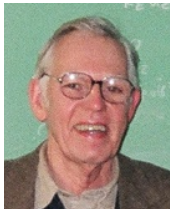
PCTS, Princeton, 9 April 2013


=85

Critical Balance as a Universal Scaling Conjecture

and its application to

Rotating Turbulence

Alexander Schekochihin
(University of Oxford)
w i t h
Sergey Nazarenko
(University of Warwick)

[work done at Institut H. Poincaré, UPMC, Paris 2009]

J. Fluid Mech. **677**, 134 (2011)

Oxford Physics™

Princeton-2008: Russell=80

Princeton-1998: Russell=70

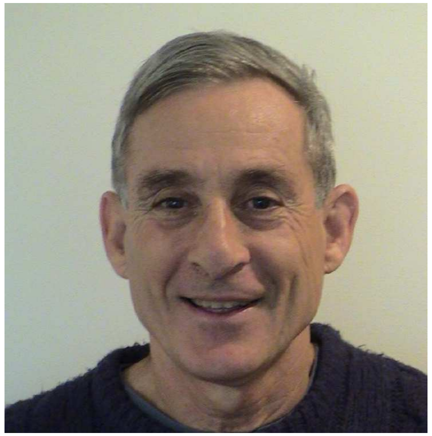
There was a celebratory conference in Jadwin Hall,
of which I don't have a picture (that was before mobile photography)…
…and at which I first heard Peter Goldreich talk about

CRITICAL BALANCE

in MHD (Alfvénic) turbulence

(those were the days when the **space & astro** community
believed, in the face of considerable evidence to the contrary,
that MHD turbulence was **isotropic** and **weak**,
so Goldreich was a rebel against orthodoxy;
interestingly, the **fusion plasma** community knew as a matter of course that
magnetised turbulence was **anisotropic**
and **gyrokinetics** was maturing as the formalism of choice,
but somehow the connection to MHD had trouble getting traction)

Critical Balance



- Assume (on evidence) MHD turbulence is **anisotropic** at all scales: $k_\parallel \ll k_\perp$
- Then there are two relevant frequencies in the problem are

$$\omega = k_\parallel v_A$$

$$\tau_{NL}^{-1} = k_\perp u_\perp$$

$\omega \gg \tau_{NL}^{-1}$ *weak (wave) turbulence*

$\omega \ll \tau_{NL}^{-1}$ *2D turbulence*

- Assume it is in between: $\omega \sim \tau_{NL}^{-1}$ *critical balance*

This removes dimensional ambiguity in the K41-style argument:

$$k_\perp E(k_\perp) \sim u_\perp^2(k_\perp) \sim \varepsilon \tau(k_\perp)$$
$$\tau^{-1} \sim \omega \sim \tau_{NL}^{-1}$$

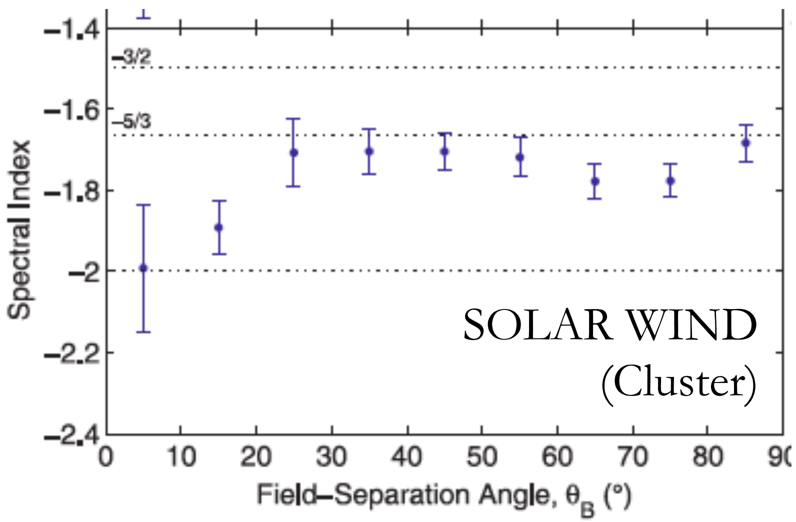
$\Rightarrow$

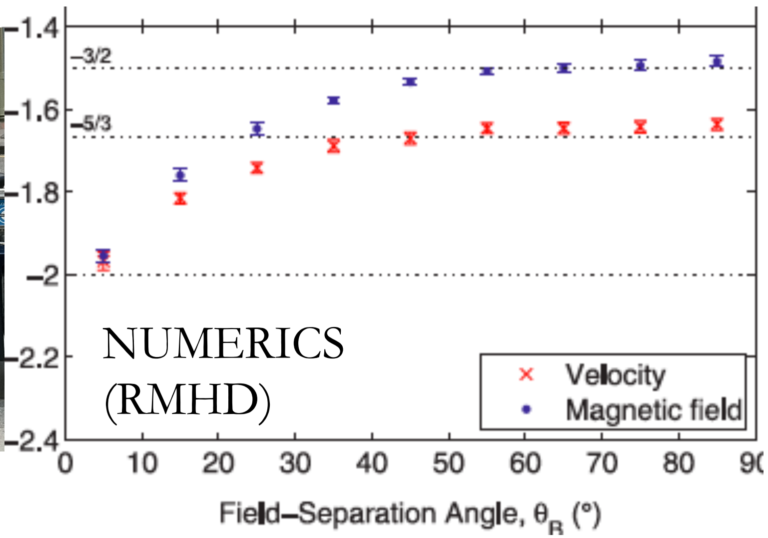
$$E(k_\perp) \sim \varepsilon^{2/3} k_\perp^{-5/3}$$
$$k_\parallel \sim \varepsilon^{1/3} v_A^{-1} k_\perp^{2/3}$$
$$E(k_\parallel) \sim \varepsilon v_A^{-1} k_\parallel^{-2}$$

[Goldreich & Sridhar 1995, 97; anticipated by Higdon 1984; amended by Boldyrev 2006]

Critical Balance

This has been quite successful in describing both simulated and real (solar wind) MHD turbulence: e.g.,





[Chen, Mallet et al. 2011, MNRAS 415, 3219]

See also talk by S. Bale and poster by R. Wicks

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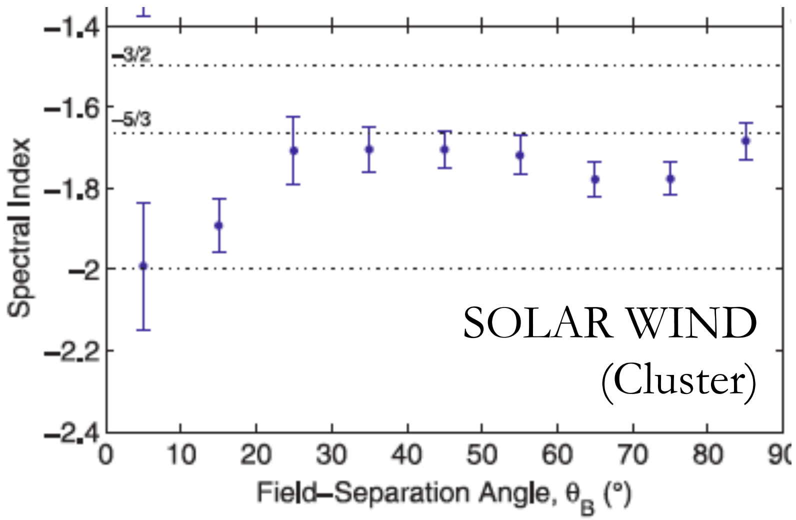
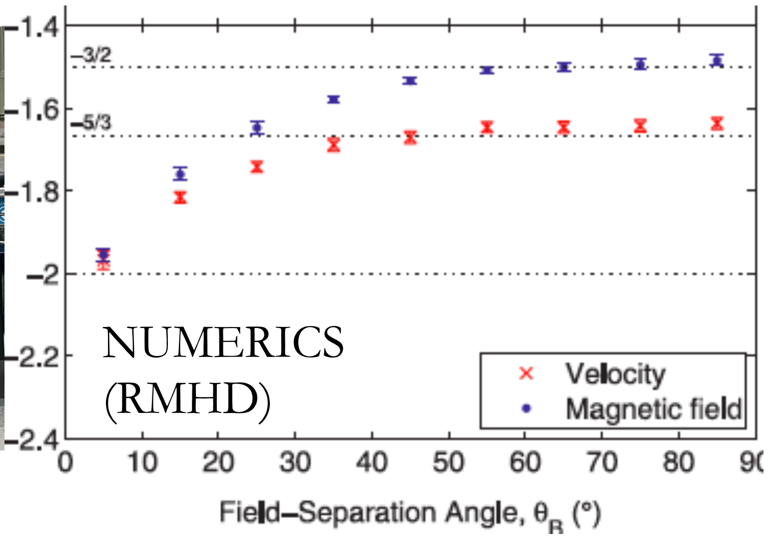
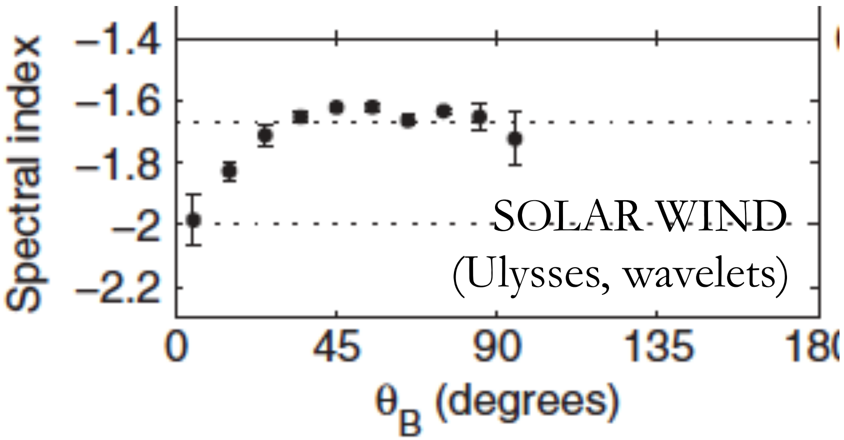




[Chen, Mallet et al. 2011, MNRAS 415, 3219]



[Horbury et al. 2008, PRL 101, 175005]

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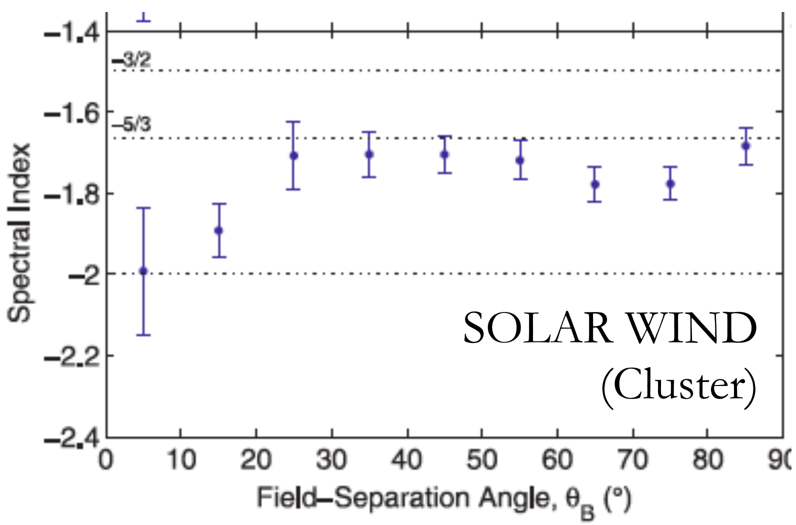
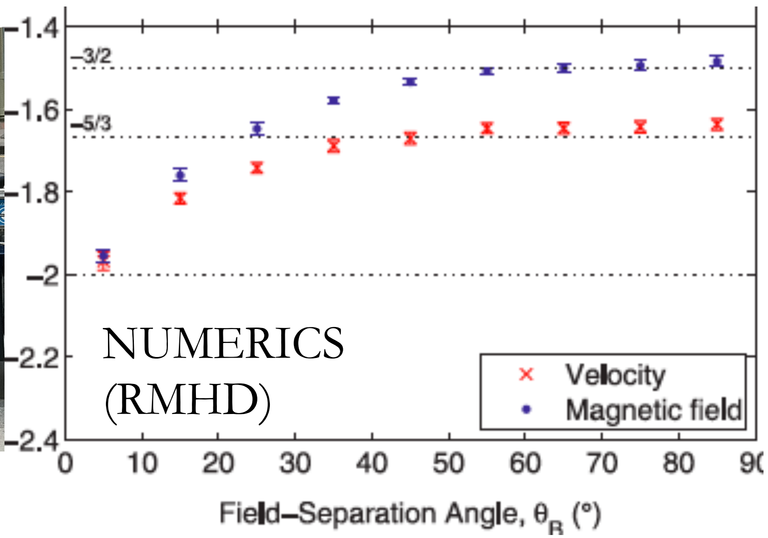




