Boundary induced amplification and nonlinear instability of interchange modes

Jupiter Bagaipo, A. B. Hassam

Institute for Research in Electronics and Applied Physics
University of Maryland, College Park

Stability, Energetics, and Turbulent Transport
in Astrophysical, Fusion, and Solar Plasmas

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Outline

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- Boundary induced instability
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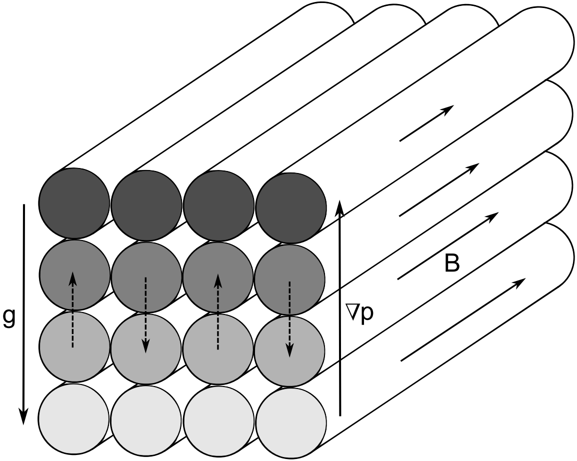
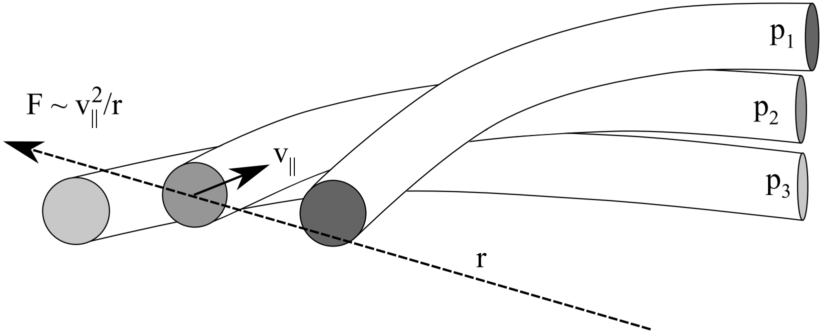
Interchange mode



Growth rate:

$$\gamma_g^2 \equiv g \frac{\rho_0'}{\rho_0}$$

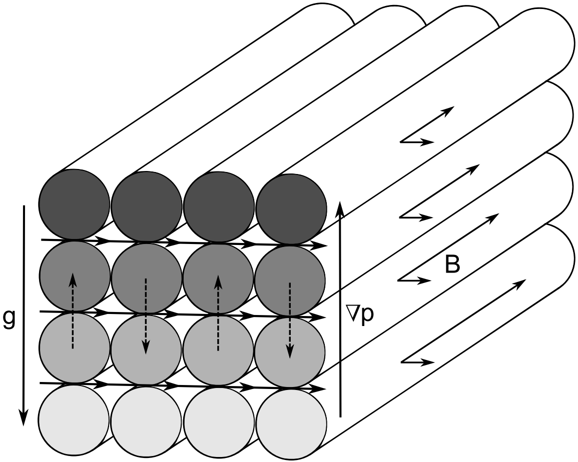
Toroidal device



Unstable when:

$$p_1 > p_2 > p_3$$

Transverse field stabilization



Dispersion relation:

$$\omega^2 = k^2 V_{Ay}^2 - \gamma_g^2$$

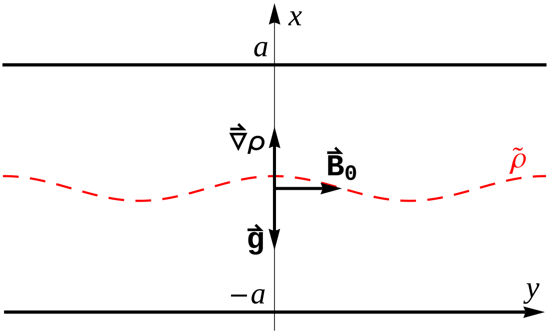
Motivation

Behaviour at marginal stability for ideal interchange modes is important in:

- Tokamaks,
- Reversed Field Pinches,
- Stellarators,
- etc...

Understanding the tradeoff between the deviation from marginality and residual convection is useful for B-field coil design in stellarators.

