frac{\delta\rho}{\rho} - \frac{a^2}{\gamma}\frac{d\ln p\rho^{-\gamma}}{dz}\xi_z + \frac{B_0}{4\pi\rho}\left(-\frac{d}{dz}\right)(\xi_z B_0)$$

$$= a^2\left[-ik_x\xi_x - \xi_z' - \xi_z\frac{d\ln\rho}{dz} - \frac{1}{\gamma}\frac{d\ln p\rho^{-\gamma}}{dz}\xi_z - \frac{V_{A0}^2}{a^2}\left(\xi_z' + \xi_z\frac{d\ln B_0}{dz}\right)\right]$$

$$= a^2\left[-ik_x\xi_x - \xi_z'\left(1+\frac{V_{A0}^2}{a^2}\right) + \frac{g}{a^2}\xi_z\right].$$

Then

$$-\omega^2\xi_x = -ik_x a^2\left[-ik_x\xi_x - \xi_z'\left(1+\frac{V_{A0}^2}{a^2}\right) + \frac{g}{a^2}\xi_z\right] + \frac{ik_x B_0}{4\pi\rho}\left[-\xi_z' B_0 - \xi_z B_0\frac{d\ln B_0}{dz}\right] + \frac{ik_x B_0}{4\pi\rho}\xi_z\frac{dB_0}{dz}$$

$$\Rightarrow (-\omega^2 + k_x^2a^2)\xi_x = ik_x a^2\xi_z'\left(1+\cancel{\frac{V_{A0}^2}{a^2}}\right) - ik_x g\xi_z - \cancel{ik_x V_{A0}^2\xi_z'} - \cancel{ik_x V_{A0}^2\frac{d\ln B_0}{dz}\xi_z} + \cancel{ik_x V_{A0}^2\frac{d\ln B_0}{dz}\xi_z}$$

$$\Rightarrow \boxed{(-\omega^2 + k_x^2a^2)\xi_x = ik_x a^2\xi_z' - ik_x\xi_z g}$$

Same as # w/o B field!

$$\Rightarrow \frac{\delta\rho}{\rho} = -\xi_z' - \xi_z\frac{d\ln\rho}{dz} - ik_x\frac{\left[ik_x a^2\xi_z' - ik_x\xi_z g\right]}{(-\omega^2 + k_x^2a^2)}$$

$$\boxed{\frac{\delta\rho}{\rho} = \frac{\omega^2 \xi_z' + \left[(\omega^2 - k_x^2 a^2)\frac{d\ln\rho}{dz} - k_x^2 g\right]\xi_z}{k_x^2 a^2 - \omega^2}}$$ same as ◎ w/o B field!

Buoyancy is fundamentally the same as in the hydro case.

Plugging all of this into the z-component of the momentum eqn. gives (with $\frac{\delta p + \delta B^2/8\pi}{p} = \frac{\omega^2}{\omega^2 - k_x^2 a^2}\left[\frac{\gamma g}{a^2}\xi_z - \gamma \xi_z'\right] - \frac{\gamma v_{A0}^2}{a^2}\xi_z'$)

$$-\omega^2 \xi_z = -\frac{1}{\rho}\frac{d}{dz}\left[\frac{p\,\omega^2}{\omega^2 - k_x^2 a^2}\left(\frac{\gamma g}{a^2}\xi_z - \gamma\xi_z'\right) - \frac{B_0^2}{4\pi}\xi_z'\right]$$

$$- g\left\{\frac{\omega^2 \xi_z' + \left[(\omega^2 - k_x^2 a^2)\frac{d\ln\rho}{dz} - k_x^2 g\right]\xi_z}{k_x^2 a^2 - \omega^2}\right\}$$

$$+ \frac{i k_x B_0}{4\pi\rho}\left(i k_x B_0 \xi_z\right)$$

$$\Rightarrow \left(-\omega^2 + k_x^2 v_{A0}^2\right)\xi_z = -\frac{a^2}{\gamma}\frac{d\ln p}{dz}\frac{\omega^2}{\omega^2 - k_x^2 a^2}\left(\frac{\gamma g \xi_z}{a^2} - \gamma\xi_z'\right)$$

$$- \frac{a^2}{\gamma}\frac{\omega^2 k_x^2 a^2}{(\omega^2 - k_x^2 a^2)^2}\frac{d\ln T}{dz}\left(\frac{\gamma g \xi_z}{a^2} - \gamma\xi_z'\right) + v_{A0}^2 \xi_z'' + \xi_z' v_{A0}^2 \frac{d\ln B_0^2}{dz}$$

$$- \frac{a^2}{\gamma}\frac{\omega^2}{\omega^2 - k_x^2 a^2}\left(\cancel{\frac{\gamma g}{a^2}\xi_z'} - \frac{\gamma g \xi_z}{a^2}\frac{d\ln T}{dz} - \gamma\xi_z''\right)$$

$$+ \frac{g}{\omega^2 - k_x^2 a^2}\left\{\cancel{\omega^2 \xi_z'} + \left[(\omega^2 - k_x^2 a^2)\frac{d\ln\rho}{dz} - k_x^2 g\right]\xi_z\right\}$$

After some straightforward algebra, we find

$$\xi_z'' \left(v_{A0}^2 + \frac{a^2\omega^2}{\omega^2 - k_x^2 a^2} \right)$$

$$+ \xi_z' \left[v_{A0}^2 \frac{d\ln B_0^2}{dz} + \frac{a^2\omega^2}{(\omega^2 - k_x^2 a^2)^2} \left(\omega^2 \frac{d\ln P}{dz} - k_x^2 a^2 \frac{d\ln \rho}{dz} \right) \right]$$

$$+ \xi_z \left[\frac{g}{\gamma} \frac{k_x^2 a^2}{\omega^2 - k_x^2 a^2} \left(\frac{d\ln p\rho^{-\gamma}}{dz} + \frac{\gamma v_{A0}^2}{a^2} \frac{d\ln B_0}{dz} \right) - \frac{g\,\omega^2 k_x^2 a^2}{(\omega^2 - k_x^2 a^2)^2} \frac{d\ln T}{dz} \right.$$
$$\left. + \omega^2 - k_x^2 v_{A0}^2 \right]$$

All extra terms are $\propto v_{A0}^2$. For $k_z \to 0$, $k_x^2 a^2 \gg 1$, this becomes

$$\boxed{\omega^2 \simeq \underbrace{k_x^2 v_{A0}^2}_{\text{magnetic tension}} + \frac{g}{\gamma} \left(\underbrace{\frac{d\ln p\rho^{-\gamma}}{dz}}_{\text{thermal buoyancy}} + \underbrace{\frac{1}{\beta} \frac{d\ln B_0^2}{dz}}_{\text{magnetic buoyancy}} \right)} \quad w/ \quad \beta = \frac{B_0^2}{8\pi p}.$$

Now, $\frac{d\ln p\rho^{-\gamma}}{dz} + \frac{1}{\beta}\frac{d\ln B_0^2}{dz} > 0$ for stability. But the physics is the same as in Schwarzschild convection — just the pressure balance is different.

One can obtain an exact solution for an isothermal atmosphere, just like in the hydro case.

$$\frac{d\ln T}{dz} = 0 \quad ; \quad \frac{d\ln B^2}{dz} = \frac{d\ln \rho}{dz} = \frac{d\ln P}{dz} = -\frac{1}{H}$$

Write $\xi_z \propto e^{\pm i k_z z + z/2H}$ and set

$$k_z^2 \equiv -\frac{1}{4H^2} + \frac{(\omega^2 - k_x^2 v_A^2)(\omega^2 - k_x^2 a^2) + k_x^2 a^2 N_{ms}^2}{\omega^2(a^2 + v_A^2) - k_x^2 a^2 v_A^2}$$

w/ $N_{ms}^2 \equiv \frac{g}{\gamma}\left(\frac{d\ln P\rho^{-\gamma}}{dz} + \frac{1}{\beta}\frac{d\ln B^2}{dz}\right)$. Dispersion relation is then

$$\omega^4 - \omega^2\left(k^2 + \frac{1}{4H^2}\right)v_{ms}^2 + k_x^2 a^2\left[k_z^2 a^2 + \frac{v_A^2}{4H^2} + N_{ms}^2\right] = 0$$

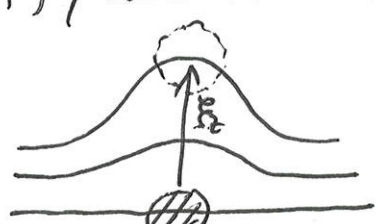
with $v_{ms}^2 \equiv a^2 + v_A^2$ the magnetosonic speed (squared).

NB: A cold atmosphere ($a^2 \to 0$) with support only from $B(z)$ has

$$\omega^4 - \omega^2\left(k^2 + \frac{1}{4H^2}\right)v_A^2 - k_x^2 v_A^2 \frac{g}{H} = 0,$$

which is unstable.

The idea is that the supporting magnetic pressure gradient wants to relax (B^2 likes being constant), and it can do so by offloading mass sinusoidally:



VI.9. Rotation

In §II.5, we wrote down the equations of hydrodynamics in a rotating frame – see (II.44). Here we do the same for the equations of MHD. With $\boldsymbol{v} = \boldsymbol{u} - R\Omega(R,z)\hat{\boldsymbol{\varphi}}$ and

$$\frac{\mathrm{D}}{\mathrm{D}t} \doteq \frac{\partial}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} + \Omega\frac{\partial}{\partial\varphi},$$

the continuity and force equations are the same,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}, \tag{VI.43}$$

$$\frac{\mathrm{D}v_R}{\mathrm{D}t} = f_R + 2\Omega v_\varphi + R\Omega^2 + \frac{v_\varphi^2}{R}, \tag{VI.44}$$

$$\frac{\mathrm{D}v_\varphi}{\mathrm{D}t} = f_\varphi - \frac{\kappa^2}{2\Omega}v_R - R\frac{\partial\Omega}{\partial z}v_z - \frac{v_R v_\varphi}{R}, \tag{VI.45}$$

$$\frac{\mathrm{D}v_z}{\mathrm{D}t} = f_z, \tag{VI.46}$$

but with the addition of the Lorentz force:

$$\boldsymbol{f} = -\frac{1}{\rho}\boldsymbol{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}B_i}{4\pi\rho}\,\hat{\boldsymbol{e}}_i + \frac{B_R B_\varphi}{4\pi\rho R}\hat{\boldsymbol{\varphi}} - \frac{B_\varphi^2}{4\pi\rho R}\hat{\boldsymbol{R}} - \boldsymbol{\nabla}\Phi. \tag{VI.47}$$

Note the additional geometric terms $\propto B^2/R$; these are tension forces associated with the bend in the magnetic-field lines as they follow the azimuthal direction. To these equations we must append the induction equation:

$$\frac{\mathrm{D}B_R}{\mathrm{D}t} = -B_R\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} + \boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}v_R, \tag{VI.48}$$

$$\frac{\mathrm{D}B_\varphi}{\mathrm{D}t} = -B_\varphi\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} + \boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}v_\varphi + \frac{\partial\Omega}{\partial\ln R}B_R + R\frac{\partial\Omega}{\partial z}B_z, \tag{VI.49}$$

$$\frac{\mathrm{D}B_z}{\mathrm{D}t} = -B_z\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} + \boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}v_z. \tag{VI.50}$$

With the exception of advection by the differential rotation, the only additions to the induction equation beyond its more customary Cartesian form appear in its azimuthal component: $+\,R\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\Omega$ on the right-hand side. This corresponds to stretching of the flux-frozen magnetic field by the differential rotation.