[Chen, Mallet et al. 2011, MNRAS 415, 3219]

[see also Wicks et al. 2010, MNRAS 401, L31; *see Rob Wick's poster*]

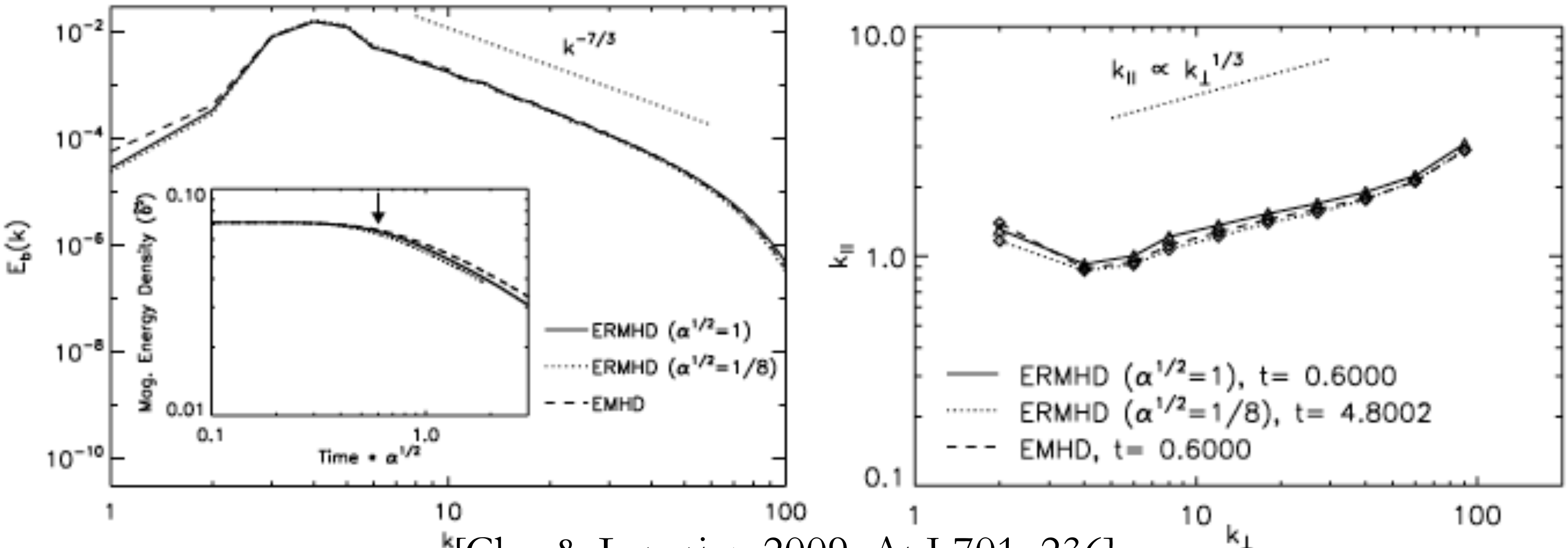
Critical Balance: Plasma Extensions

The idea of balancing the linear and nonlinear frequencies has since proved useful in a number of contexts:

✓ **KAW (~EMHD, Hall) turbulence**: $\omega \sim k_{\|} v_A k_{\perp} \rho_i \sim \tau_{NL}^{-1}$

gives $E(k_{\perp}) \propto k_{\perp}^{-7/3}$ and $k_{\|} \propto k_{\perp}^{1/3}$

[Cho & Lazarian 2004, AAS et al. 2009, ApJS 182, 310; amended by Boldyrev 2012]



[Cho & Lazarian 2009, ApJ 701, 236]

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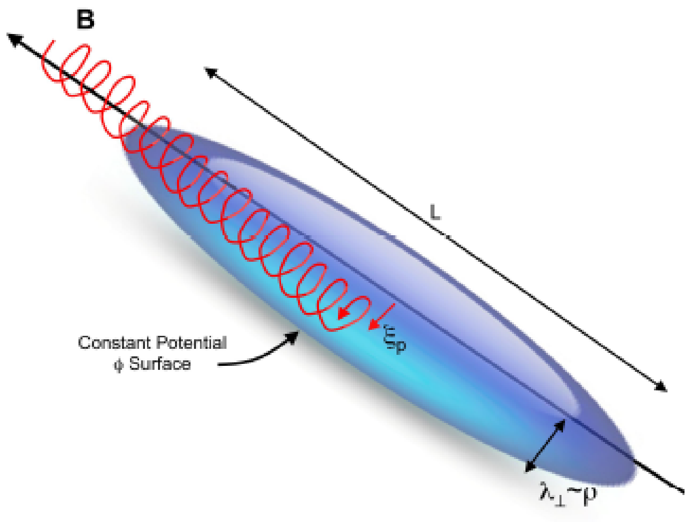
✓ Generally in magnetised (**GK**) turbulence:

$$\frac{\delta f}{F_0} \sim \frac{u_\perp}{v_{\mathrm{th}i}} \sim \frac{k_{||}}{k_\perp}$$ CB as an ordering assumption [Howes et al. 2006]



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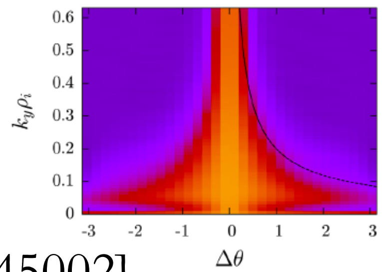
✓ In particular, in **ITG turbulence**: $\omega_* \sim k_{\parallel} v_{\mathrm{th}i} \sim \tau_{NL}^{-1}$

CB-based scaling theory fits numerical simulations

[Barnes, Parra, AAS 2011, PRL 107, 115003] *see Felix Parra's talk*



and maybe also measurements [Ghim et al. 2013, PRL 110, 145002]

CB = Universal Scaling Conjecture?

Consider a generic wave-supporting system with these properties:

✓ there is a **direction of anisotropy**: $k_\parallel \ll k_\perp$

(magnetic field, axis of rotation…)

✓ there are **parallel propagating waves**: $\omega = k_\parallel v(k_\perp)$

MHD: $v = v_A$ *Alfvén waves*

rotating systems: $v = 2\Omega / k_\perp$ *inertial waves*

GK (low-frequency magnetised plasma): *dispersion relation generally in this form*

✓ there is a $\boldsymbol{u}\cdot\nabla\boldsymbol{u} \simeq \boldsymbol{u}_\perp\cdot\nabla_\perp\boldsymbol{u}$ **nonlinearity**, so $\tau_{NL}^{-1} \sim k_\perp u_\perp(k_\perp)$

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Oxford Physics

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Rotating Turbulence: Anisotropic!

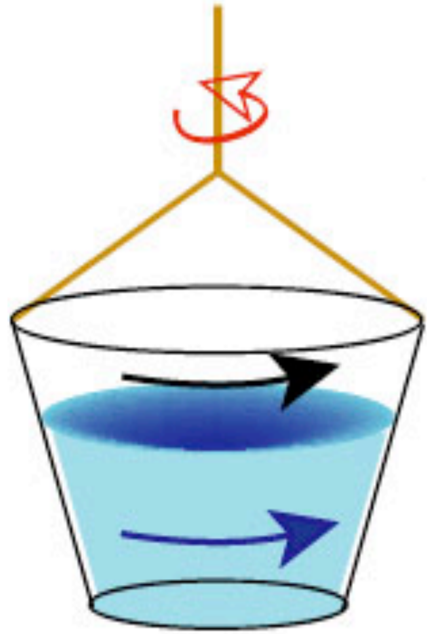
$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u}$$

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} \simeq \boldsymbol{u}_\perp\cdot\nabla_\perp\boldsymbol{u}$$

$$\omega = \pm\frac{2\Omega k_\parallel}{k} \simeq \pm\frac{2\Omega k_\parallel}{k_\perp} \qquad \boxed{v(k_\perp) = \frac{2\Omega}{k_\perp}}$$

because **turbulence is anisotropic**:

$$k_\parallel \ll k_\perp$$



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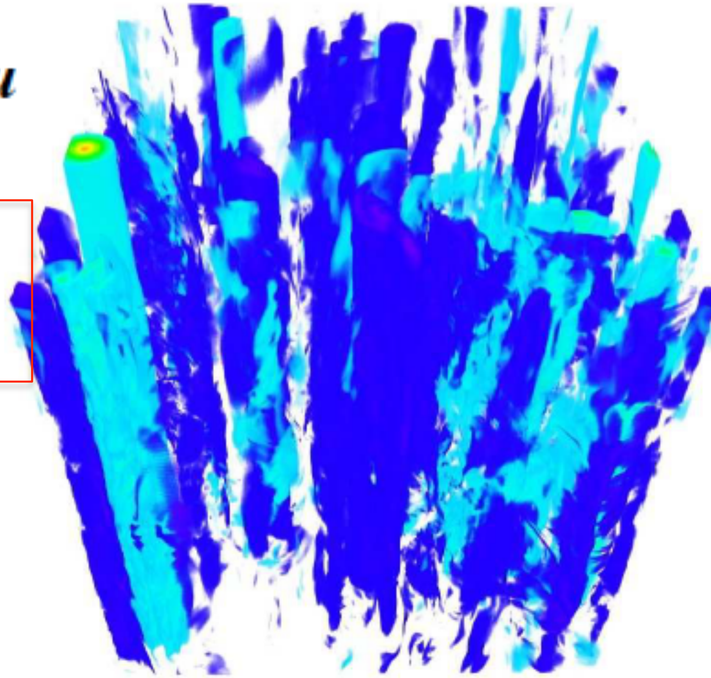

Mininni & Pouquet 2010, PoF 22, 035106

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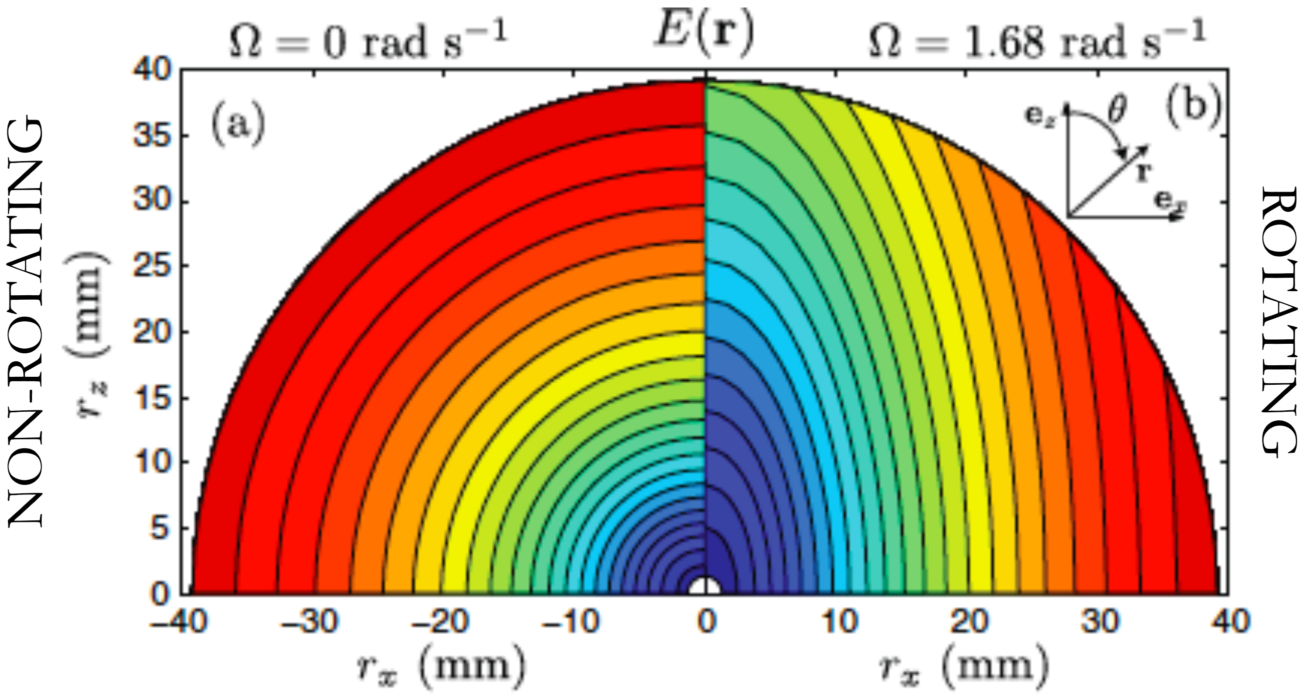

