# A nonlinear perspective on the MRI dynamo transition and subcritical turbulence in sheared plasmas


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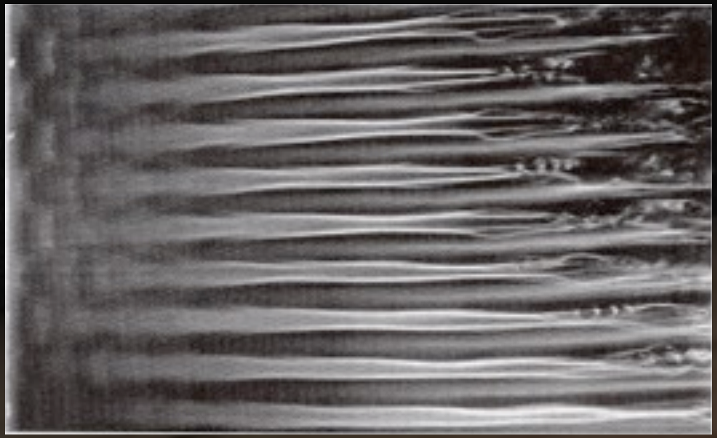
# Transitions in fluid & plasmas

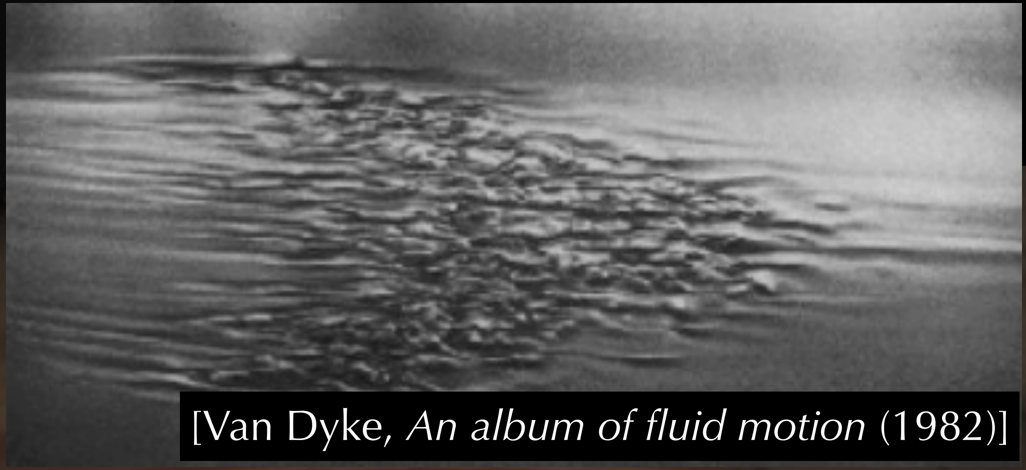
- Supercritical transitions: linear instabilities
  - Hydro: Rayleigh-Benard, Kelvin-Helmholtz, centrifugal...
  - MHD: MRI, Parker instability, magnetic RT, kinematic dynamos...
  - Plasmas: ITG, ETG...
- Subcritical transitions through nonlinear processes
  - Non-rotating, linearly stable hydrodynamic shear flows
    - Pipes, airplanes, cars...
  - Turbulence and instability-driven dynamos in astrophysical shear flows
    - Magnetorotational dynamo in accretion disks
    - Stellar dynamos
  - Sheared plasma turbulence in fusion devices

# Subcritical transitions of shear flows: a glimpse through a pipe

- Many hydrodynamic shear flows have no linear instability...
    - ...but nevertheless undergo a transition to turbulence
    - Pipe flow: a century-old problem
      [Reynolds,  Phil. Trans. R. Soc. 174, 935 (1883)]