In the hand-written pages that follow, these equations are used to describe the evolution of small fluctuations about a homogeneous, differentially rotating disk with $\Omega = \Omega(R)$, in which the centrifugal acceleration $R\Omega^2$ is balanced by gravity $-\partial\Phi/\partial R$. If the latter is dominated by that of a central point mass M, we have $\Phi = -GM/R$ and so $\Omega = (GM/R^3)^{1/2}$ – i.e., Keplerian rotation.

Before proceeding, I'll write down the linearized MHD equations written in cylindrical coordinates (R,φ,z) in a rotating frame with $\boldsymbol{\Omega} = \Omega(R,z)\hat{\boldsymbol{z}}$. The only assumptions here are that the background magnetic field is uniform, and that the equilibrium state arises from a balance between the centrifugal force and gravity plus thermal-pressure gradients (i.e., we allow for density and pressure stratification in the background state). We also neglect curvature terms of order $\sim(v_\mathrm{A}^2/R)(\delta B/B)$, as these are small compared to the other terms unless the toroidal magnetic field is super-thermal by a factor $\sim(R/H)^{1/2}$, where $H \sim c_\mathrm{s}/\Omega$ is the disk thickness and c_s is the sound speed – an atypical situation.

Without further ado…

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\varphi}\right)\delta\rho=-(\delta\boldsymbol{v}\cdot\boldsymbol{\nabla})\rho-\rho(\boldsymbol{\nabla}\cdot\delta\boldsymbol{v}),\tag{VI.51}$$

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\varphi}\right)\delta v_R=-\frac{1}{\rho}\frac{\partial}{\partial R}\left(\delta P+\frac{\boldsymbol{B}\cdot\delta\boldsymbol{B}}{4\pi}\right)+\frac{\delta\rho}{\rho^2}\frac{\partial P}{\partial R}+\frac{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\delta B_R}{4\pi\rho}-\frac{\partial\delta\Phi}{\partial R}-2\Omega\delta v_\varphi,\tag{VI.52}$$

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\varphi}\right)\delta v_\varphi=-\frac{1}{\rho R}\frac{\partial}{\partial\varphi}\left(\delta P+\frac{\boldsymbol{B}\cdot\delta\boldsymbol{B}}{4\pi}\right)+\frac{\delta\rho}{\rho}\frac{1}{\rho R}\frac{\partial P}{\partial\varphi}+\frac{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\delta B_\varphi}{4\pi\rho}-\frac{1}{R}\frac{\partial\delta\Phi}{\partial\varphi}+\frac{\kappa^2}{2\Omega}\delta v_R+R\frac{\partial\Omega}{\partial z}\delta v_\varphi,\tag{VI.53}$$

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\varphi}\right)\delta v_z=-\frac{1}{\rho}\frac{\partial}{\partial z}\left(\delta P+\frac{\boldsymbol{B}\cdot\delta\boldsymbol{B}}{4\pi}\right)+\frac{\delta\rho}{\rho^2}\frac{\partial P}{\partial z}+\frac{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\delta B_z}{4\pi\rho}-\frac{\partial\delta\Phi}{\partial z}\tag{VI.54}$$

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\varphi}\right)\delta B_R=(\boldsymbol{B}\cdot\boldsymbol{\nabla})\delta v_R-B_R(\boldsymbol{\nabla}\cdot\delta\boldsymbol{v}),\tag{VI.55}$$

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\varphi}\right)\delta B_\varphi=(\boldsymbol{B}\cdot\boldsymbol{\nabla})\delta v_\varphi-B_\varphi(\boldsymbol{\nabla}\cdot\delta\boldsymbol{v})+\frac{\partial\Omega}{\partial\ln R}\delta B_R+R\frac{\partial\Omega}{\partial z}\delta B_z,\tag{VI.56}$$

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\varphi}\right)\delta B_z=(\boldsymbol{B}\cdot\boldsymbol{\nabla})\delta v_z-B_z(\boldsymbol{\nabla}\cdot\delta\boldsymbol{v}),\tag{VI.57}$$

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\varphi}\right)\delta\sigma=-\delta v_R\frac{\partial\ln P\rho^{-\gamma}}{\partial R}-\delta v_z\frac{\partial\ln P\rho^{-\gamma}}{\partial z},\tag{VI.58}$$

where $\delta\sigma\doteq\delta P/P-\gamma\delta\rho/\rho$.

- <u>Rotational</u> and <u>magnetorotational instability</u>.

Accretion disks are ubiquitous in astrophysics, and they get their namesake by actually facilitating mass accretion onto compact objects like young protostars, neutron stars, black holes, etc. But for this to happen, angular momentum must be redistributed, and it turns out that this is frustratingly difficult in Keplerian disks. The problem is that, hydrodynamically, Keplerian flows are quite stable (we'll show below that they are linearly stable; there is no proof that they are nonlinearly stable, but experimental efforts to find nonlinear instability in hydrodynamic, differentially rotating flows have so far failed). Fluid elements do not like to give up their angular momentum. The culprit is the Coriolis force, a surprisingly strong stabilizing effect. (Indeed, planar shear flows without rotation quite easily disrupt so long as the viscosity is not too large.) Another issue is that the molecular viscosity, which might transport angular momentum purely by frictional means, is absolutely negligible in most all astrophysical fluids. One way out is to posit some anomalous viscosity via (unknown) turbulence. This is the route taken in the classic Shakura & Sunyaev

(1973) paper — assume turbulent transport, characterize it by a scalar viscosity, and take that viscous stress to be proportional to the gas pressure:

$$T_{R\varphi} = \alpha_{SS}\, p$$

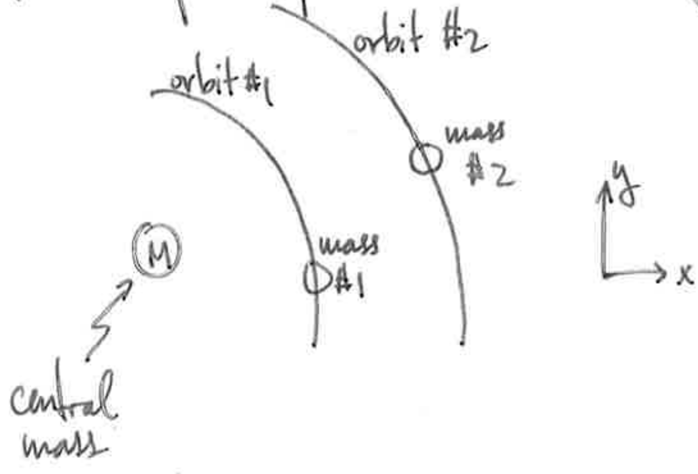
gas pressure

proportionality constant

R-φ component of the stress tensor, responsible for transporting φ momentum in the R direction.

This led to the "α-disk" framework of accretion disks, which has been extremely profitable, but woefully unsatisfying. This changed in 1991.

Let's pause here and explore the above claims a bit further. I said a Keplerian disk is hydrodynamically stable to small disturbances. Let's prove it. There are two ways to do this — using point masses in orbits, and using the full hydro eqns. in a rotating frame. Here's the first:



The eqns. of motion for these masses are

$$\ddot{x} - 2\Omega\dot{y} = -\frac{d\Omega^2}{d\ln R}\, x$$

$$\ddot{y} + 2\Omega\dot{x} = 0$$

These are called the "Hill equations" (Hill 1878). They include the Coriolis force and an extra term in the "radial" equation for the x displacement that accounts for the "tidal" force (the local difference between the centrifugal force and gravity). And they are local — note the Cartesian coordinate system with x pointing locally radial and y pointing locally azimuthal. Take solutions $x, y \sim e^{i\omega t}$ to compute the normal modes of this system:

$$\begin{bmatrix} -\omega^2 + \frac{d\Omega^2}{d\ln R} & 2\Omega i\omega \\ -2\Omega i\omega & -\omega^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \omega^2\left(\omega^2 - \frac{d\Omega^2}{d\ln R}\right) = 4\Omega^2\omega^2$$

$$\Rightarrow \omega^2 - \underbrace{\left(4\Omega^2 + \frac{d\Omega^2}{d\ln R}\right)}_{\equiv \kappa^2} = 0 \Rightarrow \boxed{\omega = \pm\kappa}$$

"epicyclic frequency"

These are epicyclic oscillations when $\kappa^2 > 0$, and exponentially growing disturbances when $\kappa^2 < 0$.

Note that $4\Omega^2 + \frac{d\Omega^2}{d\ln R} = \frac{1}{R^3}\frac{d\ell^2}{dR}$, where $\ell = \Omega R^2$ is the (specific) angular momentum. Thus,

$$\boxed{\frac{d\ell^2}{d\ln R} > 0 \Leftrightarrow \text{linear stability}}$$

"Rayleigh criterion"

The fluid way: Let's assume incompressibility for simplicity. Going back to our hydrodynamic equs., with gravity from a central point mass and $\vec{u} = \vec{v} + R\Omega\hat{\varphi}$, we have

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\right) v_R + \vec{v}\cdot\vec{\nabla} v_R - 2\Omega v_\varphi - R\Omega^2 - \frac{v_\varphi^2}{R}$$

$$= -\frac{\partial p}{\partial R}\frac{1}{\rho} + g_R,$$

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\right) v_\varphi + \vec{v}\cdot\vec{\nabla} v_\varphi + 2\Omega v_R + v_R\frac{d\Omega}{d\ln R} + \frac{v_R v_\varphi}{R}$$

$$= -\frac{1}{R}\frac{\partial p}{\partial\varphi}\frac{1}{\rho},$$

where $g_R = -\frac{GM}{R^2}$. Our equilibrium state is $\vec{v}=0$, $p=$ constant, and $g_R = -R\Omega^2 \Rightarrow \Omega^2 = \frac{GM}{R^3}$, a Keplerian orbit. Writing $\vec{v} = 0 + \vec{\delta v}$ and $p = p_0 + \delta p$, our equations to linear order in δ are

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\right)\delta v_R - 2\Omega\delta v_\varphi = -\frac{1}{\rho}\frac{\partial}{\partial R}\delta p,$$

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\right)\delta v_\varphi + 2\Omega\delta v_R + \frac{d\Omega}{d\ln R}\delta v_R = -\frac{1}{\rho}\frac{1}{R}\frac{\partial\delta p}{\partial\varphi}.$$

For simplicity, let us neglect the $\partial/\partial\varphi$ derivatives and let $\delta \sim \exp(-i\omega t + ik_R R + ik_z z)$. Then, with

$\vec{\nabla}\cdot\vec{\delta v} = 0 \Rightarrow k_R \delta v_R = -k_z \delta v_z$ and $\frac{\partial}{\partial t}\delta v_z = -\frac{1}{\rho}\frac{\partial}{\partial z}\delta p$,

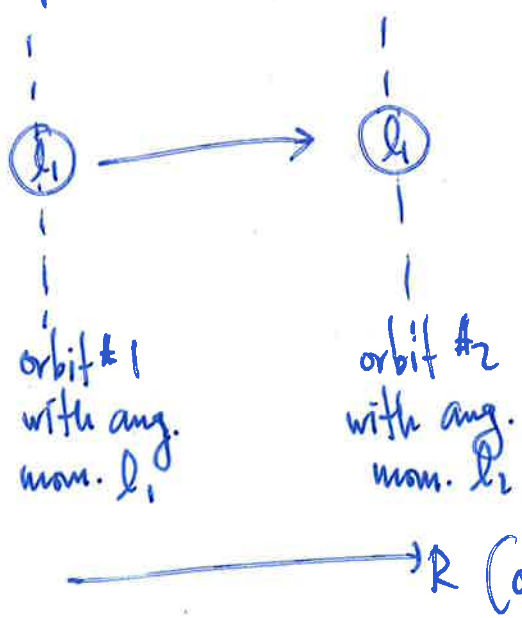
we find $\frac{\delta p}{\rho} = -\frac{k_R}{k_z^2}\,\omega\,\delta v_R$ and so

$$\begin{bmatrix} -i\omega \frac{k^2}{k_z^2} & -2\Omega \\ \underbrace{\frac{\kappa^2}{2\Omega}}_{=2\Omega + \frac{d\Omega}{d\ln R}} & -i\omega \end{bmatrix}\begin{bmatrix} \delta v_R \\ \delta v_\varphi \end{bmatrix} = 0 \Rightarrow \boxed{\omega^2 = \frac{k_z^2}{k^2}\kappa^2}$$

Same stability criterion as before, $\kappa^2 > 0$.