Lamriben et al. 2011, PRL 107, 024503

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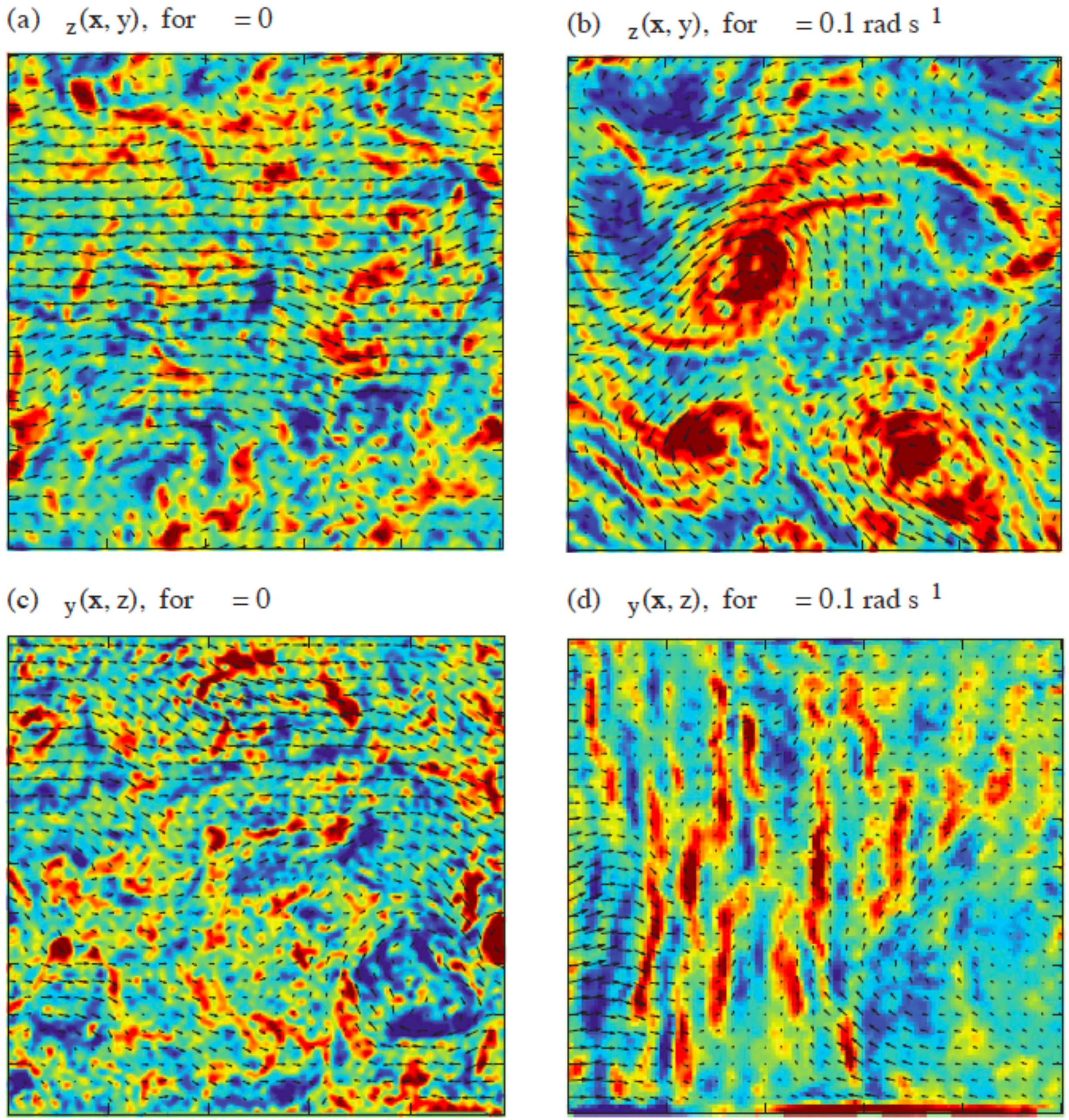

NON-ROTATING

ROTATING

Moisy et al. 2011, JFM 666, 5

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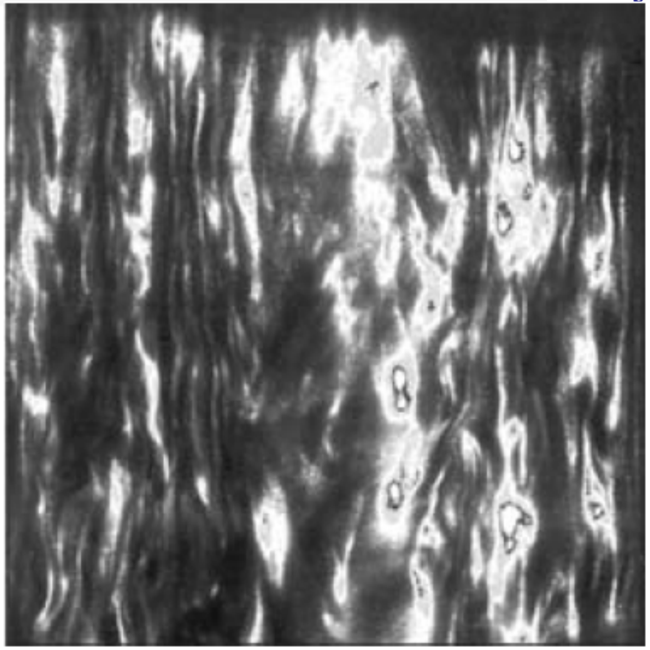

Davidson et al. 2006, JFM 557, 135

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Rotating Turbulence: CB Scalings

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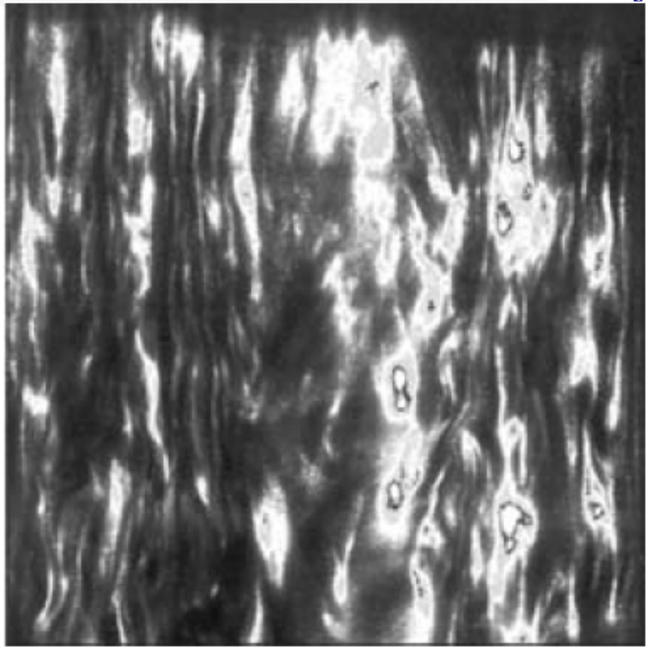

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- Parallel spectrum: $E(k_\parallel) \sim \varepsilon^{4/5}\Omega^{-2/5}k_\parallel^{-7/5}$

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Is CB Inevitable?

WEAK TURBULENCE: $\omega\tau_{NL} \gg 1$

This derivation is in fact problematic because it involves treating $k_\parallel = 0$ modes as waves. For an amended treatment (MHD), see AAS et al. 2012, PRE 85, 036406

One (weak) interaction: $\delta u_\perp \sim (\omega\tau_{NL})^{-1} u_\perp$

Cascade time: τ s.t. after $n \sim \tau\omega$ interactions, $n^{1/2}\delta u_\perp \sim u_\perp$

$$\tau \sim \omega\tau_{NL}^2$$
$$k_\perp E(k_\perp) \sim u_\perp^2(k_\perp) \sim \varepsilon\tau$$

$\Rightarrow$

$$E(k_\perp) \sim (\varepsilon k_\parallel)^{1/2} [v(k_\perp)]^{1/2} k_\perp^{-2}$$

This result only satisfies the weak interaction approximation if

$$\omega\tau_{NL} \sim k_\parallel v(k_\perp)\varepsilon^{-1/3} k_\perp^{-2/3} \gg 1$$

Thus, **weak turbulence will drive itself into a CB state** (unless $v(k_\perp)$ grows faster than $k_\perp^{2/3}$)

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$\Longrightarrow$

$$E(k_\perp) \sim (\varepsilon k_\parallel \Omega)^{1/2} k_\perp^{-5/2}$$

This result only satisfies the weak interaction approximation if

$$k_\perp \ll k_{\perp c} \sim \varepsilon^{-1/5}(k_\parallel \Omega)^{3/5} \sim k_0 Ro^{-4/5}$$

NB: Cascade only in $k_\perp$: exact for MHD, approximate for rotating

So $k_\parallel$ is an energy injection parameter [Galtier 2003]

Let us assume *isotropic forcing:* $k_\parallel \sim k_\perp \sim k_0$

and *low Rossby number* $Ro = u_{rms}k_0/\Omega \ll 1$

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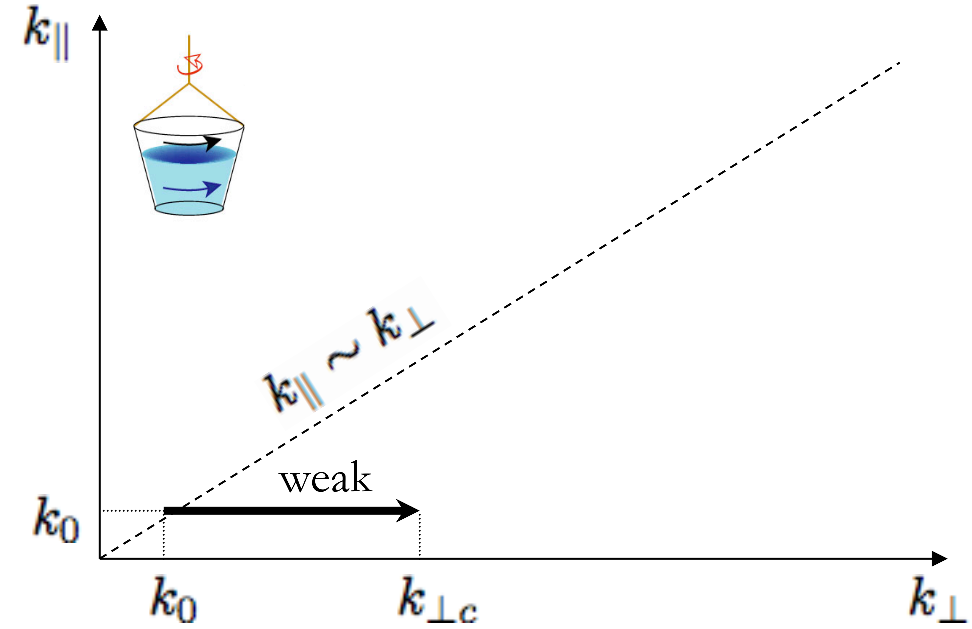
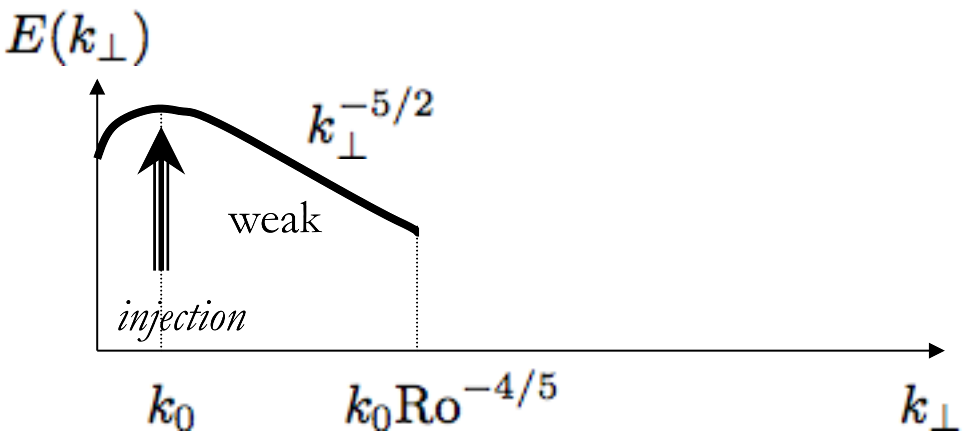
Cascade Path: From Weak Turbulence…