Previous result

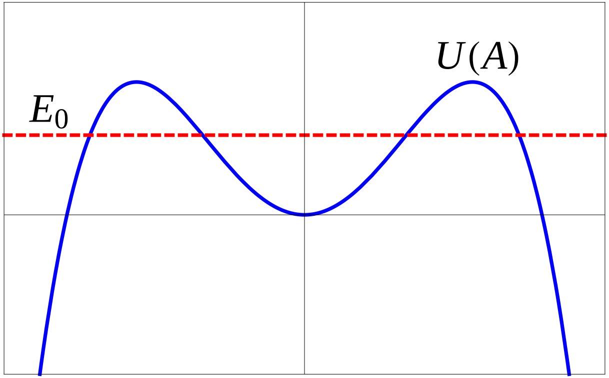


$$B_0 = B_c + b, \qquad \tilde{\rho} \propto A \sim \frac{\partial_t}{\gamma_g} \sim \left|\frac{b}{B_c}\right|^{1/2} \ll 1$$

$$\boxed{\frac{1}{k^2}\ddot{A} = -2bA + \frac{1}{4}\frac{k^2-3}{k^2+1}A^3}$$

Amplitude dependent stability

$$E_0 = \frac{1}{2k^2}\dot{A}^2 + U(A), \qquad U(A) = bA^2 - \frac{1}{16}\frac{k^2-3}{k^2+1}A^4$$



$$A_c = 2\sqrt{2}\sqrt{\frac{k^2+1}{k^2-3}} \times b^{1/2}$$

Numerical marginal conditions amplitude test

Linearly stable

$\Delta B/B_c \approx 0.04\%$

Small amplitude

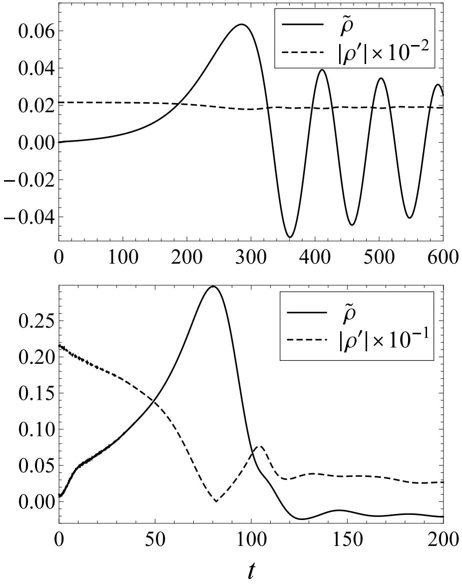
$a_0 = 10^{-4}$

Linearly stable

$\Delta B/B_c \approx 0.04\%$

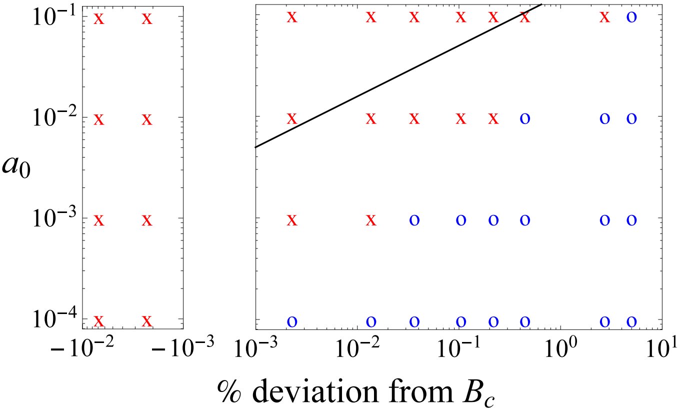
Large amplitude

$a_0 = 10^{-2}$



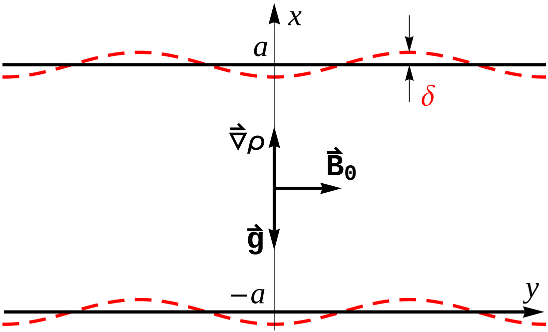
Numerical simulation result[1]

Stable runs are marked with a blue circle and unstable runs with a red 'x'. The theoretical boundary is shown as the line $a_0 \propto \sqrt{b}$.



[1] *Phys. Plasmas* **18** 122103 (2011).