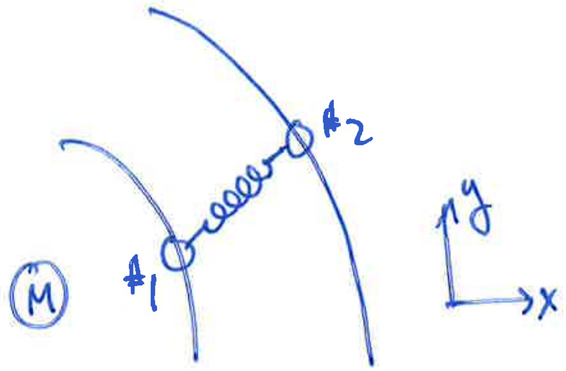
Physically, what's going on? Take a fluid element at orbit #1 and displace it outwards to orbit #2 while maintaining a constant ang. momentum. Since $l_2 > l_1$, the fluid element cannot stay in its new orbit and must return back to orbit #1.

$\Rightarrow$ STABLE.



Now, back to 1991...

Steve Balbus and John Hawley, then both at Univ. of Virginia, found by a straightforward linear analysis and clever use of 90's supercomputers, that a small but finite magnetic field is all that is required to linearly destabilize Keplerian rotation. How could this be missed? The answer is complicated. The instability — at first known as the "Balbus–Hawley instability" but now goes by the moniker "magnetorotational instability" (MRI) — appeared in a little-known Russian paper by Velikhov in 1959, and 2 years later made its way into Chandrasekhar's classic text on "hydrodynamic and hydromagnetic stability". But there it appeared in a rather odd guise, at least to an astronomer thinking about accretion disks — Couette flow, i.e., rotational flow excited by placing a (conducting) fluid between two cylindrical walls rotating at different speeds. It wasn't until B&H rediscovered it and placed it in the astrophysical context that the instability became appreciated as a possible solution to the accretion problem. What followed was an industry of linear analysis and nonlinear numerical simulations aiming to characterize the MRI in a wide variety of disk systems. But let's go back to the beginning:



Take the Hill system but attach a spring between the two masses — magnetic fields act as springs, so you can imagine this being a field line threading two fluid elements. Add Hooke's law to the eqns. of motion:

$$\ddot{x} - 2\Omega\dot{y} = -\frac{d\Omega^2}{d\ln R}x - Kx$$

w/ K = spring constant

$$\ddot{y} + 2\Omega\dot{x} = -Ky$$

$$x, y \sim e^{-i\omega t} \Rightarrow \begin{bmatrix} -\omega^2 + \frac{d\Omega^2}{d\ln R} + K & 2\Omega i\omega \\ -2\Omega i\omega & -\omega^2 + K \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \left(\omega^2 - \frac{d\Omega^2}{d\ln R} - K\right)\left(\omega^2 - K\right) = 4\Omega^2\omega^2$$

$$\Rightarrow (\omega^2 - \underbrace{K}_{\text{extra}})\left(\omega^2 - \underbrace{K}_{\text{extra}} - 4\Omega^2 - \frac{d\Omega^2}{d\ln R}\right) = 4\Omega^2 \underbrace{K}_{\text{extra}}$$

Solutions are $\omega^2 - K = \frac{\kappa^2}{2} \pm \sqrt{\left(\frac{\kappa^2}{2}\right)^2 + 4\Omega^2 K}$,

whose $(-)$ solution is unstable if $\boxed{K + \frac{d\Omega^2}{d\ln R} < 0}$

This is important, because Keplerian disks have $\frac{d\Omega^2}{d\ln R} < 0$! Note that the spring cannot be too strong here. Interestingly,

$$\boxed{\frac{dl^2}{d\ln R} > 0 \text{ for hydro stability} \longrightarrow \frac{d\Omega^2}{d\ln R} > 0 \text{ for MHD stability}}$$

One can show that the Lagrangian change in the ang. mom. of a fluid element as it is displaced is given by

$$\frac{\Delta l}{l} = \frac{x}{R}\left(\frac{\kappa^2}{2\Omega^2} - \frac{i\omega}{\Omega}\frac{y}{x}\right)$$

$$= -\frac{x}{R}\left(\frac{2K}{\omega^2 - K}\right)$$

the spring broke conservation of angular momentum!

If $K \ll |\omega^2| \sim \Omega^2$, then outward displacements $(x>0)$ gain angular momentum as they are torqued by the spring (NB: $\omega^2 < 0$ corresponds to growth, so $\Delta l \propto (x/R)$).

At max. growth (take $\frac{\partial}{\partial K}$ of disp. relation and find extrema of ω^2), the growth rate $-i\omega = \frac{1}{2}\left|\frac{d\Omega}{d\ln R}\right|$ and

$$\left.\frac{\Delta l}{l}\right|_{max} = \frac{2x}{R}\left(1 - \frac{1}{4}\left|\frac{d\Omega}{d\ln R}\right|\right).$$

Now the fluid picture: (let $\vec{B}_0 = B_0\hat{z}$ for simplicity)

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\right)\delta v_R - 2\Omega\,\delta v_\varphi = -\frac{1}{\rho}\frac{\partial}{\partial R}\left(\delta p + \frac{\vec{B}_0\cdot\delta\vec{B}}{4\pi}\right) + \frac{\vec{B}_0\cdot\vec{\nabla}\,\delta B_R}{4\pi\rho},$$

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\right)\delta v_\varphi + \frac{\kappa^2}{2\Omega}\,\delta v_R = -\frac{1}{\rho}\frac{1}{R}\frac{\partial}{\partial\varphi}\left(\delta p + \frac{\vec{B}_0\cdot\delta\vec{B}}{4\pi}\right) + \frac{\vec{B}_0\cdot\vec{\nabla}\,\delta B_\varphi}{4\pi\rho}.$$

We need equs. for δB_R and δB_φ (δB_z is determined from $\vec{\nabla}\cdot\delta\vec{B} = 0$). So, take the induction equ. and write it in cylindrical coordinates w/ $\vec{u} = \vec{v} + R\Omega\hat{\varphi}$:

$$\left(\vec{B}\cdot\vec{\nabla}\vec{u} = \hat{e}_i\,\vec{B}\cdot\vec{\nabla}v_i + \frac{v_R B_\varphi}{R}\hat{\varphi} - \frac{v_\varphi B_\varphi}{R}\hat{R} - \Omega B_\varphi\hat{R} + \hat{\varphi}B_R\left(\Omega + \frac{d\Omega}{d\ln R}\right)\right)$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\right)\delta B_R = \vec{B}_0\cdot\vec{\nabla}\,\delta v_R,$$

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\right)\delta B_\varphi = \vec{B}_0\cdot\vec{\nabla}\,\delta v_\varphi + \delta B_R\frac{d\Omega}{d\ln R}.$$

Again, take $\partial/\partial\varphi = 0$ and let $\delta \sim \exp\left(-i\omega t + ik_R R + ik_z z\right)$.

$$\vec{\nabla}\cdot\vec{\delta v} = 0 \longrightarrow \delta v_z = -\frac{k_R}{k_z}\delta v_R$$

$$\vec{\nabla}\cdot\vec{\delta B} = 0 \longrightarrow \delta B_z = -\frac{k_R}{k_z}\delta B_R$$

$$\frac{\partial}{\partial t}\delta v_z = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(\delta p + \frac{\vec{B}_0\cdot\vec{\delta B}}{4\pi}\right) + \frac{\vec{B}_0\cdot\vec{\nabla}\,\delta B_z}{4\pi\rho}$$

$$\Rightarrow \frac{\delta p}{\rho} + \frac{\vec{B}_0\cdot\vec{\delta B}}{4\pi\rho} = \frac{-\omega k_R}{k_z^2}\delta v_R - \frac{\vec{k}\cdot\vec{B}_0}{4\pi\rho}\frac{k_R}{k_z^2}\delta B_R$$

$$\Rightarrow \begin{cases} -i\omega\,\delta v_R - 2\Omega\,\delta v_\varphi = -ik_R\left[-\frac{\omega k_R}{k_z^2}\delta v_R - \frac{\vec{k}\cdot\vec{B}_0}{4\pi\rho}\frac{k_R}{k_z^2}\delta B_R\right] + \frac{i\vec{k}\cdot\vec{B}_0}{4\pi\rho}\delta B_R \\ -i\omega\,\delta v_\varphi + \frac{\kappa^2}{2\Omega}\delta v_R = \frac{i\vec{k}\cdot\vec{B}_0}{4\pi\rho}\delta B_\varphi \end{cases}$$

$$\Rightarrow \begin{cases} -i\omega\frac{k^2}{k_z^2}\delta v_R - 2\Omega\,\delta v_\varphi = \frac{i\vec{k}\cdot\vec{B}_0}{4\pi\rho}\frac{k^2}{k_z^2}\delta B_R \\ \frac{\kappa^2}{2\Omega}\delta v_R - i\omega\,\delta v_\varphi = \frac{i\vec{k}\cdot\vec{B}_0}{4\pi\rho}\delta B_\varphi \end{cases}$$

and
$$\begin{cases} -i\omega\,\delta B_R = i\vec{k}\cdot\vec{B}_0\,\delta v_R \\ -i\omega\,\delta B_\varphi = i\vec{k}\cdot\vec{B}_0\,\delta v_\varphi + \delta B_R\frac{d\Omega}{d\ln R} \end{cases}$$
solve these for $\vec{\delta v}$ and plug into these

$$\begin{bmatrix} -i\omega \frac{k^2}{k_z^2}\left(\frac{-i\omega\,\delta B_R}{i\mathbf{k}\cdot\mathbf{B}_0}\right) - 2\Omega\left(\frac{-i\omega\,\delta B_\varphi - \delta B_R\, d\Omega/d\ln R}{i\mathbf{k}\cdot\mathbf{B}_0}\right) = \frac{i\mathbf{k}\cdot\mathbf{B}_0}{4\pi\rho}\frac{k^2}{k_z^2}\,\delta B_R \\ \frac{\kappa^2}{2\Omega}\left(\frac{-i\omega\,\delta B_R}{i\mathbf{k}\cdot\mathbf{B}_0}\right) - i\omega\left(\frac{-i\omega\,\delta B_\varphi - \delta B_R\, d\Omega/d\ln R}{i\mathbf{k}\cdot\mathbf{B}_0}\right) = \frac{i\mathbf{k}\cdot\mathbf{B}_0}{4\pi\rho}\,\delta B_\varphi \end{bmatrix}$$

Cleaning up...

$$\begin{bmatrix} -\omega^2 + (\mathbf{k}\cdot\mathbf{v}_A)^2 - \frac{k_z^2}{k^2}\frac{d\Omega^2}{d\ln R} & 2\Omega i\omega\frac{k_z^2}{k^2} \\ -2\Omega i\omega & -\omega^2 + (\mathbf{k}\cdot\mathbf{v}_A)^2 \end{bmatrix}\begin{bmatrix} \delta B_R \\ \delta B_\varphi \end{bmatrix} = 0.$$