- Phenomenology
    - Streaks, turbulent bursting, regeneration cycles



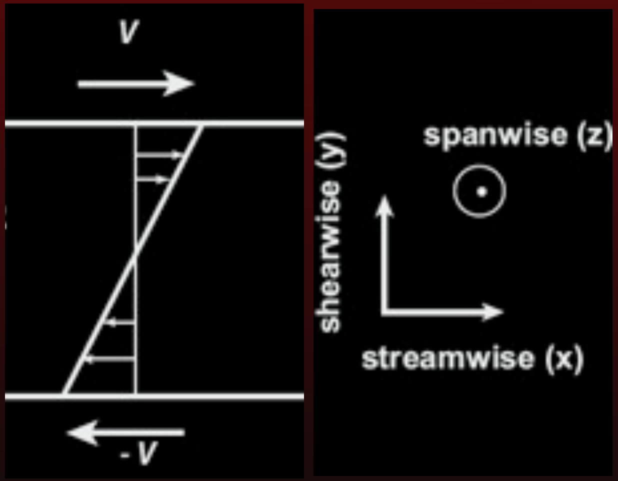


[Van Dyke, *An album of fluid motion* (1982)]

- 
    - Transition requires 3D, finite amplitude perturbations
      [Darbyshire & Mullin,  JFM 289, 83 (1995); Dauchot & Daviaud, Phys. Fluids 7, 335 (1995); Hof et al., PRL 91, 244502 (2003)]

# The 1990s: “by-pass” transitions

- Streamwise-averaged equations




$$\overline{\boldsymbol{u}}_p = \boldsymbol{\nabla} \times (\psi \boldsymbol{e_x}) \ , \quad \omega_x = -\Delta\psi$$

$$\frac{\partial \overline{u}_x}{\partial t} = -\overline{u}_y - \boldsymbol{e}_x \cdot \left(\overline{\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}}\right) + \frac{1}{\mathrm{Re}} \Delta \, \overline{u}_x \ ,$$

$$\frac{\partial \, \omega_x}{\partial t} - \frac{\partial \, (\psi, \omega)}{\partial \, (y, z)} = -\boldsymbol{e}_x \cdot \boldsymbol{\nabla} \times \left(\overline{\boldsymbol{u'} \cdot \boldsymbol{\nabla} \boldsymbol{u'}}\right) + \frac{1}{\mathrm{Re}} \Delta \, \omega_x$$

- Transient linear algebraic growth: the lift-up effect
  [Ellingsen & Palm, Phys. Fluids 18, 487 (1975); Landahl, JFM 98, 243 (1980)]
  - Vigorous amplification of streamwise-independent streaks

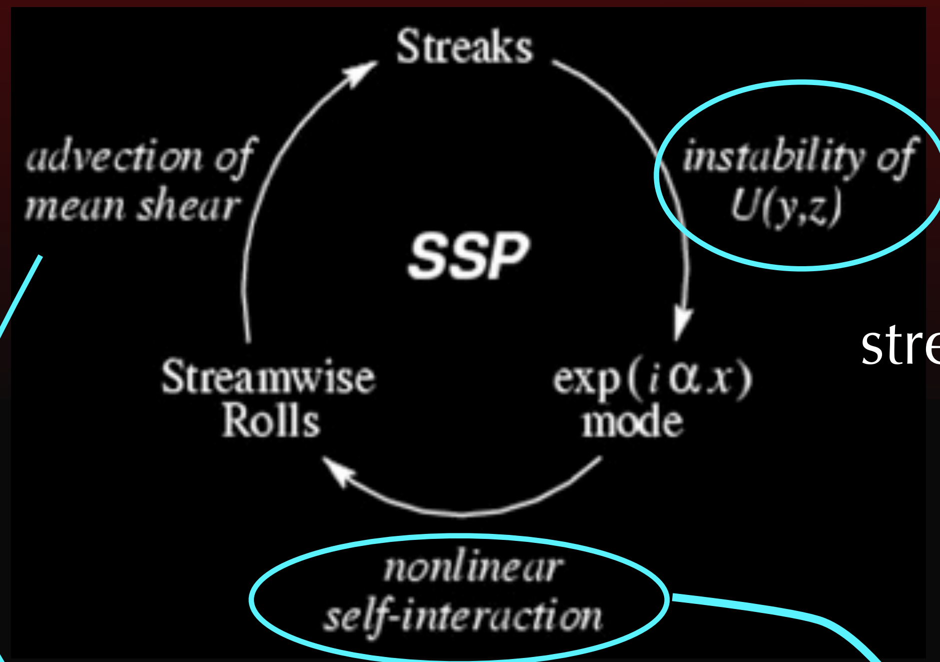
$$\partial_t \overline{u}_x + \overline{u}_y = 0 \quad \Rightarrow \quad \overline{u}_x \sim \overline{u}_y t$$

