Zonal Flow as Pattern Formation: Merging Jets and the Ultimate Jet Length Scale

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Background

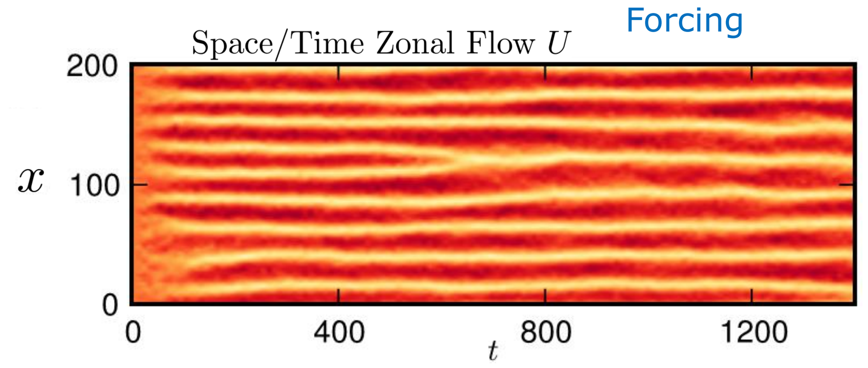
- Questions being addressed:
  - Why do zonal flows form?
  - What is their length scale?
- Model: Generalized Hasegawa-Mima equation, in the quasilinear approximation, statistically averaged
- We extend the zonostrophic instability calculation of Srinivasan & Young (2012) into the regime of self-consistent nonlinear interactions between zonal flows and fluctuations

Generalized Hasegawa-Mima Equation (Forced, 2D)

$$\partial_t w + \mathbf{v}_E \cdot \nabla w + \kappa \partial_y \phi = \underbrace{\xi}_{\text{Random Forcing}} - \underbrace{\mu w}_{\text{Friction}} - \nu(-1)^h \nabla^{2h} w$$

$$w = (\nabla^2 - \hat{\alpha})\phi$$

$$\hat{\alpha} = \begin{cases} 1 & \text{DW mode} \\ 0 & \text{ZF mode} \end{cases}$$



Merging jets are commonly observed in the transient regime of simulations

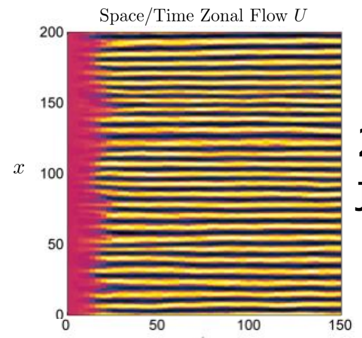
Quasilinear (QL) Approximation

- Neglect eddy self-nonlinearities in the eddy equation

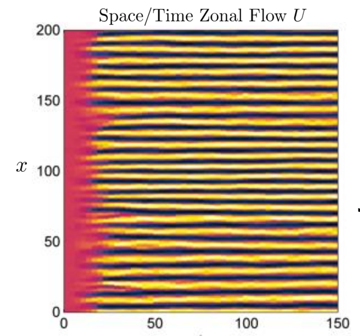
$$\partial_t U + \partial_y \overline{v'_x v'_y} = -\mu U - \nu(-1)^h \partial_y^{2h} U$$

$$\partial_t w' + U \cdot \nabla w' + \mathbf{v}'_E \cdot \nabla \overline{w} + \cancel{\mathbf{v}'_E \cdot \nabla w' - \overline{\mathbf{v}'_E \cdot \nabla w'}} + \kappa \partial_y \phi' = \xi - \mu w' - \nu(-1)^h \nabla^{2h} w'$$

- Zonal flows still form; but the jet width is nonunique (also true without the quasilinear approximation)



27 jets



24 jets

Averaging

Find an evolution equation for the two-point correlation function of the QL equations (use an ergodicity assumption); no closure problem

$$W(x_1, x_2, y, t) \equiv \frac{1}{L_y} \int_0^{L_y} d\overline{y}|_y \, w'(x_1, y_1, t) w'(x_2, y_2, t) \qquad \overline{y} = (y_1 + y_2)/2, \quad y = y_1 - y_2$$