- **Weak,** anisotropic:

$$E(k_\perp) \sim (\varepsilon k_0 \Omega)^{1/2} k_\perp^{-5/2}$$

$$k_\parallel \sim k_0$$

$$k_0 < k_\perp < k_{\perp c}$$

$$k_{\perp c} \sim \varepsilon^{-1/5} (k_\parallel \Omega)^{3/5} \sim k_0 Ro^{-4/5}$$

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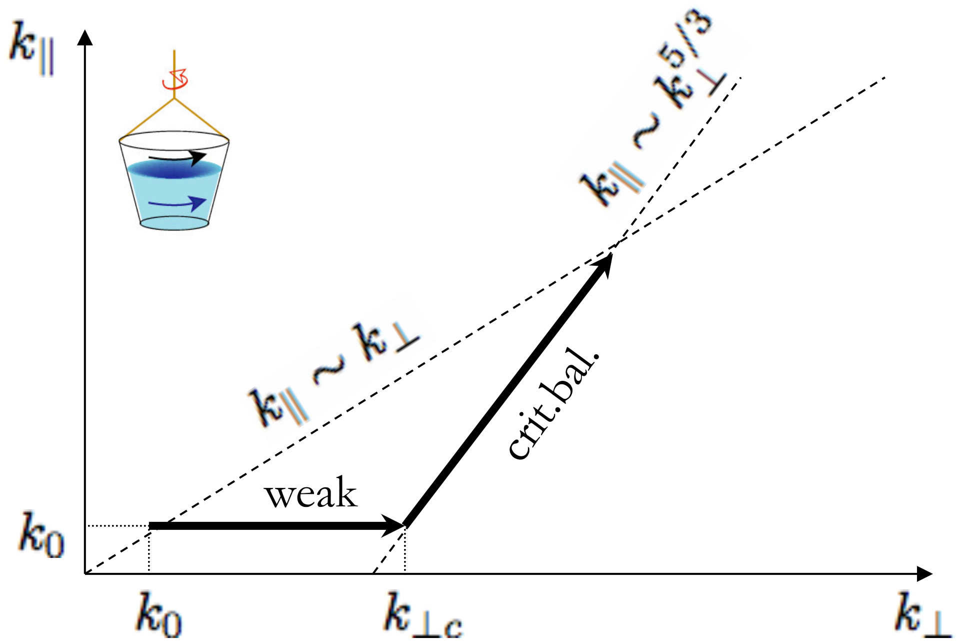
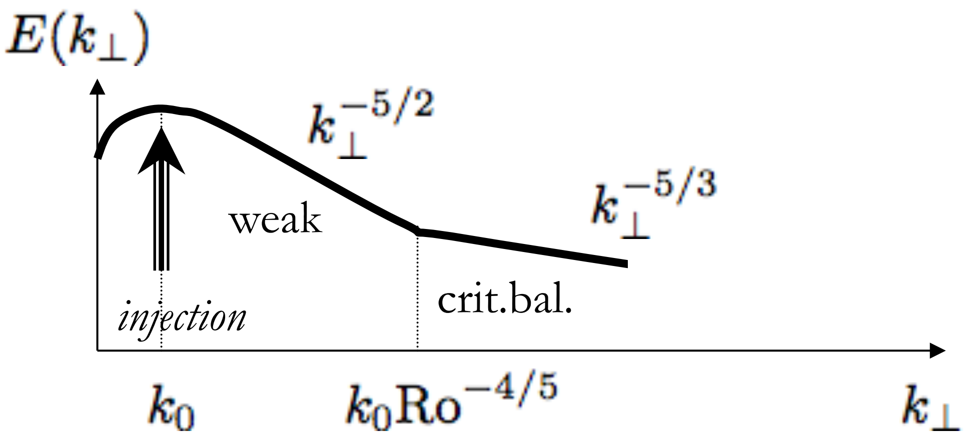
Cascade Path: …via CB…





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$$k_0 < k_{\perp} < k_{\perp c}$$
$$k_{\perp c} \sim \varepsilon^{-1/5} (k_{\|} \Omega)^{3/5} \sim k_0 Ro^{-4/5}$$

- **Critically balanced,** anisotropic:

$$E(k_{\perp}) \sim \varepsilon^{2/3} k_{\perp}^{-5/3}$$
$$k_{\|} \sim \varepsilon^{1/3} \Omega^{-1} k_{\perp}^{5/3}$$

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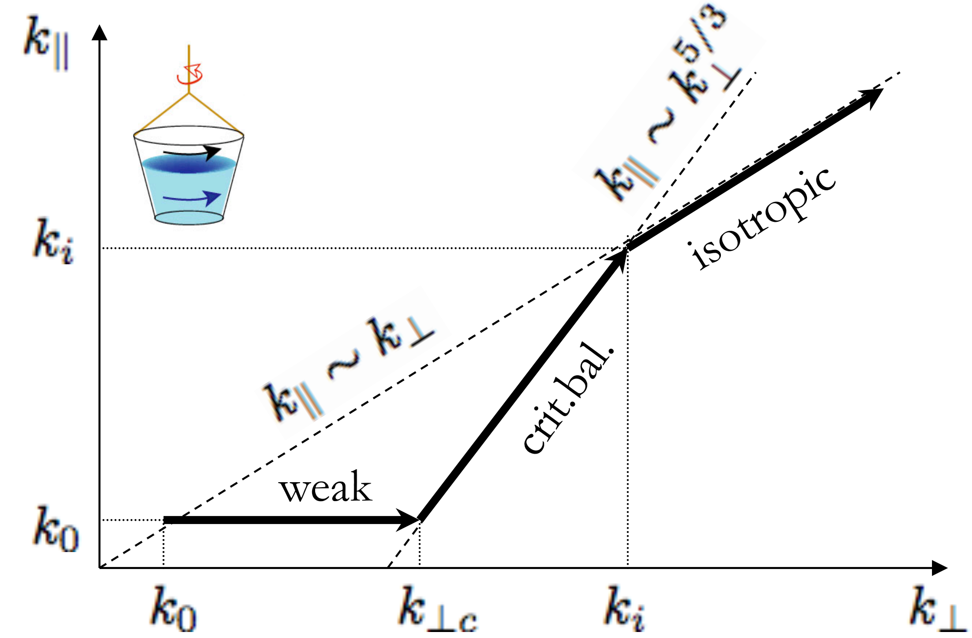
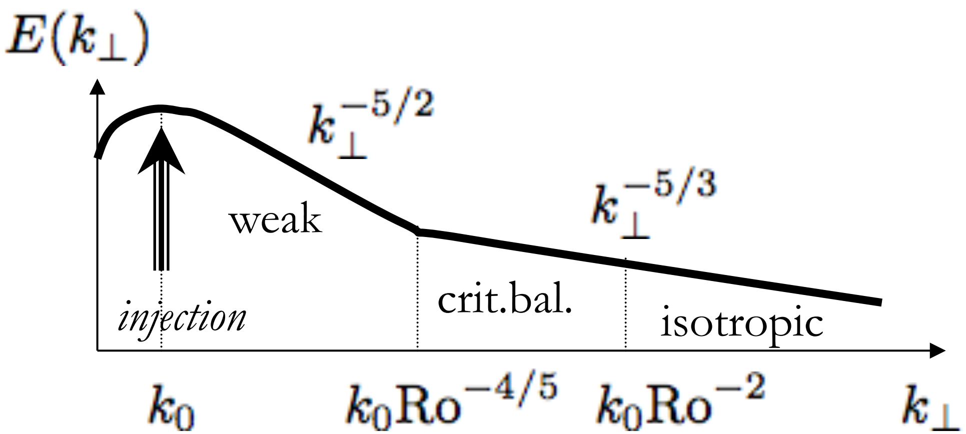
Cascade Path: Towards Isotropic State





[Isotropisation confirmed in DNS by Mininni et al. 2012, JFM 699, 263]

- **Weak,** anisotropic:

$$E(k_\perp) \sim (\varepsilon k_0 \Omega)^{1/2} k_\perp^{-5/2}$$
$$k_\parallel \sim k_0$$
$$k_0 < k_\perp < k_{\perp c}$$
$$k_{\perp c} \sim \varepsilon^{-1/5}(k_\parallel \Omega)^{3/5} \sim k_0 Ro^{-4/5}$$

- **Critically balanced,** anisotropic:

$$E(k_\perp) \sim \varepsilon^{2/3} k_\perp^{-5/3}$$
$$k_\parallel \sim \varepsilon^{1/3} \Omega^{-1} k_\perp^{5/3}$$
$$k_{\perp c} < k_\perp < k_i$$
$$k_i \sim \varepsilon^{-1/2} \Omega^{3/2} \sim k_0 Ro^{-2}$$

Zeman (1994) scale

- **Kolmogorov,** isotropic:

$$E(k) \sim \varepsilon^{2/3} k^{-5/3} \quad k_\perp \sim k_\parallel \sim k$$

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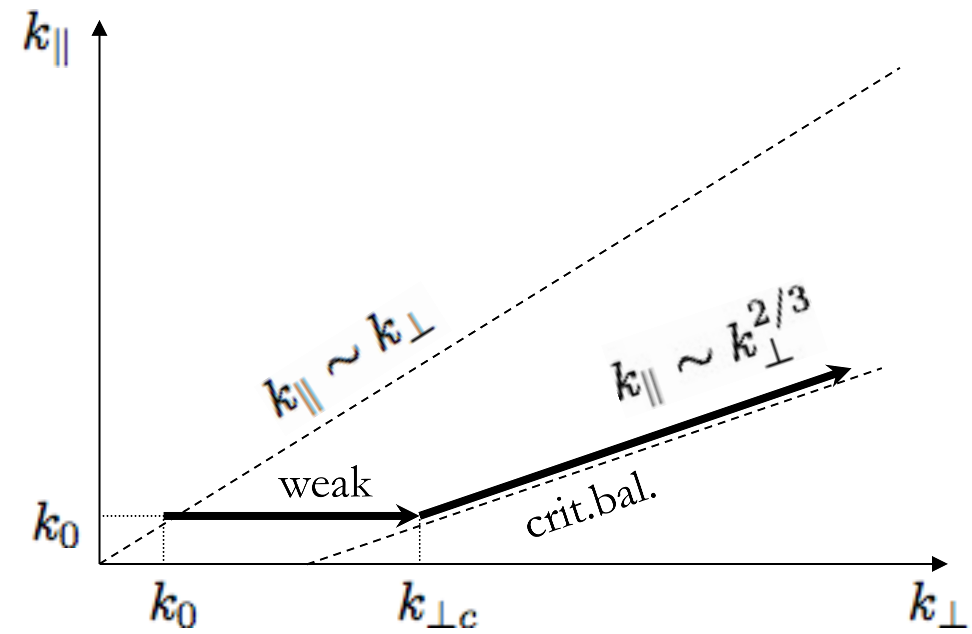
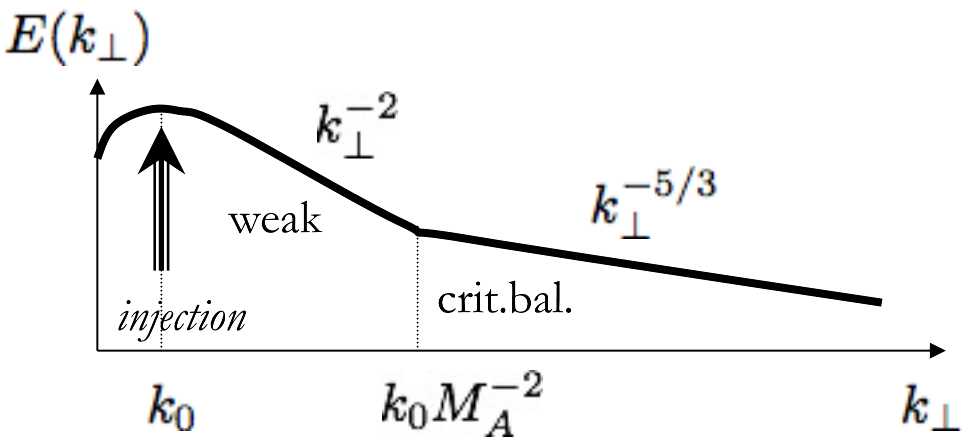
Cf. MHD: Ever More Anisotropic