Rippled boundary problem



$\rho_0' > 0$ and B_0 constant

$$\frac{\delta}{a} \ll 1$$

Reduced equations in 2D

Complete, nonlinear set:

$$\partial_t \psi = \{\psi, \varphi\}$$

$$\hat{\mathbf{z}} \cdot \vec{\nabla}_\perp \times \rho (\partial_t \vec{\mathbf{u}} + \{\varphi, \vec{\mathbf{u}}\}) = \{\psi, \nabla_\perp^2 \psi\} + g \partial_y \rho$$

$$\vec{\mathbf{u}} = \hat{\mathbf{z}} \times \nabla_\perp \varphi$$

$$\vec{\mathbf{B}}_\perp = \hat{\mathbf{z}} \times \nabla_\perp \psi$$

Where, in general, $\rho = \rho(\psi)$ and

$$\{f, h\} \equiv \partial_x f \partial_y h - \partial_y f \partial_x h.$$

"Simple-minded" linear calculation

Boundary condition:

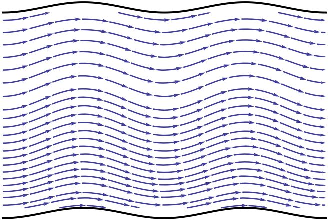
$$\text{At } x = \pm a - \delta\cos(ky)$$
$$\partial_y\psi = -\delta\, k\sin(ky)\partial_x\psi$$

Quasistatic equilibrium:

$$\psi = B_0 x + \tilde{\psi}$$
$$(B_0^2\nabla_\perp^2 + g\rho_0')\partial_y\tilde{\psi} = -B_0\{\tilde{\psi}, \nabla_\perp^2\tilde{\psi}\}$$

To lowest order in δ/a:

$$\tilde{\psi} = \frac{\delta B_0}{\cos(k_x a)}\cos(k_x x)\cos(ky)$$
$$k_x^2 = \frac{g\rho_0'}{B_0^2} - k^2$$



Boundary induced amplification

Critical condition:

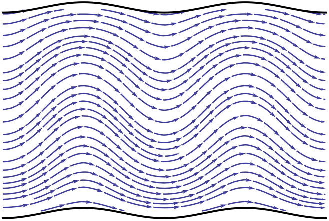
$$B_0 \to B_c \text{ when } k_x \to k_c \equiv \pi/2a$$
$$B_c^2(k_c^2 + k^2) \equiv B_c^2 k_\perp^2 = g\rho_0'$$

Amplification scaling:

$$B_0 = B_c + b \text{ and } k_x = k_c - \Delta k_x$$
$$b/B_c \sim \Delta k_x / k_c \ll 1$$

To lowest order in b/B_c:

$$\tilde{\psi} \approx \frac{\delta/a}{b/B_c} \frac{k_c}{k_\perp^2} B_c \cos(k_x x) \cos(ky)$$



Core penetration

Dirac delta gradient:

$$\rho_0'(x) = \rho_0' a \delta(x)$$

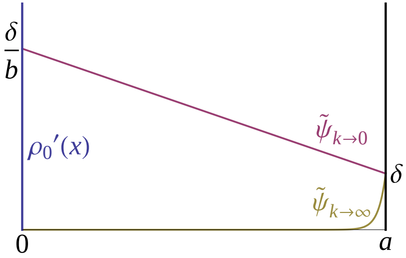
Solution:

$$\tilde{\psi} = B_0 \delta \frac{ka\cosh(kx) - \frac{1}{2}\kappa^2 a^2 \sinh(k|x|)}{ka\cosh(ka) - \frac{1}{2}\kappa^2 a^2 \sinh(ka)} \cos(ky), \quad \kappa^2 \equiv \frac{g\rho_0'}{B_0^2}$$

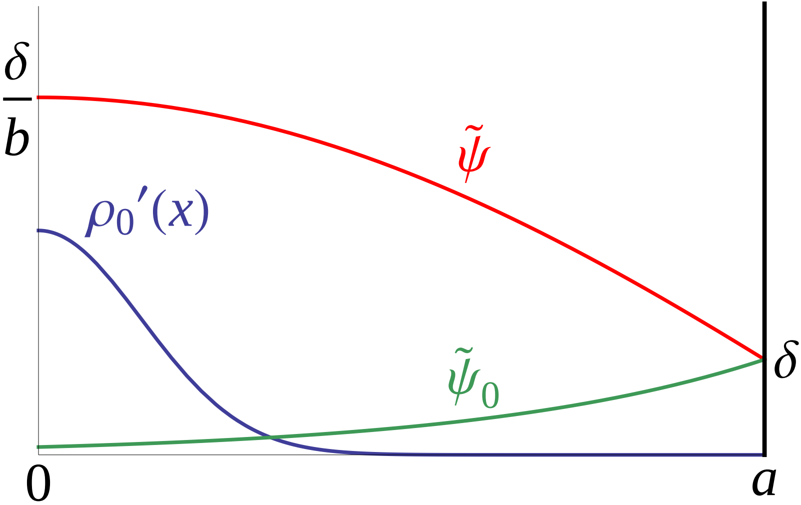
Limits:

$$\tilde{\psi}_{k\to 0} \propto B_0 \delta \frac{1 - \frac{1}{2}\kappa^2 a|x|}{1 - \frac{1}{2}\kappa^2 a^2}$$

$$\tilde{\psi}_{k\to\infty} \propto 2B_0 \delta e^{-ka} \cosh(kx)$$



Global amplification



Nonlinear scaling and expansion

Previous scaling:

$$|\tilde{\psi}/\psi_0| \sim \epsilon \equiv (b/B_c)^{1/2}$$
$$\Rightarrow \frac{\delta}{a} \sim \epsilon^3$$