This approach has been recently used in geophysical fluid dynamics

CE2 (Turbulence Statistics and Zonal Flow) Equations

*Derived in convenient form for analytic work by Srinivasan & Young (2012)

$$\partial_t W(x, y \mid \overline{x}, t) + (U_1 - U_2)\partial_y W - (U_1'' - U_2'')\left(\nabla^2 + \frac{1}{4}\partial_{\overline{x}}^2\right)\partial_y C$$
$$- [2\kappa - (U_1'' + U_2'')]\partial_{\overline{x}} \partial_y \partial_x C = \underbrace{F}_{\text{Forcing}} - \underbrace{D[W]}_{\text{Dissipation}},$$

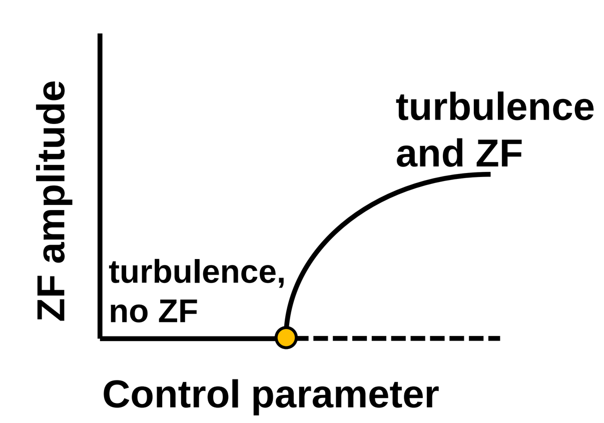
$$\partial_t U(\overline{x}, t) + \partial_{\overline{x}}[\underbrace{\partial_y \partial_x C(0, 0, \overline{x}, t)}_{\text{Reynolds stress}}] = -\mu U + \nu \partial_{\overline{x}}^{2h} U$$

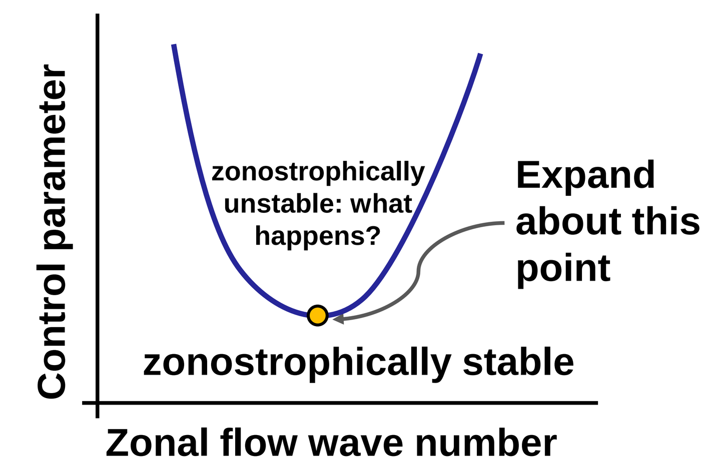
$U_{1,2} = U(\overline{y} \pm y/2)$
Z,Psi ~ eddy covariance
U ~ zonal flow velocity

Zonostrophic Instability

- Supercritical bifurcation between a state of *homogeneous* turbulence, *without* zonal flows, to a state of *inhomogeneous* turbulence, *with* zonal flows.