Look familiar? $K \to (\mathbf{k}\cdot\mathbf{v}_A)^2$! Magnetic tension is a spring. Dispersion relation:

$$\left[\omega^2 - (\mathbf{k}\cdot\mathbf{v}_A)^2\right]\left[\omega^2 - (\mathbf{k}\cdot\mathbf{v}_A)^2 - \kappa^2\frac{k_z^2}{k^2}\right] = 4\Omega^2(\mathbf{k}\cdot\mathbf{v}_A)^2$$

$$\Rightarrow \quad \omega^2 - (\mathbf{k}\cdot\mathbf{v}_A)^2 = \frac{\kappa^2}{2} \pm \sqrt{\left(\frac{\kappa^2}{2}\right)^2 + 4\Omega^2(\mathbf{k}\cdot\mathbf{v}_A)^2}$$

unstable if $\boxed{(\mathbf{k}\cdot\mathbf{v}_A)^2 + \frac{d\Omega^2}{d\ln R} < 0}$

Note: Can write discriminant as $\left[\frac{\kappa^2}{2} + (\mathbf{k}\cdot\mathbf{v}_A)^2\right]^2 - (\mathbf{k}\cdot\mathbf{v}_A)^2\left[(\mathbf{k}\cdot\mathbf{v}_A)^2 + \kappa^2 - 4\Omega^2\right]$

$$= \left[\frac{\kappa^2}{2} + (\mathbf{k}\cdot\mathbf{v}_A)^2\right]^2 - (\mathbf{k}\cdot\mathbf{v}_A)^2\left[(\mathbf{k}\cdot\mathbf{v}_A)^2 + \frac{d\Omega^2}{d\ln R}\right]$$

This means Keplerian disks are linearly unstable, provided the magnetic field isn't so strong that all the wavenumbers $k_z = 2\pi/\lambda_z$ that can fit within the height of the disk satisfy $k_z^2 v_{Az}^2 > \left|\frac{d\Omega^2}{d\ln R}\right|$ — then tension stabilizes all relevant modes.

See Balbus & Hawley 1998 Rev. Mod. Phys. for more.

PART VII

Magnetic reconnection

Magnetic reconnection refers to the topological rearrangement of magnetic-field lines that converts magnetic energy to plasma energy. In these lecture notes, we will assume that such a rearrangement is facilitated by a spatially constant Ohmic resistivity, as might occur in a well-ionized collisional fluid:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}.$$

This assumption is obviously not warranted in hot, dilute astrophysical systems, such as the collisionless solar wind, or in poorly ionized systems, like molecular clouds and pre-stellar cores. But let us assume this anyhow, knowing that (i) the physics of reconnection in even the simplest of systems is surprisingly rich and complex, and (ii) there is a huge amount of literature on all aspects of magnetic reconnection in a wide variety of environments. This part of the lecture notes is not intended as a replacement of that literature, nor a synopsis of current research in the field (particularly in the laboratory and the Earth's magnetosheath). What follows is an incomplete presentation of a few key highlights in the theory of magnetic reconnection, which will hopefully provide enough pedagogical value and inspiration to encourage you to dig into the literature further. For that, I recommend that you start with the excellent review articles by Zweibel & Yamada (2009), Yamada *et al.* (2010), and Loureiro & Uzdensky (2016).

VII.1. Tearing instability

VII.1.1. *Formulation of the problem*

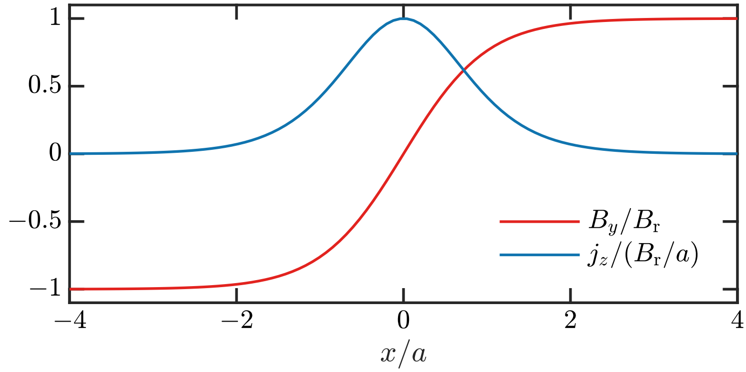
We begin by analyzing the stability of a simple stationary equilibrium in which the magnetic field reverses across $x = 0$:

$$\boldsymbol{B}_0 = B_y(x)\hat{\boldsymbol{y}} + B_{\rm g}\hat{\boldsymbol{z}}, \tag{VII.1}$$

where $B_y(x)$ is an odd function and $B_{\rm g} = \text{const}$ denotes the guide field. A oft-employed profile for $B_y(x)$ is the Harris (1962) sheet:

$$B_y(x) = B_{\rm r} \tanh\left(\frac{x}{a}\right), \tag{VII.2}$$

where $B_{\rm r}$ is the asymptotic value of the reconnecting field and a is the characteristic scale length of the current sheet. Its profile, and the associated current density $j_z = (B_{\rm r}/a)\,\mathrm{sech}^2(x/a)$, are shown in the figure below:



To have a stationary equilibrium, we require something to balance the magnetic pressure gradient implied by (VII.1). One option is to allow the thermal pressure in the background to vary so that $P(x) + B_y^2(x)/8\pi = \text{const}$. Another option is to make the guide field also depend upon x and arrange for the total magnetic-field strength to be constant; for example, if the reconnecting field satisfies (VII.2), then the guide field should satisfy $B_{\rm g}(x) = B_{\rm r}\,\mathrm{sech}(x/a)$. Either way, the total pressure $P + B^2/8\pi$ in the equilibrium should be constant. We might consider allowing the background magnetic pressure gradient to be unbalanced, which would be fine if the tearing instability were to grow much faster than the current sheet would evolve globally from being initialized out of equilibrium. But the latter would occur on the Alfvén-crossing time of the sheet, $\sim a/v_{\rm A,r}$, which we will find is actually *shorter* than the characteristic growth time of the fastest-growing tearing mode. Many presentations of the tearing instability conveniently omit this point.

We start by linearizing the momentum and non-ideal induction equations about the x-dependent, pressure-balanced equilibrium (cf. (VI.5)):

$$\rho\frac{\partial\delta\boldsymbol{u}}{\partial t} = -\boldsymbol{\nabla}\left(\delta P + \frac{B_{\rm g}\delta B_z}{4\pi} + \frac{B_y\delta B_y}{4\pi}\right) + \frac{B_{\rm g}}{4\pi}\frac{\partial\delta\boldsymbol{B}}{\partial z} + \frac{B_y}{4\pi}\frac{\partial\delta\boldsymbol{B}}{\partial y} + \frac{{\rm d}B_y}{{\rm d}x}\frac{\delta B_x}{4\pi}\hat{\boldsymbol{y}}, \quad \text{(VII.3}a\text{)}$$

$$\frac{\partial\delta\boldsymbol{B}}{\partial t} = -\delta u_x\frac{{\rm d}B_y}{{\rm d}x}\hat{\boldsymbol{y}} + B_{\rm g}\frac{\partial\delta\boldsymbol{u}}{\partial z} + B_y\frac{\partial\delta\boldsymbol{u}}{\partial y} + \eta\nabla^2\delta\boldsymbol{B}. \quad \text{(VII.3}b\text{)}$$

Here we have taken the plasma to be incompressible, $\boldsymbol{\nabla}\cdot\delta\boldsymbol{u} = 0$; note further that the mass density ρ in (VII.3a) refers only to its time-independent background value. As a result, we don't need the continuity equation to close our system of equations, and the energy equation is replaced by the requirement that the divergence of $(1/\rho)\times$ (VII.3a) vanish (which constrains δP). We have also ignored resistive diffusion of the background current-sheet profile, which should be fine as long as the growth rate of the tearing mode is $\gg \eta B_{\rm r}/a^2$; this will amount to the condition that the Lundquist number of the current sheet, $S_a \doteq v_{\rm A,r}a/\eta$ where $v_{\rm A,r} \doteq B_{\rm r}/\sqrt{4\pi\rho_0}$ is the Alfvén speed associated with the reconnecting field and ρ_0 is the mass density at the center of the reconnecting layer $(x=0)$, satisfies $S_a^{1/2} \gg 1$.

The next step, which is not at all necessary but is standard and simplifies this first pass through the analysis, is to assume that the perturbations have no z component and do not vary in the z direction, thereby reducing the problem completely to 2D. In this case, the guide field $B_{\rm g}$ disappears from the analysis, and incompressibility and the solenoidality constraint on the magnetic field allow us to write the perturbed velocity and magnetic field in terms of scalar potentials whose gradients lie in the x-y plane:

$$\delta\boldsymbol{u} = \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\phi, \qquad \frac{\delta\boldsymbol{B}}{\sqrt{4\pi\rho_0}} = \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\psi, \quad \text{(VII.4)}$$

Likewise, we may associate the background reconnecting field B_y with a scalar potential: $B_y(x)/\sqrt{4\pi\rho_0} = \varPsi'(x)$, where the prime denotes differentiation with respect to x. For example, if B_y is taken to be the Harris-sheet profile (VII.2), then $\varPsi(x) = v_{\mathrm{A,r}} a \ln[\cosh(x/a)]$. Making these simplifications in (VII.3), substituting in (VII.4), and simplifying leads to

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\phi\right) = -\boldsymbol{\nabla}\left(\frac{\delta P}{\rho_0} + \varPsi'\frac{\partial\psi}{\partial x}\right) + \varPsi'\frac{\partial}{\partial y}\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\psi - \varPsi''\frac{\partial\psi}{\partial y}\hat{\boldsymbol{y}}, \tag{VII.5a}$$

$$\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\frac{\partial}{\partial t}\psi = \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\left(\varPsi'\frac{\partial\phi}{\partial y} + \eta\nabla^2\psi\right). \tag{VII.5b}$$

We may remove the $\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}$ from both sides of the latter equation without consequence. Finally, to eliminate $\boldsymbol{\nabla}\delta P$ from (VII.5a), we take the curl of (VII.5a) and use $\boldsymbol{\nabla}\times\boldsymbol{\nabla} = 0$. After the use of some vector identities and rearranging, our final equations are

$$\frac{\rho}{\rho_0}\frac{\partial}{\partial t}\left(\nabla^2\phi + \frac{\mathrm{d}\ln\rho}{\mathrm{d}x}\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = \varPsi'\frac{\partial}{\partial y}\nabla^2\psi - \varPsi'''\frac{\partial\psi}{\partial y}, \tag{VII.6a}$$

$$\frac{\partial\psi}{\partial t} - \varPsi'\frac{\partial\phi}{\partial y} = \eta\nabla^2\psi. \tag{VII.6b}$$

The former equation describes the evolution of the fluid vorticity, $\boldsymbol{\nabla}\times\boldsymbol{u} = \hat{\boldsymbol{z}}\nabla^2\psi$.