[Goldreich & Sridhar 1995, 97]

- **Weak,** anisotropic:

$$E(k_\perp) \sim (\varepsilon k_{\|} v_A)^{1/2} k_\perp^{-2}$$
$$k_{\|} \sim k_0$$
$$k_0 < k_\perp < k_{\perp c}$$
$$k_{\perp c} \sim \varepsilon^{-1/2} (k_{\|} v_A)^{3/2} \sim k_0 M_A^{-2}$$

- **Critically balanced,** anisotropic:

$$E(k_\perp) \sim \varepsilon^{2/3} k_\perp^{-5/3}$$
$$k_{\|} \sim \varepsilon^{1/3} v_A^{-1} k_\perp^{2/3}$$
$$k_\perp > k_{\perp c}$$

Note: KAW turbulence is similar to MHD:
gets more anisotropic $k_{\|} \sim k_\perp^{1/3}$
[Cho & Lazarian 2004]
ITG to rotating:
gets (a bit) more isotropic $k_{\|} \sim k_\perp^{4/3}$
[Barnes, Parra & AAS 2011]

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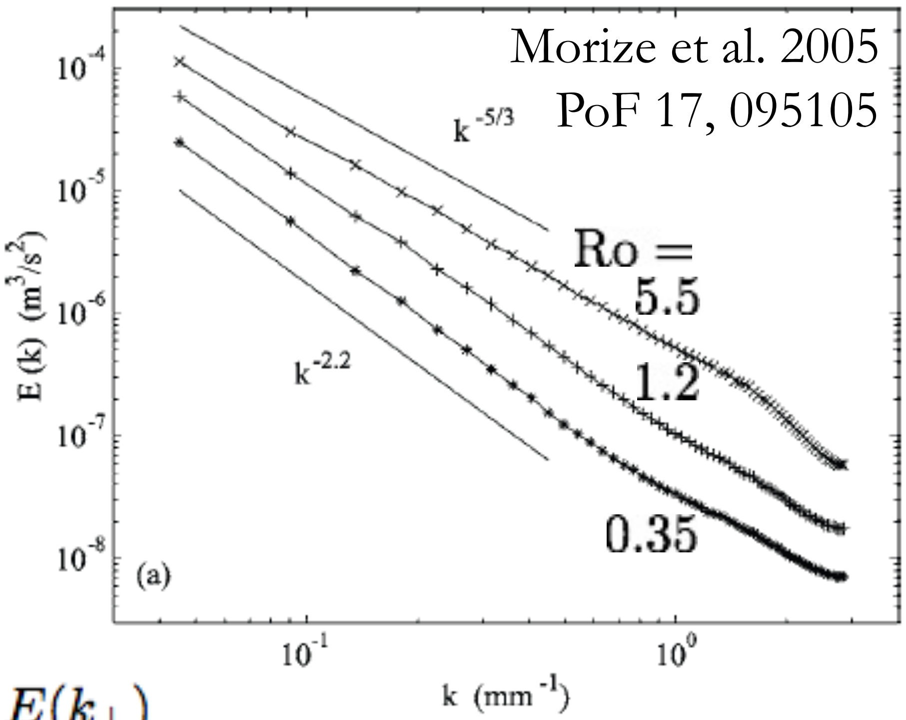
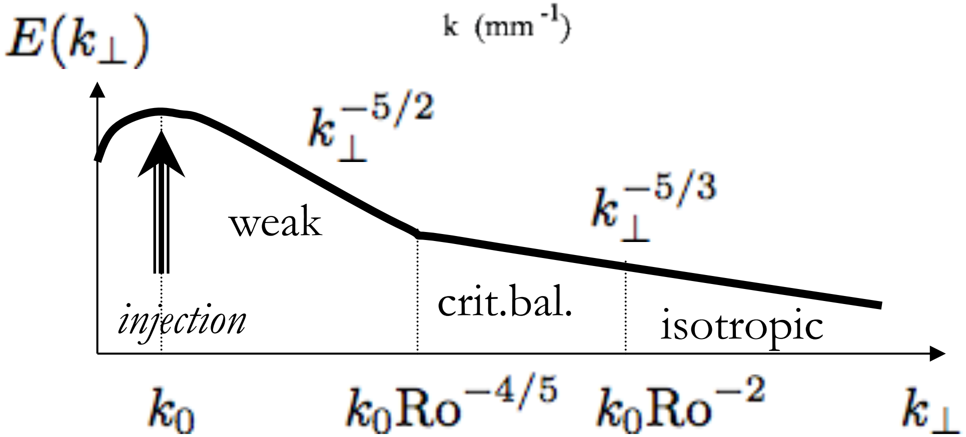
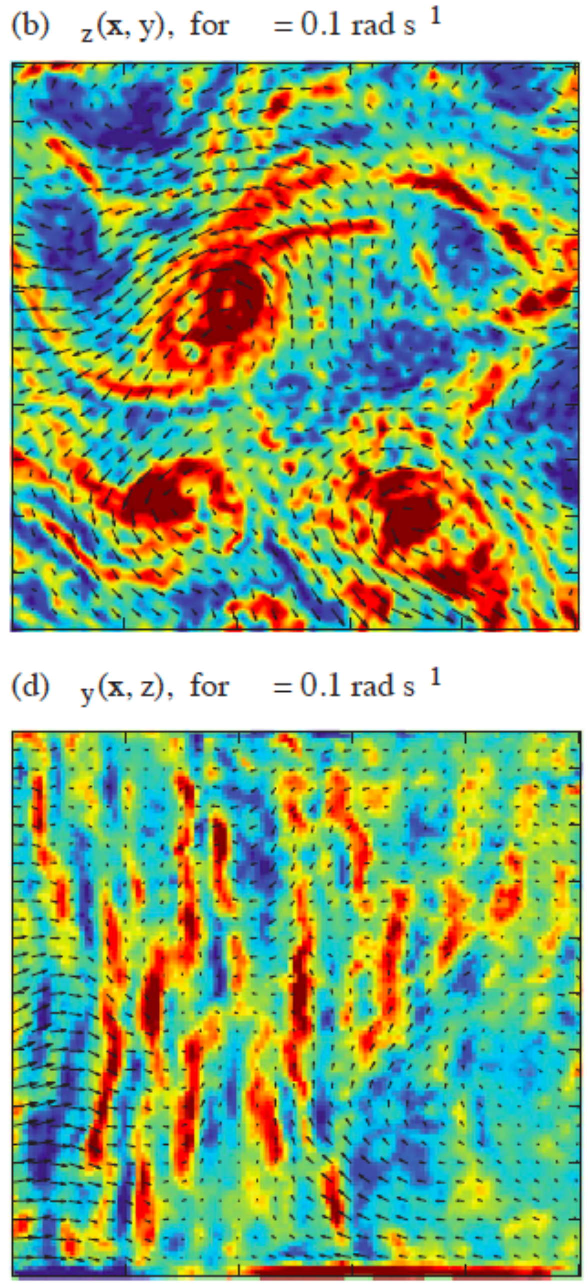
Reality Check: Weak to Strong in the Lab?







Moisy et al. 2011, JFM 666, 5 (later paper, the same group)

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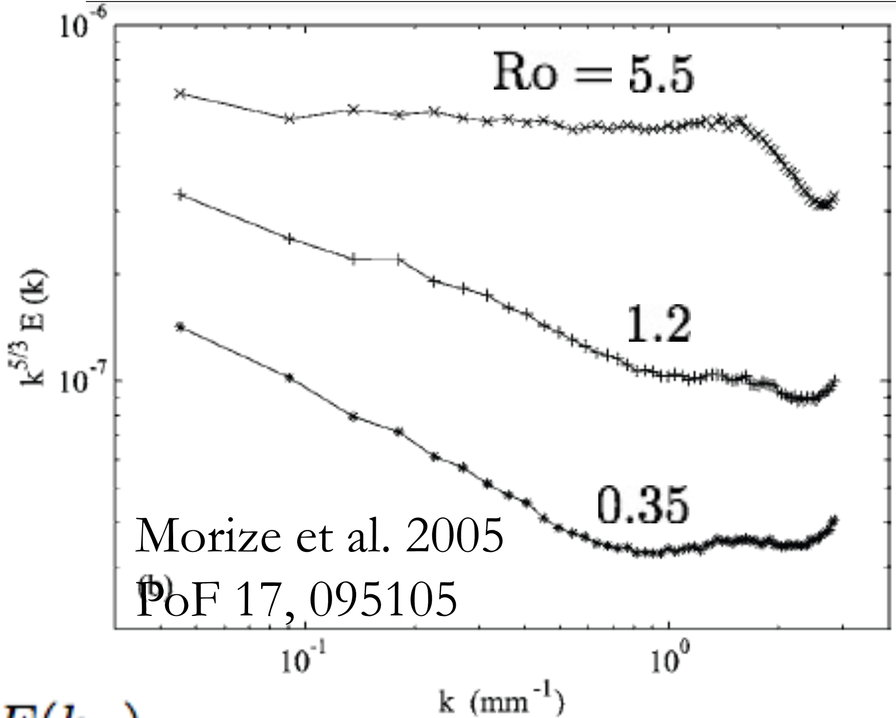
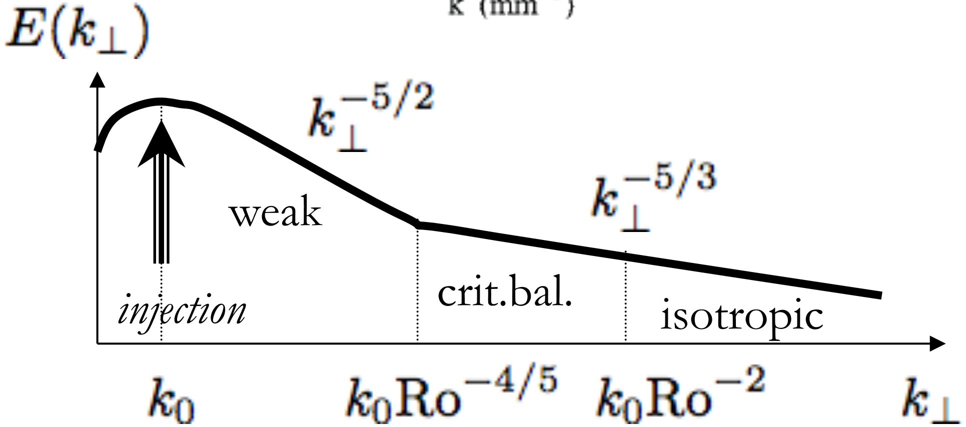
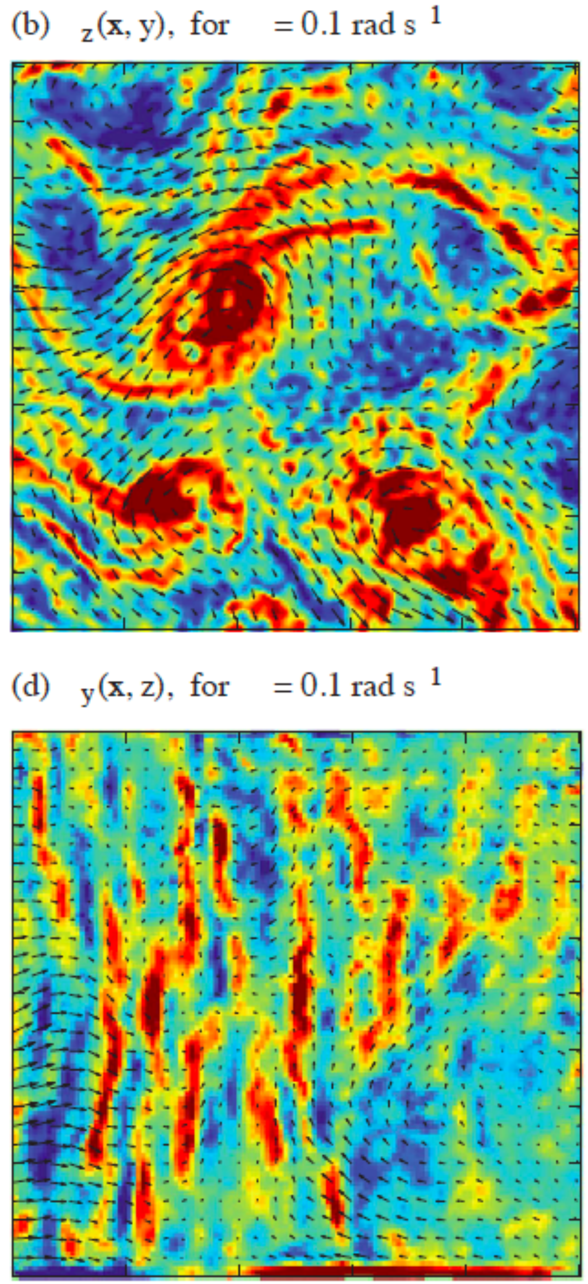