Bifurcation Analysis



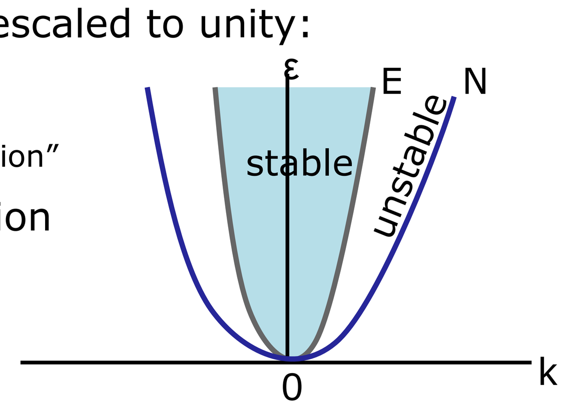


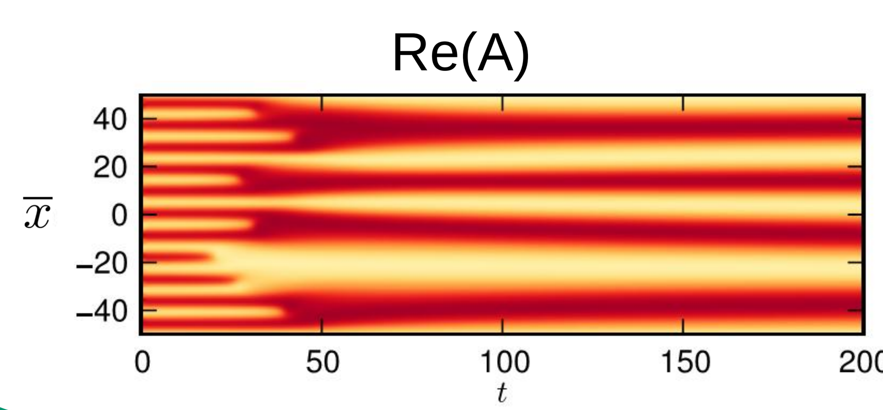
- Multiscale perturbation expansion of CE2 gives an equation for the spatially varying, complex amplitude of the bifurcating mode
- Equation is completely constrained by symmetries to take a universal form. With most coefficients rescaled to unity:

$$\partial_t A(\overline{x}, t) = \varepsilon A + \partial_{\overline{x}}^2 A - |A|^2 A$$

"Amplitude Equation", "Real Ginzburg-Landau Equation"

- For any $k^2 < \varepsilon$, $A = \sqrt{\varepsilon - k^2} e^{ik\overline{x}}$ is a solution
- Only solutions with $k^2 < \varepsilon/3$ are stable





Merging behavior seen in the amplitude equation starting from a slightly perturbed unstable equilibrium (ε=1)

Calculation of Equilibrium & Stability

Equilibrium

- Assume a given zonal flow wavenumber *q*
- Expand solution as a Fourier—Galerkin series with coefficients U_p, W_{mnp} to be determined.

$$U(\overline{x}) = \sum_{p=-P}^{P} U_p e^{ipq\overline{x}}$$

$$W(x, y \mid \overline{x}) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \sum_{p=-P}^{P} W_{mnp} e^{imax} e^{inby} e^{ipq\overline{x}}$$

- Project onto the basis functions to generate a set of nonlinear algebraic equations
- Then use Newton's method to solve for the equilibrium
- Repeat for multiple values of *q* (multiple solutions)