Because (VII.6) are linear in the perturbation amplitudes, and because the background only depends upon x, we are allowed to adopt the solutions

$$\psi(t,x,y) = \psi(x)\,\mathrm{e}^{\mathrm{i}ky+\gamma t} \qquad \text{and} \qquad \phi(t,x,y) = \phi(x)\,\mathrm{e}^{\mathrm{i}ky+\gamma t}, \tag{VII.7}$$

where k is the wavenumber and γ is the rate at which the perturbations will grow or decay. In this case, $\partial/\partial t \to \gamma$ and $\partial/\partial y \to \mathrm{i}k$, leaving us with

$$\gamma\frac{\rho}{\rho_0}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - k^2 + \frac{\mathrm{d}\ln\rho}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\right)\phi = \mathrm{i}k\varPsi'\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - k^2\right)\psi - \mathrm{i}k\varPsi'''\psi, \tag{VII.8a}$$

$$\gamma\psi - \mathrm{i}k\varPsi'\phi = \eta\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - k^2\right)\psi. \tag{VII.8b}$$

The trick to solving this set of equations is to realize that, as η tends towards zero, the derivative on the right-hand side of (VII.8b) must grow to balance the terms on the left-hand side. In other words, a boundary layer forms about $x = 0$, outside of which the system satisfies the ideal-MHD equations and inside of which the resistivity is important. The width of this boundary layer is customarily denoted δ_{in}, and much of reconnection theory rests on determining its size given the various attributes of the host plasma. To do so, we will first solve (VII.8a) and (VII.8b) in the "outer region", where the resistivity is negligible and the system behaves as though it were ideal. Then they will be solved in the "inner region", where the resistivity dominates and $k \sim a^{-1} \ll \mathrm{d}/\mathrm{d}x \sim \delta_{\mathrm{in}}^{-1}$. The two solutions must asymptotically join onto one another; this matching, along with boundary conditions at $x = 0$ and $\pm\infty$, will determine the full solution.

Before proceeding with this program, it will be advantageous to define the resistive and Alfvén timescales,

$$\tau_\eta \doteq \frac{a^2}{\eta} \quad \text{and} \quad \tau_{\mathrm{A}} \doteq \frac{1}{ka\varPsi''(0)} = \frac{1}{kv_{\mathrm{A,r}}}, \tag{VII.9}$$

respectively. We further assume $\tau_\eta^{-1} \ll \gamma \ll \tau_{\mathrm{A}}^{-1}$, i.e. the tearing mode grows faster than it takes for the entirety of the current sheet to resistively diffuse but slower than it takes for an Alfvén wave to cross k^{-1}. Physically, this implies that the outer solution results from neglecting the plasma's inertia and Ohmic resistivity.

VII.1.2. *Outer equation*

Adopting the ordering $\tau_\eta^{-1} \ll \gamma \ll \tau_{\rm A}^{-1}$, equations (VII.8*a*) and (VII.8*b*) reduce to

$$0 = \left(\frac{{\rm d}^2}{{\rm d}x^2} - k^2 - \frac{\Psi'''}{\Psi'} \right) \psi_{\rm out} \quad \text{and} \quad \phi_{\rm out} = \frac{\gamma}{{\rm i}k\Psi'} \psi_{\rm out}. \tag{VII.10}$$

Note that $\Psi'''/\Psi' = B_y''/B_y$ measures the gradient of the current density, and so different current-sheet profiles will result in different solutions to (VII.10). Regardless of the exact current-sheet profile, however, both $\phi_{\rm out}$ and $\psi_{\rm out}$ must tend to zero as $x \to \pm\infty$. Also, since the y-component of the perturbed magnetic field must reverse direction at $x = 0$, $\psi_{\rm out}$ must have a discontinuous derivative there, corresponding to a singular current. Indeed, it is this discontinuity that characterizes the free energy available to reconnect, quantified by the tearing-instability parameter

$$\varDelta' \doteq \frac{1}{\psi_{\rm out}(0)} \frac{{\rm d}\psi_{\rm out}}{{\rm d}x} \bigg|_{-0}^{+0}, \tag{VII.11}$$

and that ultimately warrants consideration of a resistive inner layer.

VII.1.3. *Inner equation*

In the inner region where $k \ll {\rm d}/{\rm d}x \sim \delta_{\rm in}^{-1}$, the dominant terms in (VII.8*a*) and (VII.8*b*) are

$$\gamma \frac{\rho}{\rho_0} \frac{{\rm d}^2\phi_{\rm in}}{{\rm d}x^2} = {\rm i}k\Psi' \frac{{\rm d}^2\psi_{\rm in}}{{\rm d}x^2}, \tag{VII.12}$$

$$\gamma\psi_{\rm in} - {\rm i}k\phi_{\rm in}\Psi' = \eta \frac{{\rm d}^2\psi_{\rm in}}{{\rm d}x^2}. \tag{VII.13}$$

Note that ρ, whose gradient length scale is a, may be taken as constant over the inner-layer thickness $\delta_{\rm in} \ll a$, and so the pre-factor ρ/ρ_0 in (VII.12) is $\simeq$1. These equations may be solved analytically provided some amenable form of Ψ'. Because we are deep within the current sheet, the leading-order term in a Taylor expansion will suffice, *viz.*, $\Psi' \approx \Psi''(0)x = v_{\rm A,r}(x/a)$. Then (VII.12) and (VII.13) may be straightforwardly combined to obtain

$$\frac{{\rm d}^2\psi_{\rm in}}{{\rm d}x^2} = -\left[\frac{\gamma}{k\Psi''(0)} \right]^2 \frac{1}{x} \frac{{\rm d}^2}{{\rm d}x^2} \left[\frac{1}{x} \left(1 - \frac{\eta}{\gamma} \frac{{\rm d}^2}{{\rm d}x^2} \right) \psi_{\rm in} \right]. \tag{VII.14}$$

With some effort, this equation can actually be solved for $\psi_{\rm in}$ analytically. I'll show you how below. But even without that effort, equation (VII.14) may be used to estimate the width of the boundary layer, $\delta_{\rm in}$:

$$1 \sim (\gamma a \tau_{\rm A})^2 \frac{\eta}{\gamma \delta_{\rm in}^4} \quad \Longrightarrow \quad \frac{\delta_{\rm in}}{a} \sim \left(\frac{\gamma \tau_{\rm A}^2}{\tau_\eta} \right)^{1/4}. \tag{VII.15}$$

Note that $\delta_{\rm in}$ depends on k – each tearing mode k has a different boundary-layer width; because of this, each k will correspond to a different $\varDelta'$.

Normalizing lengthscales to $\delta_{\rm in}$ by introducing $\xi \doteq x/\delta_{\rm in}$, equation (VII.14) may be written as

$$\frac{{\rm d}^2\psi_{\rm in}}{{\rm d}\xi^2} = -\frac{1}{\xi} \frac{{\rm d}^2}{{\rm d}\xi^2} \left[\frac{1}{\xi} \left(\varLambda - \frac{{\rm d}^2}{{\rm d}\xi^2} \right) \psi_{\rm in} \right], \tag{VII.16}$$

where the eigenvalue $\varLambda \doteq \gamma^{3/2}\tau_{\rm A}\tau_\eta^{1/2} = \gamma\delta_{\rm in}^2/\eta$ is the growth rate of the tearing mode normalized by the rate of resistive diffusion across a layer of width $\delta_{\rm in}$. Provided we can solve (VII.16), the solution $\psi_{\rm in}$ must be matched onto the outer solution $\psi_{\rm out}$. This is done by equating the discontinuity in $\psi_{\rm out}$, quantified by $\varDelta'$ (see (VII.11)), to the total change in $\mathrm{d}\psi_{\rm in}/\mathrm{d}x$ across the inner region, *viz.*,

$$\varDelta' = \frac{2}{\delta_{\rm in}}\int_0^1 \mathrm{d}\xi\, \frac{1}{\psi_{\rm in}(0)}\frac{\mathrm{d}^2\psi_{\rm in}}{\mathrm{d}\xi^2}.$$

(The factor of 2 is because the solution is odd, and so the total change across the $x=0$ surface is twice the change measured for $x>0$.) The upper limit on the integral can be extended to $+\infty$ by committing only a $\sim$10% error:

$$\varDelta' = \frac{2}{\delta_{\rm in}}\int_0^\infty \mathrm{d}\xi\, \frac{1}{\psi_{\rm in}(0)}\frac{\mathrm{d}^2\psi_{\rm in}}{\mathrm{d}\xi^2}. \tag{VII.17}$$

So, find $\psi(\xi)$ by solving the inner equation (VII.16), compute the integral in (VII.17), and invert the answer to obtain the growth rate in terms of $\varDelta'$.

Before carrying out that program, it will be useful to further simply (VII.16) by introducing

$$\chi(\xi) \doteq \xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{\psi_{\rm in}(\xi)}{\xi}\right], \tag{VII.18}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{\xi^2}\frac{\mathrm{d}\chi}{\mathrm{d}\xi}\right) - \left(1+\frac{\varLambda}{\xi^2}\right)\chi\right] = 0. \tag{VII.19}$$

Integrating this equation once and, for reasons that will eventually become apparent, setting the integration constant to $-\chi_\infty$, we find

$$\xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{\xi^2}\frac{\mathrm{d}\chi}{\mathrm{d}\xi}\right) - \left(\xi^2+\varLambda\right)\chi = -\chi_\infty\xi^2. \tag{VII.20}$$

Once this equation is solved, the inner solution is obtained using (cf. (VII.18))

$$\psi_{\rm in}(\xi) = -\xi\int_\xi^\infty \mathrm{d}x\,\frac{\chi(x)}{x^2} = -\xi\int_\xi^\infty \mathrm{d}x\,\frac{\chi'(x)}{x} - \chi(\xi), \tag{VII.21}$$

which may then be plugged into (VII.17) to compute $\varDelta'$.

VII.1.4. *Approximate solutions*

There are a few ways to solve (VII.10) and (VII.20), none of which are particularly obvious. However, it's possible to obtain scaling laws for $\varDelta'$ and the tearing-mode growth rate γ without actually doing so. In fact, the answers obtained in this way differ from those obtained by a more mathematically rigorous solution (see §VII.1.5) by only order-unity coefficients. Nice.

We start with (VII.10), the outer equation. With some knowledge that the fastest-growing modes occur at long wavelengths ($ka \ll 1$), we can make some progress by simply dropping the middle term in (VII.10). Then, so long as B_y varies faster within $|x| \lesssim a$ than it does at $|x| \gg a$, we can estimate

$$\varDelta' \sim \frac{1}{ka^2}. \tag{VII.22}$$

(This scaling is exact for the Harris-sheet profile, solved for in §VII.1.5.) One may formalize this estimate somewhat (Loureiro *et al.* 2007, 2013) by quantifying what "varies faster within $|x| \lesssim a$ than it does at $|x| \gg a$" means, but not much is gained intuitively by going that route, and the estimate (VII.22) will suffice.

As for the inner equation (VII.16), we know from (VII.20) that, whatever its solution, $\psi_{\rm in}(\xi)$ only depends on the parameter Λ. Thus, equation (VII.17) may be written as

$$\Delta' \delta_{\rm in} = f(\Lambda) \tag{VII.23}$$

for some function $f(\Lambda)$. Combining (VII.22) and (VII.23) yields an expression for the growth rate, provided we can invert $f(\Lambda)$. Fortunately, we can, at least in certain limits.

The first limit is the so-called "constant-ψ approximation" or "FKR regime", which corresponds to $f(\Lambda) \sim \Lambda \ll 1$ (Furth *et al.* 1963). Then (VII.23) gives $\Delta' \delta_{\rm in} \sim \Lambda$, so that

$$\boxed{\gamma_{\rm FKR} \sim \tau_{\rm A}^{-2/5} \tau_\eta^{-3/5} (\Delta' a)^{4/5}\,, \quad \frac{\delta_{\rm in}}{a} \sim \left(\frac{\tau_{\rm A}}{\tau_\eta} \right)^{2/5} (\Delta' a)^{1/5}} \tag{VII.24}$$

With $\Delta' \sim 1/ka^2$ (see (VII.22)), these become

$$\frac{\gamma_{\rm FKR}}{v_{\rm A,r}/a} \sim (ka)^{-2/5} S_a^{-3/5}\,, \quad \frac{\delta_{\rm in}}{a} \sim (ka)^{-3/5} S_a^{-2/5}, \tag{VII.25}$$

where we have introduced the *Lundquist number*

$$S_a \doteq \frac{a v_{\rm A,r}}{\eta}. \tag{VII.26}$$

Note that longer wavelengths have faster growth rates (the divergence as $k \to 0$ will be cured in the "Coppi" regime, in which the small-Δ' assumption breaks down – see below). This approximation results from setting $\psi_{\rm in} = \psi_{\rm in}(0)$ on the left-hand side of (VII.13), so that the inner equation (VII.13) becomes

$$\gamma \psi_{\rm in}(0) - {\rm i}k\phi_{\rm in} \Psi''(0) x = \eta \frac{{\rm d}^2 \psi_{\rm in}}{{\rm d}x^2}, \tag{VII.27}$$

and so (cf. (VII.20))

$$\xi^2 \frac{\rm d}{{\rm d}\xi} \left(\frac{1}{\xi^2} \frac{{\rm d}\chi}{{\rm d}\xi} \right) - \xi^2 (\chi - \chi_\infty) = -\Lambda \psi_{\rm in}(0). \tag{VII.28}$$

In effect, we are assuming that the resistive diffusion time across the inner-layer thickness is much shorter than the instability growth time, i.e., $\gamma \ll \eta/\delta_{\rm in}^2$, so that $\psi_{\rm in}$ can be approximated as constant on the dynamical time scale. Using (VII.25) in this inequality requires $S_a \gg (\Delta' a)^4$. This is sometimes called the "small-Δ' regime".