Morize et al. 2005
PoF 17, 095105





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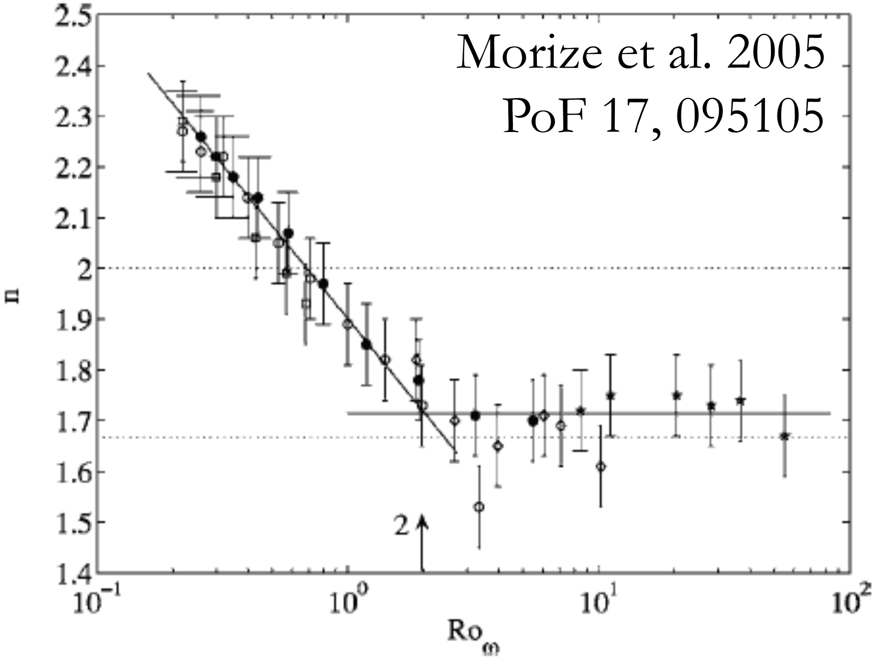
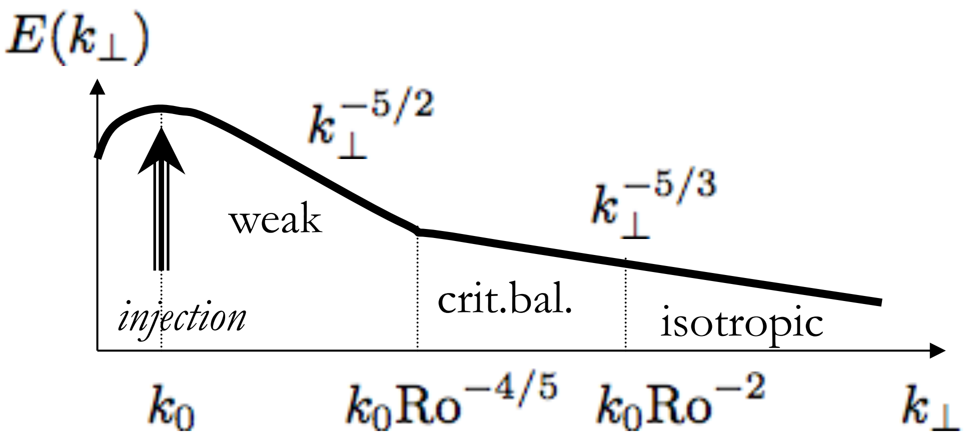
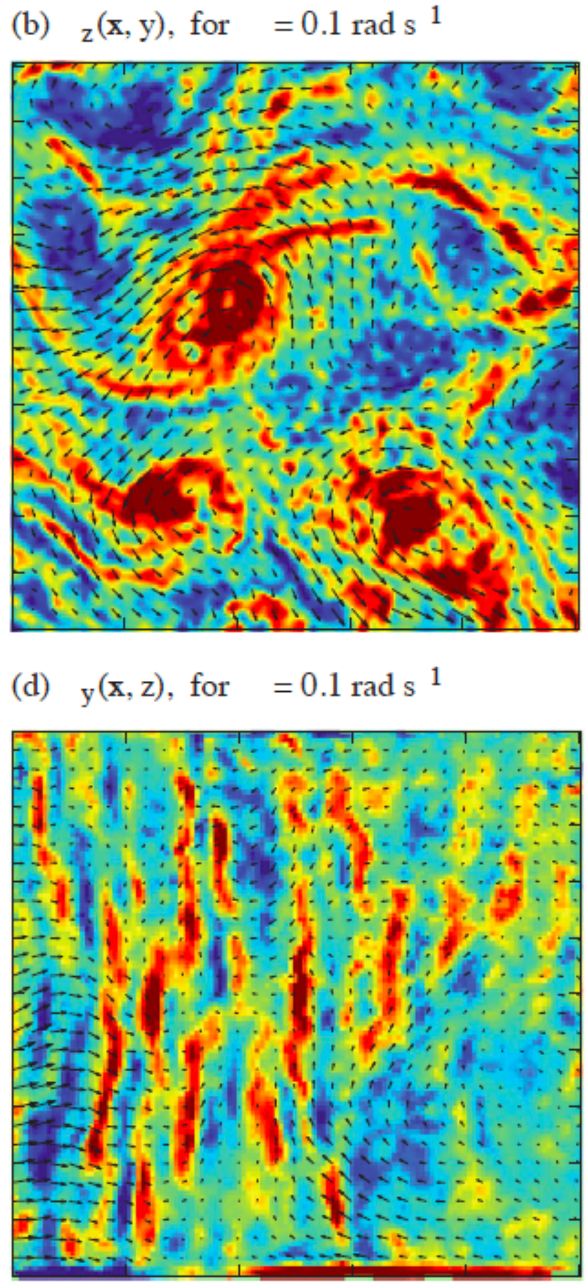




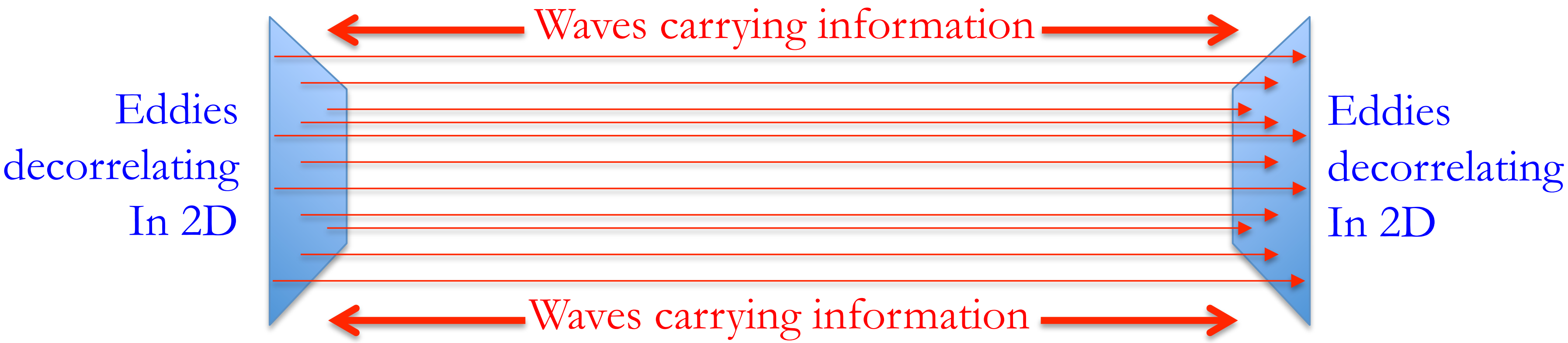


Moisy et al. 2011, JFM 666, 5 (later paper, the same group)

Is CB Inevitable?

2D TURBULENCE: $\omega\tau_{NL} \ll 1$

- **General causality argument:**



Max corr. distance: $l_\| \sim v\tau_{NL} \sim v l_\perp / u_\perp \Rightarrow k_\| v(k_\perp) \sim k_\perp u_\perp$
critical balance

- **Inverse cascade (for rotating turbulence)**

In 2D, the system does not feel Coriolis force ➔**2D hydro**
So inverse cascade will carry energy to larger $l_\perp$, where
eddies turn over slower and so $\tau_{NL} \sim l_\| / v \sim \omega^{-1}$ *critical balance*

NB: *System must be large enough in the parallel direction!*

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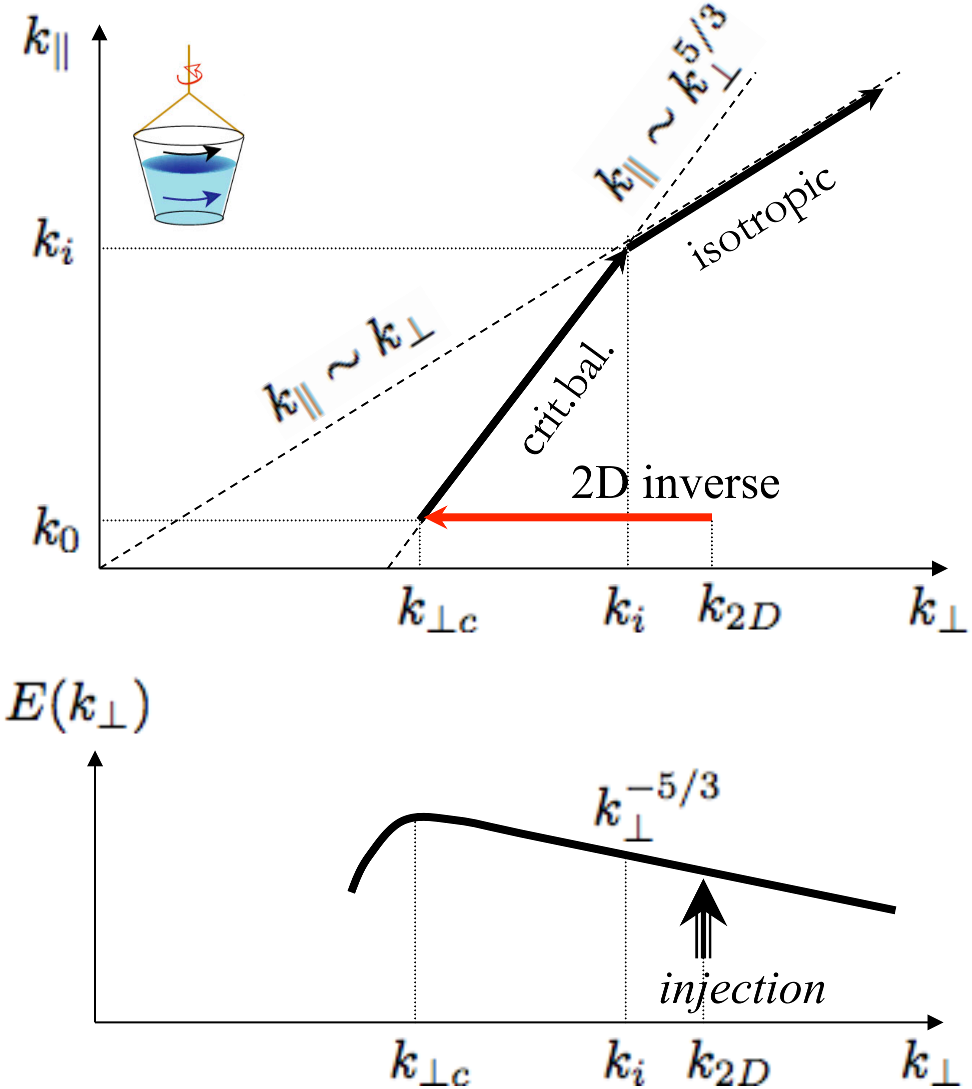
Cascade Path: 2D→CB→K41



- **2D**:

$$E(k_\perp) \sim \varepsilon^{2/3} k_\perp^{-5/3}$$

$$k_\parallel \sim k_0$$

$$k_{2D} > k_\perp > k_{\perp c}$$

$$k_{\perp c} \sim \varepsilon^{-1/5} (k_\parallel \Omega)^{3/5}$$

- **Critically balanced,** anisotropic:

$$E(k_\perp) \sim \varepsilon^{2/3} k_\perp^{-5/3}$$

$$k_\parallel \sim \varepsilon^{1/3} \Omega^{-1} k_\perp^{5/3}$$

$$k_{\perp c} < k_\perp < k_i$$

$$k_i \sim \varepsilon^{-1/2} \Omega^{3/2}$$

Zeman (1994) scale

- **Kolmogorov,** isotropic:

$$E(k) \sim \varepsilon^{2/3} k^{-5/3} \quad k_\perp \sim k_\parallel \sim k$$

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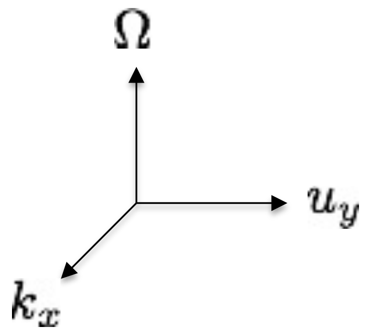
Polarisation Alignment?