Stability

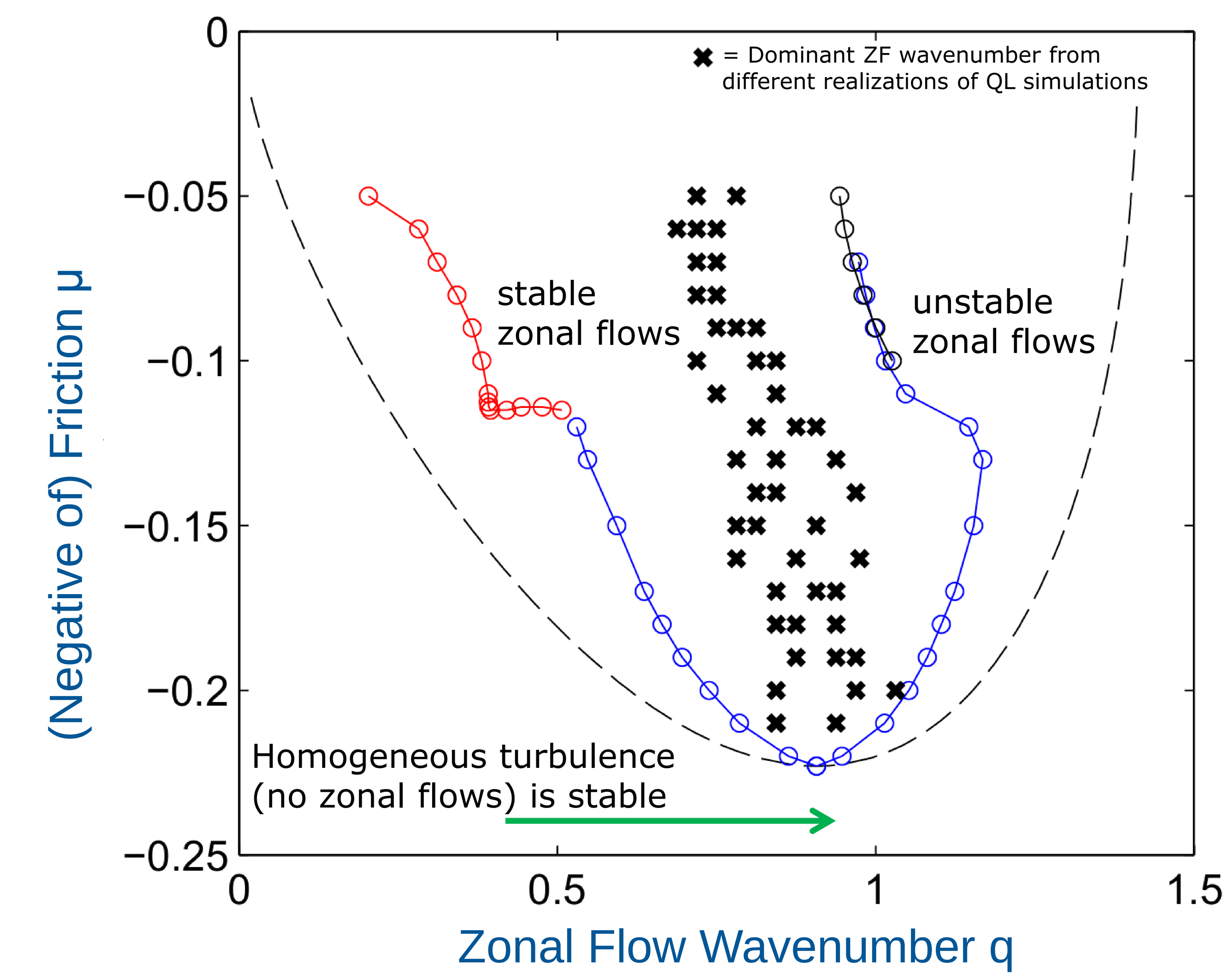
- Consider perturbations $\delta W(x, y \mid \overline{x}, t)$, $\delta U(\overline{x}, t)$ about an equilibrium
- Equilibrium is periodic in $\overline{x}$: expand perturbation as a Bloch state

$$\delta U(\overline{x}, t) = e^{\sigma t} e^{iQ\overline{x}} \sum_p \delta U_p e^{ipq\overline{x}}$$

$$\delta W(x, y \mid \overline{x}, t) = e^{\sigma t} e^{iQ\overline{x}} \sum_{mnp} \delta W_{mnp} e^{imax} e^{inby} e^{ipq\overline{x}}$$

- Equilibrium is unstable if there exists an eigenvalue σ with positive real part (for any Q)

Stability Balloon for CE2



Idealized, infinite system: many zonal flow wavelengths correspond to a stable solution

Zonal flow wavenumbers in the unstable region must evolve to get inside the stability balloon – manifests as merging jets

Conclusions

- Zonal flows constitute pattern formation amid a turbulent bath.
  - -Zonal flows are mathematically analogous to convection rolls in Rayleigh-Benard convection
- Explained theoretically why there is a range of allowed zonal flow wavelengths. Observable zonal flow wavenumbers are governed by stability conditions, calculated here for the CE2 system.
- Merging of jets is explained as the means of attaining a stable wavenumber.
- Crucial elements which should persist in more realistic models (where the eddy self-nonlinearities are retained) and guarantee a pattern formation description:
  - Reflection and translation symmetry of statistically averaged equation
  - Supercritical zonostrophic instability

References

Parker, J. B. and Krommes, J. A. http://arxiv.org/abs/1301.5059

Srinivasan, K. and Young, W. R. J. Atmos. Sci. 69, 1633–1656 (2012).

Farrell, B. F. and Ioannou, P. J. J. Atmos. Sci. 64, 3652–3665 (2007).

Tobias, S. M., Dagon, K., and Marston, J. B. Astrophys. J. 727, 127 (2011).

Cross, M. C. and Hohenberg, P. C. Rev. Mod. Phys. 65, 851 (1993)

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