The second limit is the "Coppi regime" or "large-Δ' regime", in which the constant-ψ approximation breaks down and $\gamma \sim \eta/\delta_{\rm in}^2$. This occurs for $\Lambda \sim 1^-$, at which $f(\Lambda) \to \infty$. The growth rate then becomes independent of Δ' and we have

$$\boxed{\gamma_{\rm Coppi} \sim \tau_{\rm A}^{-2/3} \tau_\eta^{-1/3}\,, \quad \frac{\delta_{\rm in}}{a} \sim \left(\frac{\tau_{\rm A}}{\tau_\eta} \right)^{1/3}} \tag{VII.29}$$

In terms of the tearing-mode wavenumber k and the Lundquist number S_a,

$$\frac{\gamma_{\rm Coppi}}{v_{\rm A,r}/a} \sim (ka)^{2/3} S_a^{-1/3}\,, \quad \frac{\delta_{\rm in}}{a} \sim (ka)^{-1/3} S_a^{-1/3}. \tag{VII.30}$$

In this limit, the shorter wavelengths have faster growth rates, opposite to the FKR scaling (VII.25). This suggests a maximally growing mode, whose growth rate $\gamma_{\rm max}$ and wavenumber $k_{\rm max}$ may be estimated by matching the FKR solution (VII.25) to the Coppi one (VII.30):

$$\gamma_{\rm FKR} \sim \gamma_{\rm Coppi} \quad \Longrightarrow \quad k_{\rm max} a \sim S_a^{-1/4}, \quad \frac{\gamma_{\rm max}}{v_{\rm A,r}/a} \sim S_a^{-1/2}, \quad \frac{\delta_{\rm in}}{a} \sim S_a^{-1/4}. \tag{VII.31}$$

Note that the FKR (Coppi) regime corresponds to $k > k_{\rm max}$ ($k < k_{\rm max}$).

Of course, all of these scalings make sense only if the modes can fit into the current sheet, i.e., $kL \gtrsim 1$, where L is the length of the current sheet. For the maximally growing mode to be viable thus requires a current-sheet aspect ratio of $L/a \gtrsim S_a^{1/4}$. If this inequality is not satisfied, then the fastest-growing mode will be the FKR mode (VII.25) with the smallest possible allowed wavenumber, $kL \sim 1$. Thus, low-aspect-ratio sheets with $L/a \ll S_a^{1/4}$ will develop tearing perturbations comprising just one or two islands; the high-aspect-ratio sheets, in which the Coppi regime is accessible, will instead spawn whole chains comprising $\sim k_{\rm max} L$ islands.

VII.1.5. *Exact solution for a Harris sheet*

With some (read: a lot of) effort, one can be more precise than the solutions obtained in the previous section. For that task, let us adopt the equilibrium flux function $\Psi = a v_{\rm A,r} \ln[\cosh(x/a)]$, corresponding to the Harris-sheet profile (VII.2). Then (VII.10) becomes

$$\left[\frac{{\rm d}^2}{{\rm d}x^2} - k^2 + \frac{2}{a^2}\,{\rm sech}^2\Bigl(\frac{x}{a}\Bigr)\right]\psi_{\rm out} = 0 \quad \text{and} \quad \phi_{\rm out} = -{\rm i}\gamma\tau_{\rm A} \coth\Bigl(\frac{x}{a}\Bigr)\psi_{\rm out}. \tag{VII.32}$$

The former equation can be solved by changing variables to $\mu = \tanh(x/a)$, so that ${\rm sech}^2(x/a) = (1-\mu^2)^{-1}$ and

$$\frac{\rm d}{{\rm d}x} = \frac{1-\mu^2}{a}\frac{\rm d}{{\rm d}\mu}, \quad \frac{{\rm d}^2}{{\rm d}x^2} = \frac{1-\mu^2}{a}\frac{\rm d}{{\rm d}\mu}\frac{1-\mu^2}{a}\frac{\rm d}{{\rm d}\mu}.$$

Then (VII.32) becomes

$$\left[\frac{\rm d}{{\rm d}\mu}(1-\mu^2)\frac{\rm d}{{\rm d}\mu} + 2 - \frac{k^2a^2}{1-\mu^2}\right]\psi_{\rm out} = 0 \quad \text{and} \quad \phi_{\rm out} = -{\rm i}\gamma\tau_{\rm A}\frac{\psi_{\rm out}}{\mu}, \tag{VII.33}$$

the first of which you might recognize as the associated Legendre equation

$$\left[\frac{\rm d}{{\rm d}\mu}(1-\mu^2)\frac{\rm d}{{\rm d}\mu} + \ell(\ell+1) - \frac{m^2}{1-\mu^2}\right]P_\ell^m(\mu) = 0$$

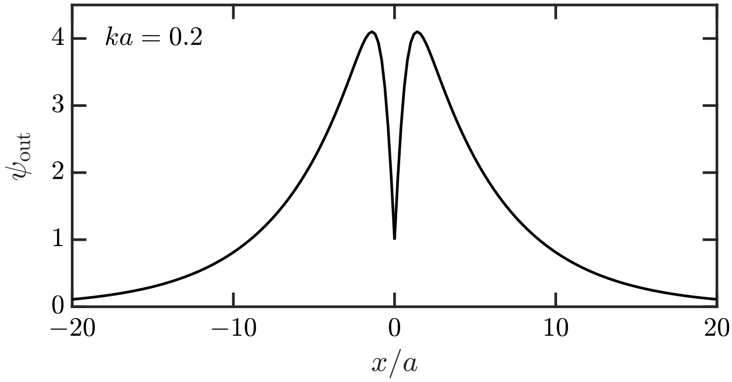
with $\ell = 1$ and $m = ka$. Transforming the boundary conditions $\psi(\pm\infty) = 0$ into $\psi(\mu = \pm 1) = 0$ and enforcing $\psi(\mu) = \psi(-\mu)$, the solution to (VII.33) is thus

$$\psi_{\rm out} = C_{1m} P_1^m(\mu), \tag{VII.34}$$

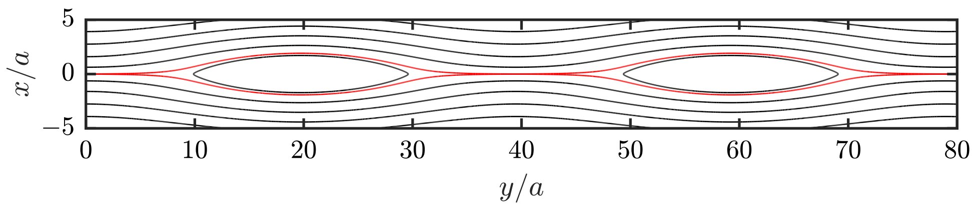
with $C_{1m} = {\rm const}$. If you can't picture in your head what the first associated Legendre polynomial with non-integer m looks like – I know I can't – you may like to know that the outer solution may be equivalently written as

$$\psi_{\rm out}(x) = C'_{1m}{\rm e}^{-kx}\left[1 + \frac{1}{ka}\tanh\Bigl(\frac{x}{a}\Bigr)\right] \tag{VII.35}$$

for $\xi \geqslant 0$, where $C'_{1m} = {\rm const}$. (Note that $\psi_{\rm out}(-\xi) = \psi_{\rm out}(\xi)$.) Visually:



Recall that Δ' measures the discontinuity of $\mathrm{d}\psi_{\rm out}/\mathrm{d}x$ at $x = 0$ (see (VII.11)). Restoring the $\cos(ky)$ dependence of $\psi_{\rm out}$ provides the following isocontours of $\psi_{\rm out}$, which are equivalently the magnetic-field lines:



For ease of visualization, I plotted two y wavelengths using $ka = (2\pi)^{-1}$ and set the tearing-mode amplitude to 0.2 (which in a realistic system would be well outside of the linear regime). The red line is called the *separatrix*; it serves as the boundary between the inside of each magnetic island and the surrounding field lines, and runs through the "X-points" that lie at $y = 0, 2\pi/k, 4\pi/k$, etc. To determine the island width w, we follow the isocontour that starts from the X-point at, say, $(x, y) = (0, 0)$ and set its x location when $y = \pi/k$ equal to the half-width $w/2$:

$$\cancelto{0}{\Psi(0)} + \underbrace{\psi(0,0)}_{=\,\psi_{\rm out}(0)} = \Psi(w/2) + \psi(w/2,\pi/k) \approx \frac{1}{2}\Psi''(0)\frac{w^2}{4} + \underbrace{\psi(w/2,\pi/k)}_{=\,-\psi_{\rm out}(w/2)},$$

where we've used $\cos(\pi) = -1$ and approximated $\Psi(x)$ a distance x away from the neutral line by its Taylor expansion, $(1/2)\Psi''(0)x^2$. If we then take ψ to be approximately constant within the island, *viz.* $\psi_{\rm out}(w/2) \approx \psi_{\rm out}(0)$, we find that island width satisfies

$$w \approx 4\sqrt{\frac{\psi(0)}{\Psi''(0)}}.$$

Note that $\Psi''(0) = v_{\rm A,r}/a$ for the Harris-sheet profile. Solving for the mode amplitude C_{1m} (or C'_{1m}) requires matching onto the inner solution, but even before doing that we can compute $\varDelta'$ using $\psi_{\rm out} \propto P_1^m(\mu)$ in (VII.11):[9]

$$\varDelta' a = \frac{1}{P_1^m(0)} \frac{\mathrm{d}P_1^m}{\mathrm{d}\mu}\bigg|_{-0}^{+0} = \frac{2}{P_1^m(0)} \frac{\mathrm{d}P_1^m}{\mathrm{d}\mu}\bigg|_{\mu=0} = 2\left(\frac{1}{m} - m\right)$$
$$= 2\left(\frac{1}{ka} - ka\right). \qquad \text{(VII.36)}$$

Note that $\varDelta' > 0$ requires $ka < 1$ – any unstable mode must have an extent at least as large as the current-sheet thickness. This places an upper limit on the wavenumber of the FKR modes (VII.25).