Inertial waves have $\delta\boldsymbol{\omega} = \mp k\boldsymbol{u}$, so they are nonlinear solutions

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \delta\boldsymbol{\omega}\times\boldsymbol{u} + \nabla u^2/2 = 0$$

In an inertial wave, $k_y = 0, u_x = 0$



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In an inertial wave, $k_y = 0, u_x = 0$

Is there a dynamical (or statistical) tendency for velocity and vorticity to align, i.e., for fluctuations to look like inertial waves?

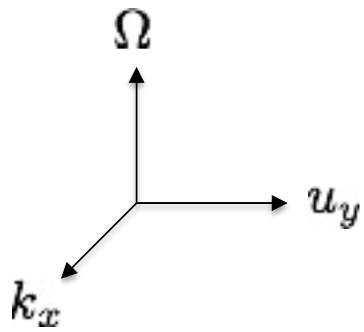
Boldyrev 2006 suggested a similar thing for Alfvén waves in MHD (also exact solutions)

See his talk!

He argues this tendency to alignment between $\mathbf{u}_\perp$ and $\delta\mathbf{B}_\perp$ has to do with **cross-helicity** $\langle \mathbf{u}_\perp \cdot \delta\mathbf{B}_\perp \rangle$ conservation.

The analog argument for rotating turbulence would invoke **helicity** $\langle \mathbf{u} \cdot \delta\omega \rangle$

[see extensive studies by Mininni & Pouquet 2009, 2010, 2012]



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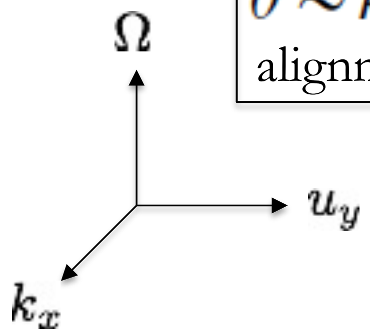
In an inertial wave, $k_y = 0, u_x = 0$

Is there a dynamical (or statistical) tendency for velocity and vorticity to align, i.e., for fluctuations to look like inertial waves?

✧ Suppose $k_x \gg k_y \gg k_\parallel$ ➔ $u_x \sim (k_y/k_x)u_y \ll u_y$ ➔

$k_\perp \sim k_x$
$u_\perp \sim u_y$
$\theta \sim k_y/k_x \ll 1$
alignment angle

➢ Then $\tau_{NL}^{-1} \sim k_y u_y \sim k_\perp u_\perp(k_\perp)\theta(k_\perp)$



[Boldyrev 2006 did this for MHD; *see his talk!*]

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Polarisation Alignment?

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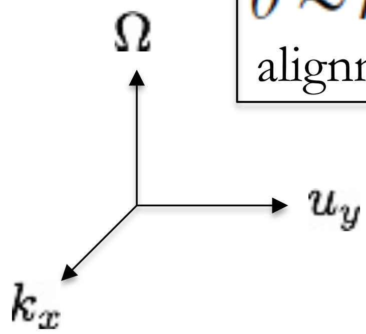

✧ Conjecture $\theta \sim \dfrac{\delta\omega_\perp}{\Omega} \sim \dfrac{u_y}{v}$

(max alignment ~ angular uncertainty)

➢ Then $\tau_{NL}^{-1} \sim k_\perp\,[u_\perp(k_\perp)]^2\,[v(k_\perp)]^{-1}$

[Boldyrev 2006 did this for MHD; *see his talk!*]

J. Fluid Mech. **677**, 134 (2011)

Polarisation Alignment?

Inertial waves have $\delta\boldsymbol{\omega} = \mp k\boldsymbol{u}$, so they are nonlinear solutions

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \delta\boldsymbol{\omega}\times\boldsymbol{u} + \nabla u^2/2 = 0$$

In an inertial wave, $k_y = 0, u_x = 0$

Is there a dynamical (or statistical) tendency for velocity and vorticity to align, i.e., for fluctuations to look like inertial waves?

$k_\perp \sim k_x$

$u_\perp \sim u_y$

$\theta \sim k_y/k_x \ll 1$

alignment angle

✧ Suppose $k_x \gg k_y \gg k_\parallel$ ➔ $u_x \sim (k_y/k_x)u_y \ll u_y$ ➔

➢ Then $\tau_{NL}^{-1} \sim k_y u_y \sim k_\perp u_\perp(k_\perp)\theta(k_\perp)$

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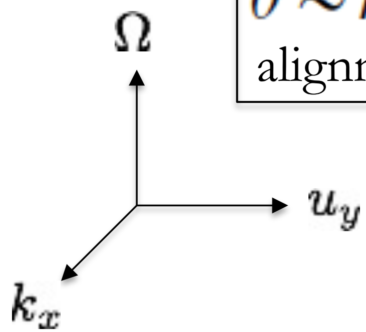

➢ Then $\tau_{NL}^{-1} \sim k_\perp [u_\perp(k_\perp)]^2 [v(k_\perp)]^{-1}$

➢ Using CB etc., $E(k_\perp) \sim [\varepsilon v(k_\perp)]^{1/2} k_\perp^{-3/2} \sim (\varepsilon\Omega)^{1/2} k_\perp^{-2}$

(for $v(k_\perp) = v_A$, get Boldyrev's $k_\perp^{-3/2}$ for Alfvén waves)

[Boldyrev 2006 did this for MHD; *see his talk!*]

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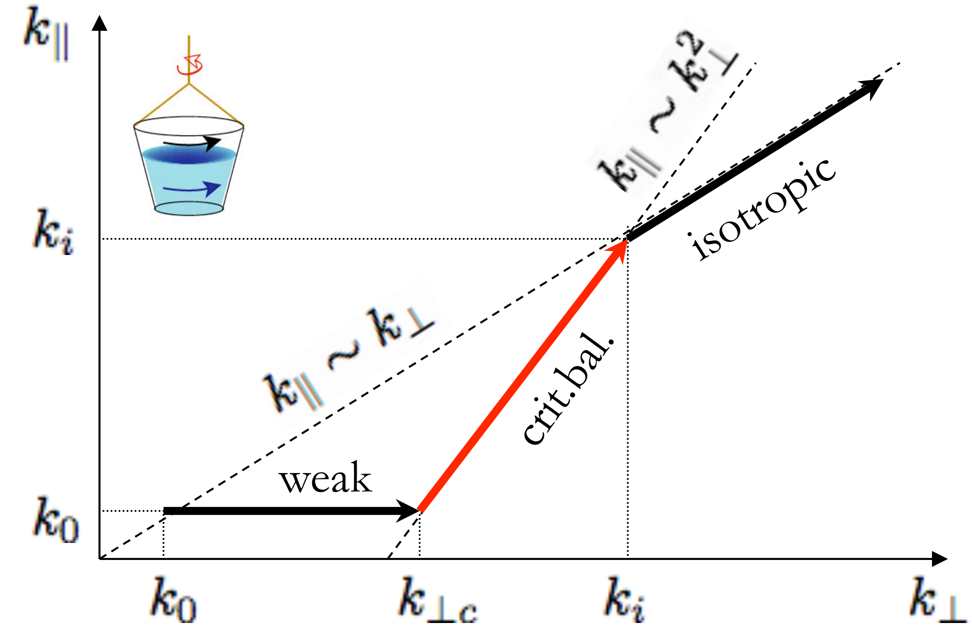
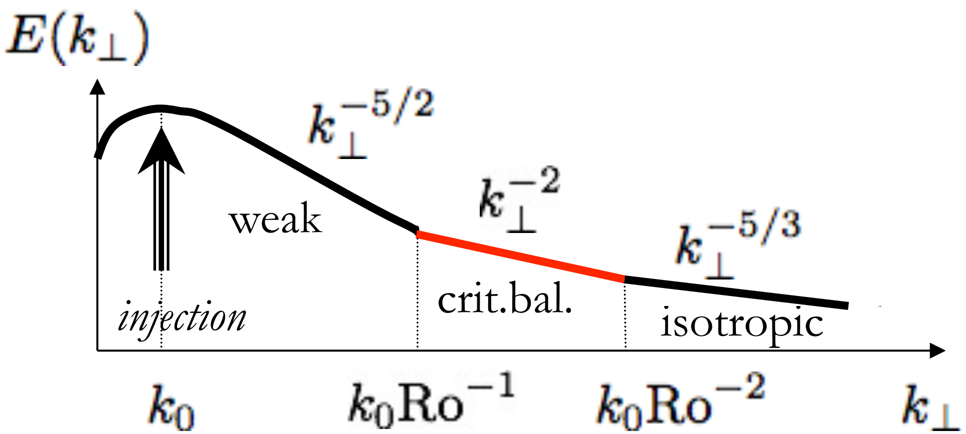
Cascade Path (Amended for Alignment)





- **Weak,** anisotropic:

$$E(k_\perp) \sim (\varepsilon k_0 \Omega)^{1/2} k_\perp^{-5/2}$$

$$k_\parallel \sim k_0$$

$$k_0 < k_\perp < k_{\perp c}$$

$$k_{\perp c} \sim \varepsilon^{-1/4} k_\parallel^{1/2} \Omega^{3/4} \sim k_0 Ro^{-1}$$