Expand ψ in ϵ:

$$\psi = \psi_0 + \psi_1 + \psi_2 + \psi_3 + \cdots$$
$$\psi_0 = (B_c + b)x$$
$$\frac{\psi_{i+1}}{\psi_i} \sim \epsilon$$

First order

Equation:

$$B_c^2(\nabla_\perp^2 + k_\perp^2)\partial_y \psi_1 = 0$$
$$\partial_y \psi_1|_{x=\pm a} = 0$$

Solution:

$$\psi_1 = \left[A + \frac{\delta/a}{b/B_c}\frac{k_c}{k_\perp^2}B_c\right]\cos(k_c x)\cos(ky)$$

Where A is the plasma response.

$$\frac{A}{B_c/k_c} \sim \epsilon$$

Second order

Equation:

$$B_c^2(\nabla_\perp^2 + k_\perp^2)\partial_y\psi_2 = -B_c\{\psi_1, \nabla_\perp^2\psi_1\}$$
$$= 0$$

$$\partial_y\psi_2|_{x=\pm a} = 0$$

Solution:

$$\psi_2 = -\frac{1}{4}\frac{k_c}{B_c}\left[A + \frac{\delta/a}{b/B_c}\frac{k_c}{k_\perp^2}B_c\right]^2\sin(2k_c x)$$

Third order

Equation:

$$B_c^2(\nabla_\perp^2 + k_\perp^2)\partial_y\psi_3 + 2B_c b\nabla_\perp^2\partial_y\psi_1 = -B_c\left(\{\psi_1, \nabla_\perp^2\psi_2\} + \{\psi_2, \nabla_\perp^2\psi_1\}\right)$$

$$\partial_y\psi_3|_{x=\pm a} = -\delta B_c k\sin(ky)$$

Secular term:

$$\psi_3 = B_c\frac{\delta}{a}x\sin(k_c x)\cos(ky)$$

$$-2B_c bA + \frac{k_c^2}{4}\frac{k^2 - 3k_c^2}{k_\perp^2}\left[A + \frac{\delta/a}{b/B_c}\frac{k_c}{k_\perp^2}B_c\right]^3 = 0$$

Time evolution

Time dependence:

$$A = A(t) \text{ where } \tau_A \partial_t \sim \epsilon$$

$$\text{keep } \dot{\delta} \text{ ``small''}$$

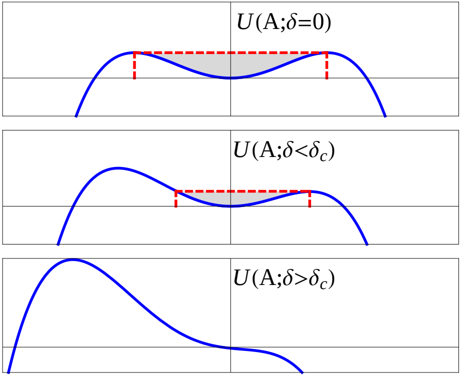
Result:

$$\boxed{\frac{1}{k^2}\ddot{A} = -2bA + \frac{1}{4}\frac{k^2-3}{k^2+1}\left[A + \frac{2}{\pi}\frac{\delta}{b}\frac{1}{k^2+1}\right]^3}$$

Normalized with k_c, B_c, and V_{Ac} equal to 1.

Boundary induced nonlinear instability[2]

$$U(A;\delta) = bA^2 - \frac{1}{16}\frac{k^2-3}{k^2+1}\left[A + \frac{2}{\pi}\frac{\delta}{b}\frac{1}{k^2+1}\right]^4$$



$$\delta_c = \frac{\pi}{2}\sqrt{\frac{32}{27}\frac{(k^2+1)^3}{k^2-3}} \times b^{3/2}$$

[2] *Phys. Plasmas (Letters)* **20** 020704 (2013).

Summary and conclusions

In marginally stable interchange mode systems boundary perturbations...

1. ...are amplified:

$$\tilde{\psi} \propto \delta/b$$

2. ...induce nonlinear instability.

Therefore, marginal system is highly sensitive to boundary perturbations:

$$\delta_c \propto b^{3/2}$$

Future work

Energy principle generalization:

- δW with nonlinear effects
- δW with boundary perturbations

Realistic geometry (shear, curvature, etc)