As for the inner equation, let us use its compact form (VII.20), repeated here for convenience:

$$\xi^2 \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{\xi^2}\frac{\mathrm{d}\chi}{\mathrm{d}\xi}\right) - \left(\xi^2 + \varLambda\right)\chi = -\chi_\infty \xi^2, \qquad \text{(VII.37)}$$

where $\varLambda \doteq \gamma^{3/2}\tau_{\rm A}\tau_\eta^{1/2}$. There are a few ways to solve (VII.37), none of which are particularly obvious. One way, explained in Appendix A of Ara *et al.* (1978), is as follows. Write

$$\chi = \chi_\infty \sum_{n=0}^{\infty} a_n L_n^{(-3/2)}(\xi^2)\, \mathrm{e}^{-\xi^2/2}, \qquad \text{(VII.38)}$$

where $L_n^\alpha(z)$ are the associated Laguerre (or "Sonine") polynomials satisfying

$$z\frac{\mathrm{d}^2 L_n^{(\alpha)}}{\mathrm{d}z^2} + (\alpha + 1 - z)\frac{\mathrm{d}L_n^{(\alpha)}}{\mathrm{d}z} + nL_n^{(\alpha)} = 0. \qquad \text{(VII.39)}$$

Substitute this decomposition into (VII.20) and use the recursion relations

$$\frac{\mathrm{d}L_n^\alpha}{\mathrm{d}z} = -L_{n-1}^{\alpha+1}(z) \text{ if } 1 \leqslant n \text{ } (= 0 \text{ otherwise}),$$
$$nL_n^{(-3/2)}(z) = -\left(z + \frac{1}{2}\right)L_{n-1}^{(-1/2)}(z) - zL_{n-2}^{(1/2)}(z),$$

to obtain

$$\sum_{n=0}^{\infty} a_n\, \xi^{-2}\, \mathrm{e}^{-\xi^2/2} L_n^{(-3/2)}(\xi^2)\big(4n + \varLambda - 1\big) = 1. \qquad \text{(VII.40)}$$

Multiply this by $\mathrm{e}^{-\xi^2/2}\xi^{-1}L_m^{-3/2}$, integrate, and use the orthogonality relation

$$\int_0^\infty \mathrm{d}z\, \mathrm{e}^{-z} z^\alpha L_m^\alpha L_n^\alpha = \delta_{mn}\frac{\varGamma(n+\alpha+1)}{\varGamma(n+1)}$$

to find that

$$a_n \frac{(n-3/2)!}{n!}(4n + \varLambda - 1) = \int_0^\infty \mathrm{d}z\, z^{-1/2}\mathrm{e}^{-z/2}L_n^{-3/2}$$
$$= \int_0^\infty \mathrm{d}z\, z^{-1/2}\mathrm{e}^{-z/2}(L_n^{-1/2} - L_{n-1}^{-1/2})$$
$$= \sqrt{2}(-1)^n\left[\frac{\varGamma(n+1/2)}{\varGamma(n+1)} + \frac{\varGamma(n-1/2)}{\varGamma(n)}\right]$$
$$\Longrightarrow \quad a_n = \frac{(-1)^n}{\sqrt{2}}\frac{4n-1}{4n+\varLambda-1}.$$

[9]See https://dlmf.nist.gov/14.5 for information on $P_\ell^m(0)$ and $\mathrm{d}P_\ell^m/\mathrm{d}\mu|_{\mu=0}$.

Thus, equation (VII.38) becomes[10]

$$\chi = \frac{\chi_\infty}{\sqrt{2}}\,\mathrm{e}^{-\xi^2/2}\sum_{n=0}^{\infty}(-1)^n L_n^{-3/2}(\xi^2)\frac{4n-1}{4n+\Lambda-1} = \xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\psi_{\rm in}}{\xi}, \tag{VII.41}$$

which may be solved for $\psi_{\rm in}$ following (VII.21).

Actually doing so and plugging the solution into (VII.17) to compute Δ' ain't easy, as it involves a lot of non-standard math. I may LaTeX those steps up one day, but, for now, I'll just skip to the answer:

$$\Delta'\delta_{\rm in} = f(\Lambda) \doteq \frac{\pi}{2}\frac{\Gamma[(\Lambda+3)/4]}{\Gamma[(\Lambda+5)/4]}\frac{\Lambda}{1-\Lambda}. \tag{VII.42}$$

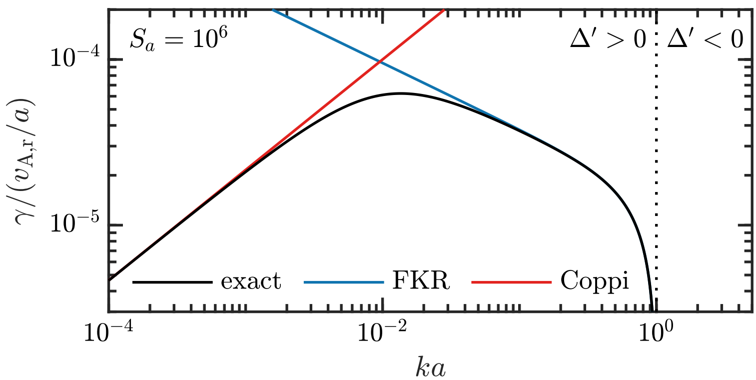
This is an implicit equation for Γ, which may be solved numerically (see figure below). But it's possible to recover our approximate results (VII.24) and (VII.29) in their respective limits. For $\Lambda \ll 1$,

$$f(\Lambda) \approx \frac{\pi}{2}\frac{\Gamma(3/4)}{\Gamma(5/4)}\Lambda \simeq 2.124\,\Lambda \quad\Longrightarrow\quad \gamma \approx 0.547\,\tau_{\rm A}^{-2/5}\tau_\eta^{-3/5}(\Delta' a)^{4/5}. \tag{VII.43}$$

Our approximate result for this FKR regime, equation (VII.24), is off by only a factor of 0.547 – not too bad. For $\Lambda = 1^-$,

$$f(\Lambda) \approx \frac{\pi}{2}\frac{\Gamma(1)}{\Gamma(3/2)}\frac{1}{1-\Lambda} = \frac{\sqrt{\pi}}{1-\Lambda} \quad\Longrightarrow\quad \gamma \approx \tau_{\rm A}^{-2/3}\tau_\eta^{-1/3} - \mathcal{O}\!\left(\frac{kv_{\rm A,r}}{\Delta' a}\right). \tag{VII.44}$$

This matches our Coppi-regime estimate, (VII.29). These asymptotic solutions actually do rather well across the full range of wavenumbers:



It also appears that we are well justified in estimating the maximally growing mode by matching the FKR and Coppi expressions (as in (VII.31)). These regimes also occur where we anticipated, with $f(\Lambda) = \Delta'\delta_{\rm in}$ being $\ll 1$ ($\gg 1$) in the FKR (Coppi) regime:

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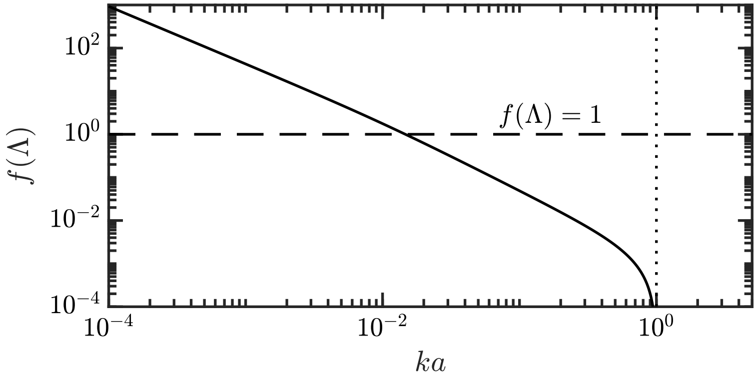
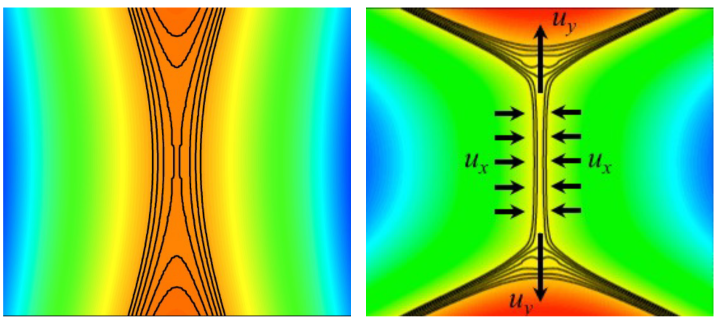
[10]Note that we cannot use the expansion (VII.38) if $\Lambda = 1$.



Thus the "small-Δ'" / "large-Δ'" phraseology.

VII.1.6. *Nonlinear evolution and X-point collapse*

How long does this linear phase, in which the tearing modes grow exponentially, last? That depends on the Δ' of the mode. If the Coppi regime is accessible – i.e., if the maximally growing wavenumber $k_{\max}$ (see (VII.31)) that results in $\Delta'\delta_{\rm in} \gtrsim 1$ also satisfies $k_{\max}a < 1$ – then X-point collapse is essentially instantaneous once the width $w = 4\sqrt{-\psi(0)/\Psi''(0)}$ of the exponentially growing island reaches $\delta_{\rm in}$. At this moment, $w\Delta'$ is also $\sim$1, and so the deformations of the current sheet by the nonlinear islands have driven the regions between the X-points to marginal stability. If the fastest-growing available modes are instead FKR-like, then there is a gap between when the nonlinear regime begins ($w \sim \delta_{\rm in}$) and when it ends ($w\Delta' \sim 1$). In between occurs a period of secular growth called the Rutherford (1973) stage, in which $\dot{w} \sim \eta\Delta'(w)$, the argument of Δ' indicating that the logarithmic derivative of $\psi_{\rm out}$ is to be taken across the island (rather than across the inner-layer width).[11] During this slow growth stage, the initially unstable current profile flattens and conditions are set up for the collapse of the inter-island X points (Waelbroeck 1993; Loureiro *et al.* 2005). The figure below, adapted from Loureiro *et al.* (2005), shows contours of ψ at the beginning of X-point collapse (left) and the formation of an embedded, high-aspect ratio current sheet (right):

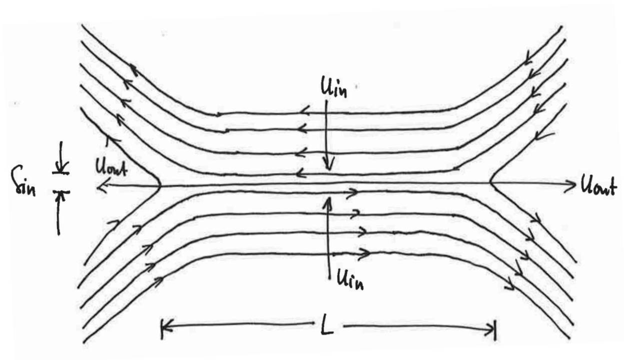


This current sheet is reminiscent of the now-famous Sweet–Parker configuration.

[11] Rutherford (1973) did not predict a saturation amplitude for the algebraically growing nonlinear tearing mode. Subsequent papers by Militello & Porcelli (2004) and Escande & Ottaviani (2004) ("POEM") derived a modified equation for the Rutherford stage, $\dot{w} \sim \eta(\Delta' - \alpha w/a^2)$ with α being a constant dependent upon the initial current-sheet geometry, thus predicting a saturated amplitude $w \sim \Delta' a^2$.