- **CB,** anisotropic, aligned:

$$E(k_\perp) \sim (\varepsilon \Omega)^{1/2} k_\perp^{-2}$$

$$k_\parallel \sim \varepsilon^{1/2} \Omega^{-3/2} k_\perp^2$$

$$k_{\perp c} < k_\perp < k_i$$

$$k_i \sim \varepsilon^{-1/2} \Omega^{3/2} \sim k_0 Ro^{-2}$$

Zeman (1994) scale

- **Kolmogorov,** isotropic:

$$E(k) \sim \varepsilon^{2/3} k^{-5/3} \quad k_\perp \sim k_\parallel \sim k$$

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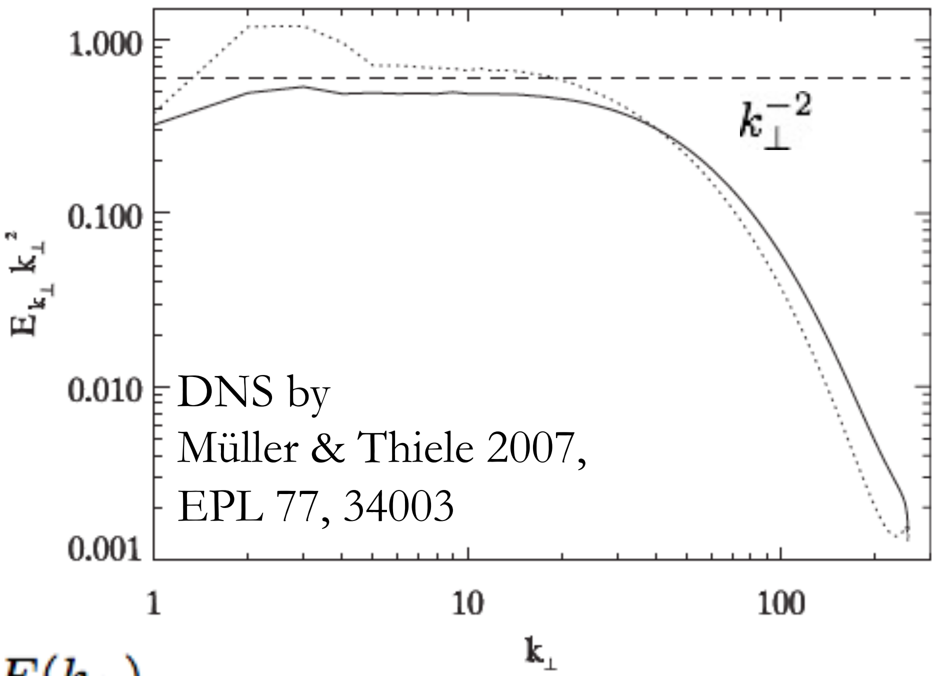
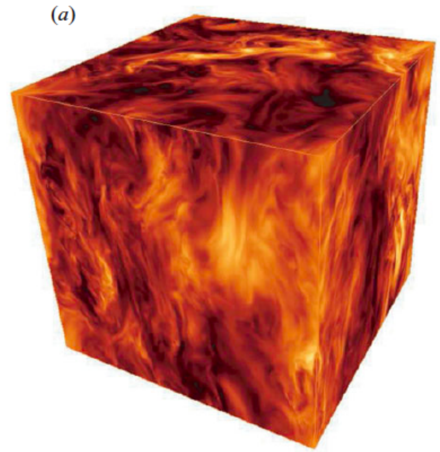
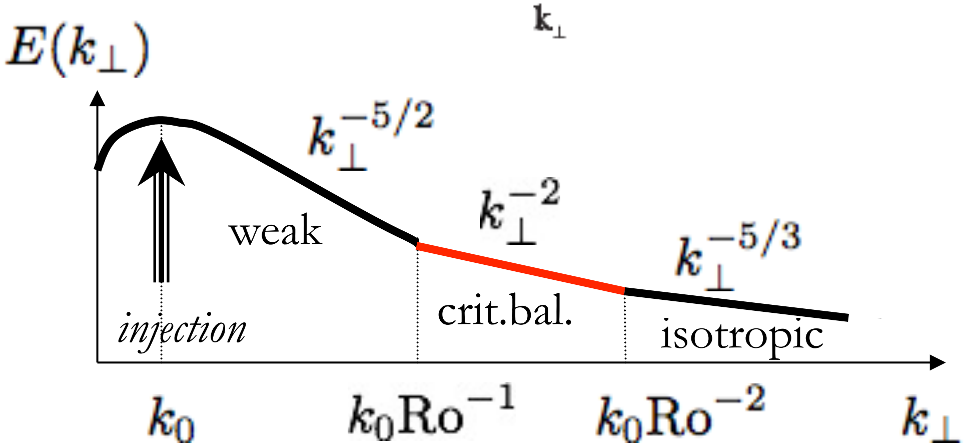




Thiele & Müller 2009,
JFM 637, 425



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DOES THIS HAPPEN IN (VIRTUAL) REALITY?

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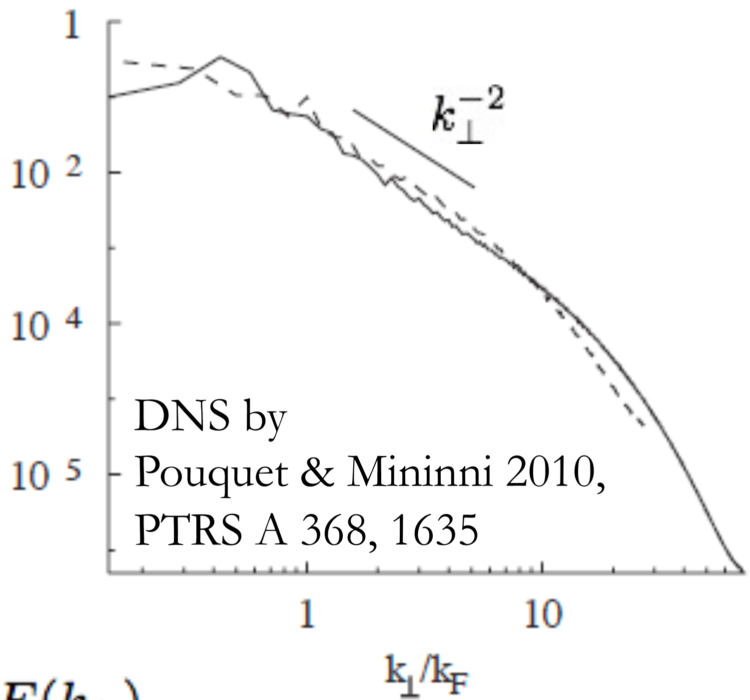
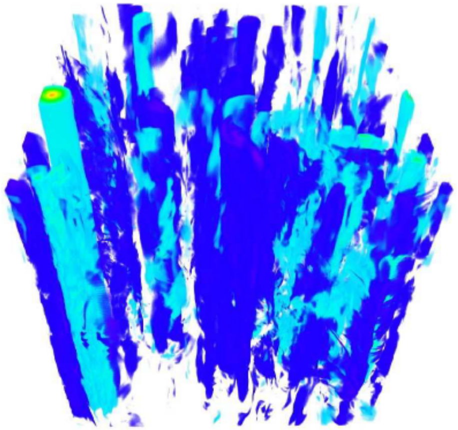
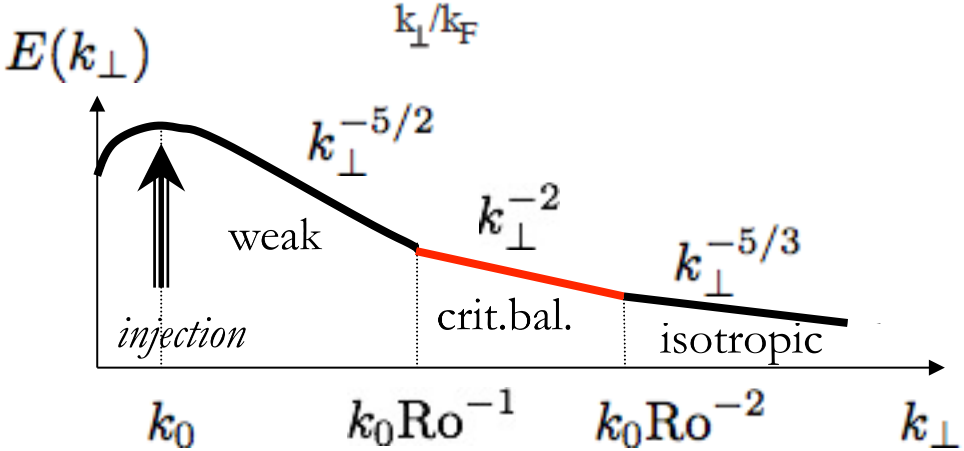




Mininni & Pouquet 2010,
PoF 22, 035106



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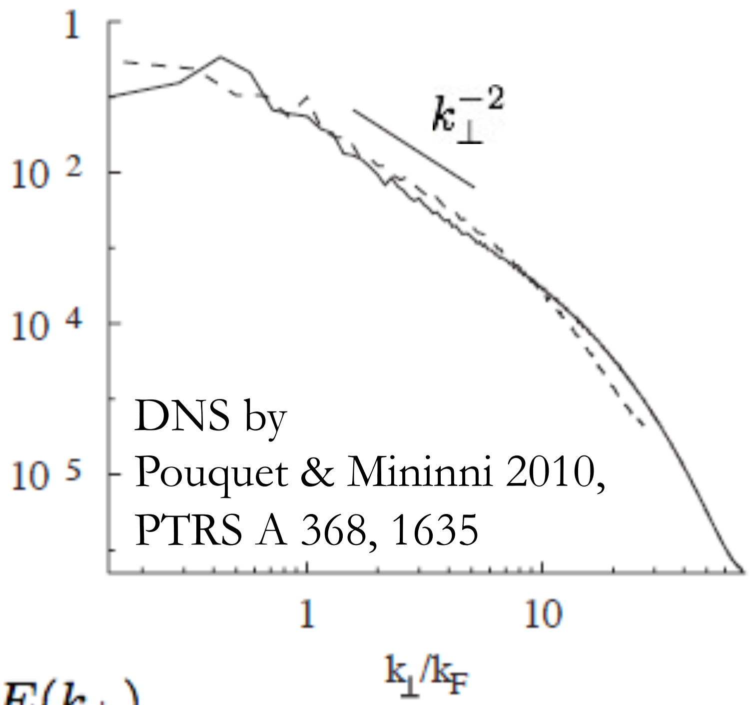

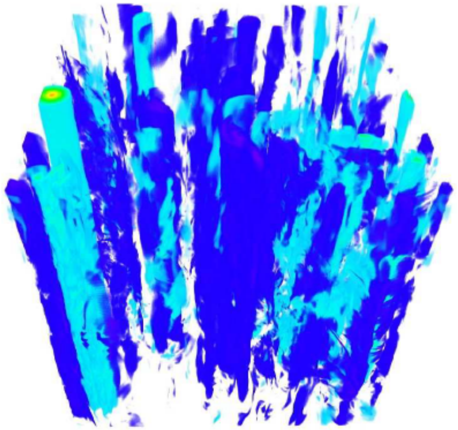

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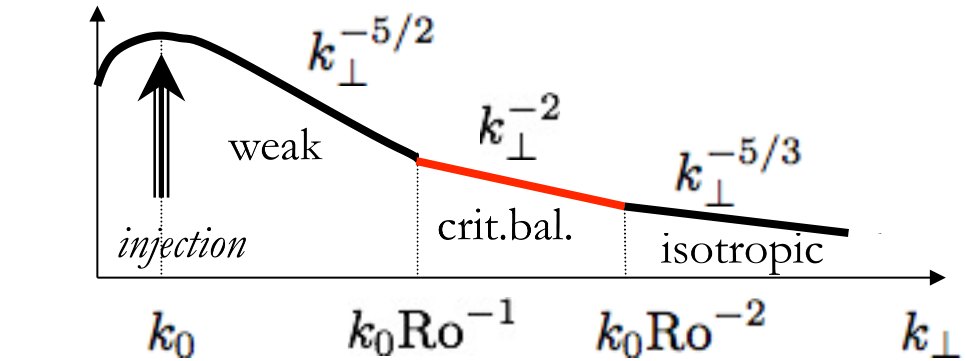

Dealignement: $\theta \sim (k_\perp/k_i)^{1/2}$

(➔decreasing relative helicity)

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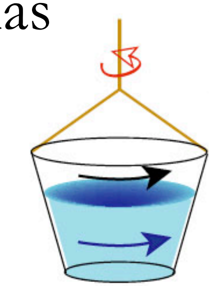
Conclusion

- **Critical balance as a universal scaling conjecture:**
  - ✧ Physically sensible
  - ✧ Aesthetically appealing
  - ✧ So far has stood the test of measurement & simulations
  - ✧ Originated from MHD, has spread to GK & Hall plasmas

    see talks by S. Bale, C. Chen, R. Wicks, S. Boldyrev, F. Parra



- **A novel cascade scenario for rotating turbulence:**
  - ✓ Makes sense
  - ✓ Describes strong anisotropic turbulence at low Rossby number
  - ✓ Naturally implies isotropisation at Zeman scale (had to happen!)
  - ✓ The alignment principle might also be universal

- Other interesting (hydro) examples (but **careful** with some analogies!):
  - Stratified turbulence: CB leads to known/observed spectra

    [Nazarenko & AAS 2011; cf. Dewan 1997, Billant & Chomaz 2001, Lindborg 2006]
  - Shallow water waves [Phillips 1958, Newell & Zakharov 2008]
  - Rossby waves in beta plane [Rhines 1975; *see talks by J. Krommes, J. Parker on ZFs*]
  - Kelvin waves in superfluids [Proment, Nazarenko & Onorato 2009]

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