VII.2. Sweet–Parker reconnection

Peter Sweet (Sweet 1958) and Eugene Parker (Parker 1957) provided the first quantitative model of magnetic reconnection, envisioning it to be a steady-state process in which a two-dimensional, incompressible flow advects magnetic flux into a current sheet of length L and thickness $\delta_{\rm SP} \ll L$. It is through the latter dimension that plasma, accelerated in the direction along the current sheet by magnetic tension, is expelled in the form of an outflow:

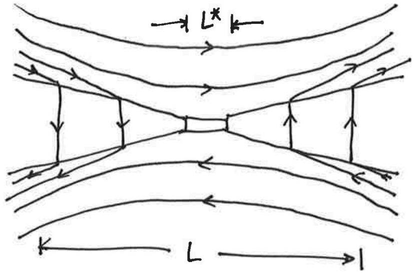


Steady state is achieved by (i) balancing the inflow velocity $u_{\rm in}$ and the outflow velocity $u_{\rm out}$ using mass conservation, $u_{\rm in}L \sim u_{\rm out}\delta_{\rm SP}$; (ii) balancing the advective and resistive electric fields so that all the inflowing magnetic flux is resistively destroyed, $u_{\rm in}v_{\rm A,r} \sim \eta j_z \sim \eta v_{\rm A,r}/\delta_{\rm SP}$; and (iii) stipulating that the outflows are Alfvénic, $u_{\rm out} \sim v_{\rm A,r}$. (This final ingredient follows from conservation of energy, with the magnetic energy flux into the sheet balancing the kinetic energy flux out of the sheet.) The result is

$$\frac{u_{\rm in}}{v_{\rm A,r}} \sim \frac{\delta_{\rm SP}}{L} \sim \left(\frac{v_{\rm A,r}L}{\eta}\right)^{-1/2} \doteq S^{-1/2}, \tag{VII.45}$$

where S is the Lundquist number (using the current-sheet length L as the normalizing lengthscale). In the solar corona, $S \sim 10^{12}$–10^{14}; in the Earth's magnetotail, $S \sim 10^{15}$–10^{16}; and in a modern tokamak like JET, $S \sim 10^{6}$–10^{8}. You can see that $S^{-1/2}$ is typically a very small number, and so Sweet–Parker (SP) reconnection is *slow* – not as slow as pure resistive diffusion, but slow in the sense that the reconnection rate $\tau_{\rm r}^{-1} \doteq u_{\rm in}/L \sim (v_{\rm A,r}/L)\, S^{-1/2}$ tends towards zero as $S \rightarrow \infty$. For example, the SP model predicts that a reconnection-driven solar flare in a $S \sim 10^{14}$ part of the solar corona should last ∼2 mths; instead, flares are observed to last between 15 min and 1 hr. Not good.

This mismatch between theory and observation was immediately appreciated, and spawned several attempts to formulate a model in which fast reconnection occurs. The culprit is the smallness of the resistive layer: the fact that it must be thin enough to make the current density large also means that the outflowing mass must pass through too small of an opening. One particularly notorious attempt to circumvent this constraint was proposed by Petschek (1964) (later revisited and amended by Kulsrud (2001)), in which the current-sheet length L was shortened at the expense of introducing four standing slow-mode shocks emanating from a central diffusion region:



The result is a logarithmic dependence of the reconnection rate on S, $\tau_{\rm r}^{-1} \sim (v_{\rm A,r}/L)\ln S$. Unfortunately, no convincing evidence for this type of reconnection has been found (Park *et al.* 1984; Biskamp 1986; Uzdensky & Kulsrud 2000; Malyshkin *et al.* 2005; Loureiro *et al.* 2005), even when Petschek's solution is used as an initial condition (Uzdensky & Kulsrud 2000).[12]

It is worth emphasizing that the failure of the SP model to explain magnetic reconnection as it occurs in nature is not due to any shortcoming of the theory itself. There are no obvious mistakes in the theory, which has been put on a rigorous footing (e.g., Uzdensky & Kulsrud 2000). Indeed, both numerical simulations (e.g., see figure 4(b) of Loureiro *et al.* 2005) and laboratory experiments (e.g., Ji *et al.* 1998) have measured reconnection rates in excellent agreement with the SP scalings (VII.45). What, then, is the issue?

VII.3. Plasmoid instability

Let us suspend judgement for the meantime and suppose that the SP model is correct. With tearing-mode theory in hand, let us ask the intriguing question of whether or not the steady-state SP current sheet is stable to tearing instabilities. One could of course go the route of rigorously doing the linear tearing theory using the SP solution as the background state, as Loureiro *et al.* (2007) did in a now-classic paper, but for our purposes it will be sufficient to simply replace the current-sheet thickness a in the tearing-mode theory of §VII.1 with $\delta_{\rm SP} \sim S^{-1/2}L$ (Tajima & Shibata 1997; Bhattacharjee *et al.* 2009; Loureiro *et al.* 2013). Focusing on the maximally growing tearing mode (VII.31),

$$k_{\rm max}L \sim \frac{L}{a}S_a^{-1/4} \longrightarrow \frac{L}{\delta_{\rm SP}}\left(\frac{v_{\rm A,r}\delta_{\rm SP}}{\eta}\right)^{-1/4} \sim S^{3/8}, \qquad \text{(VII.46}a\text{)}$$

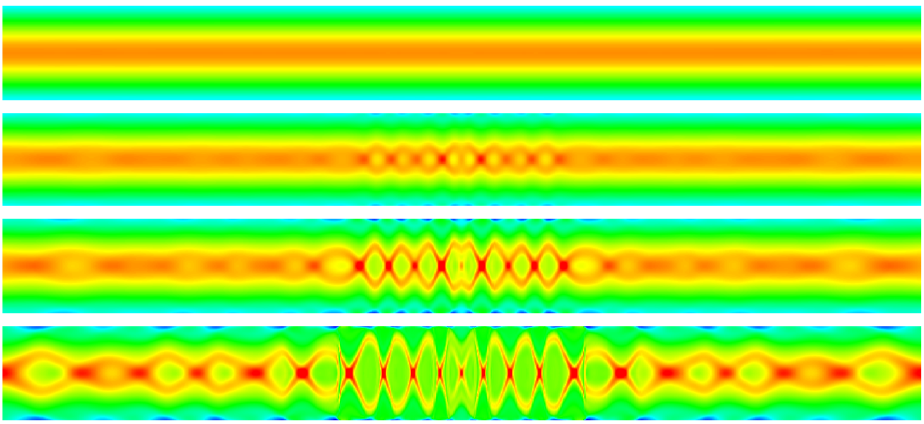
$$\frac{\gamma_{\rm max}}{v_{\rm A,r}/L} \sim \frac{L}{a}S_a^{-1/2} \longrightarrow \frac{L}{\delta_{\rm SP}}\left(\frac{v_{\rm A,r}\delta_{\rm SP}}{\eta}\right)^{-1/2} \sim S^{1/4}, \qquad \text{(VII.46}b\text{)}$$

$$\frac{\delta_{\rm in}}{L} \sim \frac{a}{L}S_a^{-1/4} \longrightarrow \frac{\delta_{\rm SP}}{L}\left(\frac{v_{\rm A,r}\delta_{\rm SP}}{\eta}\right)^{-1/4} \sim S^{-5/8}. \qquad \text{(VII.46}c\text{)}$$

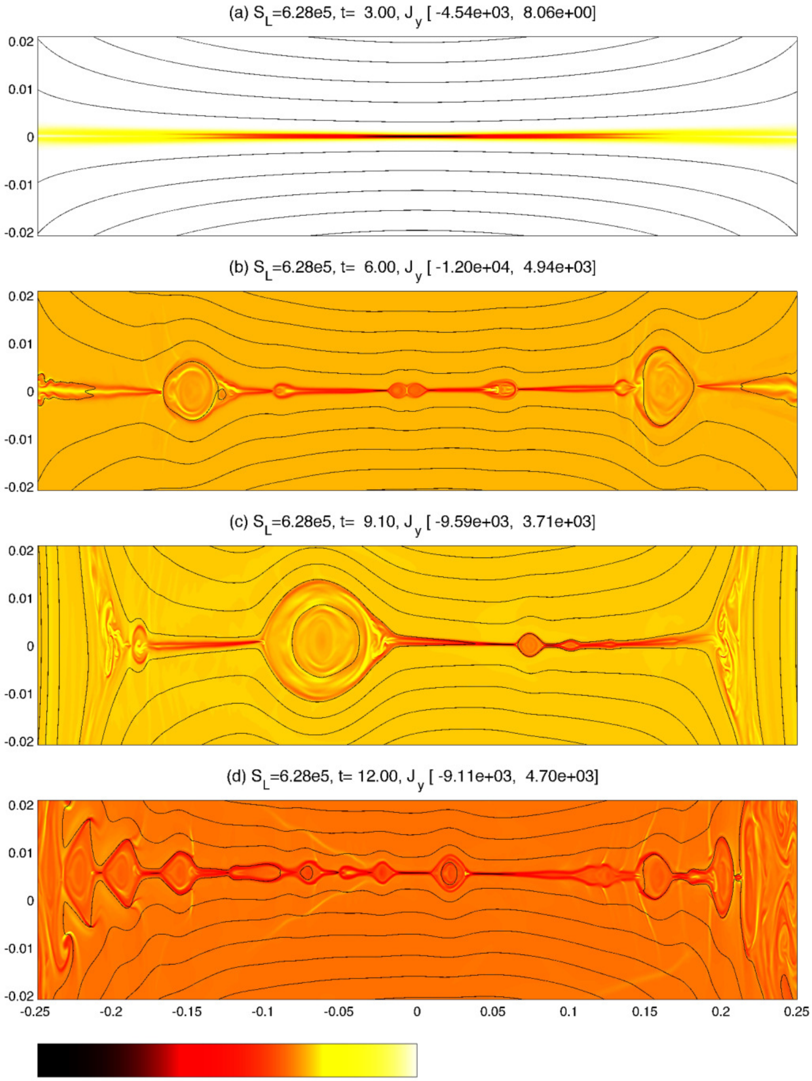
This is the *plasmoid instability* – essentially, the tearing instability of a SP current layer. Of course, the situation in question is very different than that obtained using the stationary equilibrium Harris sheet, perhaps most obviously because the former has background flows. These flows can be stabilizing in the tearing calculation, a possibility we have ignored in making the estimates in (VII.46). This may be circumvented, however, by demanding that $\gamma \gg v_{\rm A,r}/L$, $k_{\rm max}L \gg 1$, and $\delta_{\rm in}/\delta_{\rm SP} \ll 1$ – demands that may be satisfied if $S \gtrsim 10^4$. Indeed, it is at this critical Lundquist number that the plasmoid

[12]Petschek-like configurations do emerge when strongly localized (anomalous) resistivity profiles are used (Malyshkin *et al.* 2005; Sato & Hayashi 1979; Ugai 1995; Scholer 1989; Erkaev *et al.* 2000, 2001; Biskamp & Schwarz 2001), as might occur under collisionless conditions.

instability is (now routinely) observed to occur in numerical simulations of reconnection (e.g., Samtaney *et al.* 2009; Daughton *et al.* 2009; Bhattacharjee *et al.* 2009; Ni *et al.* 2010; Huang & Bhattacharjee 2010; Loureiro *et al.* 2012, 2013). The example below is taken from a resistive-MHD numerical simulation by Samtaney *et al.* (2009), showing the evolution of the current density (color) in the central $x = [-\delta_{\rm SP}, \delta_{\rm SP}]$ region of a SP current sheet with $S = 10^7$:



Below is another example, taken from Bhattacharjee *et al.* (2009) using $S = 2\pi \times 10^5$:



Since then, simulations of plasmoid-dominated reconnection has become an industry.

Given that large-aspect-ratio SP current sheets are violently unstable to the plasmoid instability, it is worth asking whether we should expect them to exist in nature at all. Indeed, Lundquist numbers of typical space and astrophysical plasmas are absurdly large, with $S \sim 10^{13}$ or so in the solar corona implying a plasmoid-instability time scale less than 0.06% of the dynamical time scale. Why would a nice SP current sheet ever be realized under these conditions? See Pucci & Velli (2014) and Uzdensky & Loureiro (2016) for more.[13]

[13] You may also wish to see Alt & Kunz (2019) and Winarto & Kunz (2022) for reasons why a relatively large-scale, smoothly varying current layer (e.g., a Harris sheet) should not be expected to occur in a weakly collisional, high-β plasma.

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