- And lots of debate on the nonlinear breakdown ensued...
  - “Linear non-normality vs nonlinear normality”

[Trefethen et al., Science 261, 578 (1993); Baggett et al., Phys. Fluids 7, 833 (1995); Waleffe, Phys. Fluids 7, 3060 (1995); Stud. Appl. Math. 95, 319 (1995)]

# The self-sustaining process

- Streaks instability and nonlinear feedback are key




Inflexional, streamwise-dependent

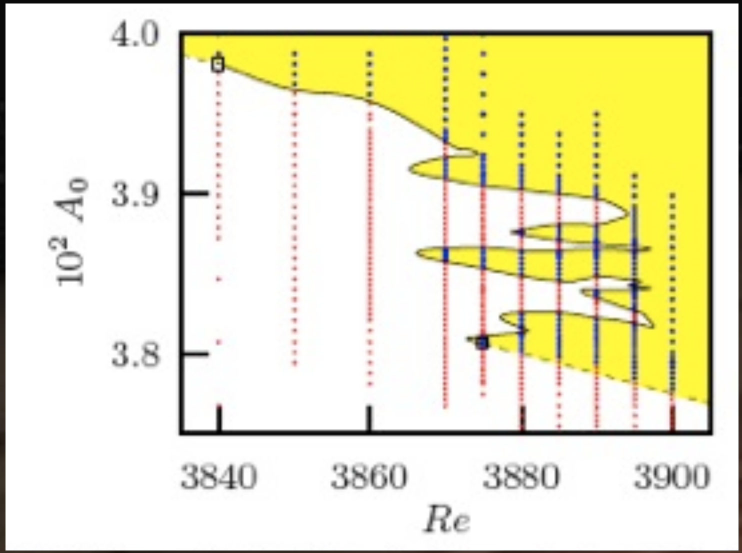
$$\frac{\partial \overline{u}_x}{\partial t} = -\overline{u}_y - \boldsymbol{e}_x \cdot \left(\overline{\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}}\right) + \frac{1}{\mathrm{Re}} \Delta\, \overline{u}_x \ ,$$

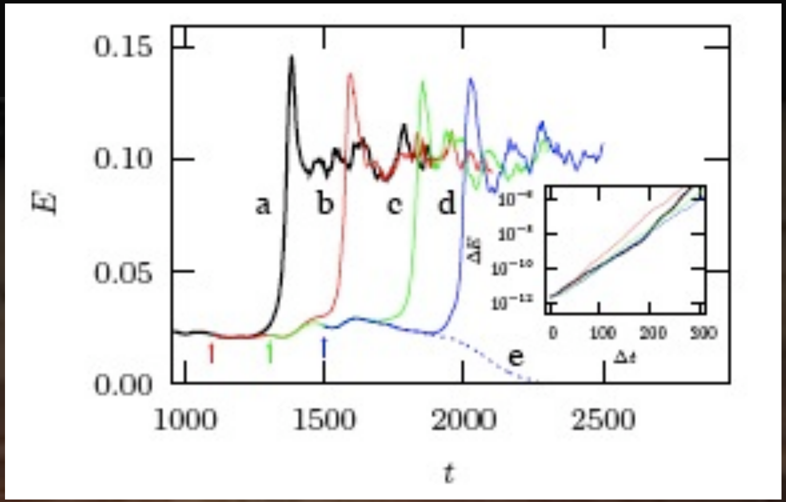
$$\frac{\partial\, \omega_x}{\partial t} - \frac{\partial\,(\psi, \omega)}{\partial\,(y, z)} = -\boldsymbol{e}_x \cdot \boldsymbol{\nabla} \times \left(\overline{\boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}'}\right) + \frac{1}{\mathrm{Re}} \Delta\, \omega_x$$

[Hamilton et al, JFM 287, 317 (1995); Waleffe, Phys. Fluids 9, 883 (1997)]

# Transitional dynamics

- Sensitive dependence on initial conditions
- Finite lifetime statistics, growing exponentially with Re
  [Faisst and Eckhardt, JFM 504, 343 (2004); Hof et al., Nat. 443, 59 (2006); PRL 101, 214501 (2008)]
- "Edge of chaos"
  - Transition to long-lasting bursty, turbulent behaviour on one side
  - Smooth decay to the laminar solution on the other side

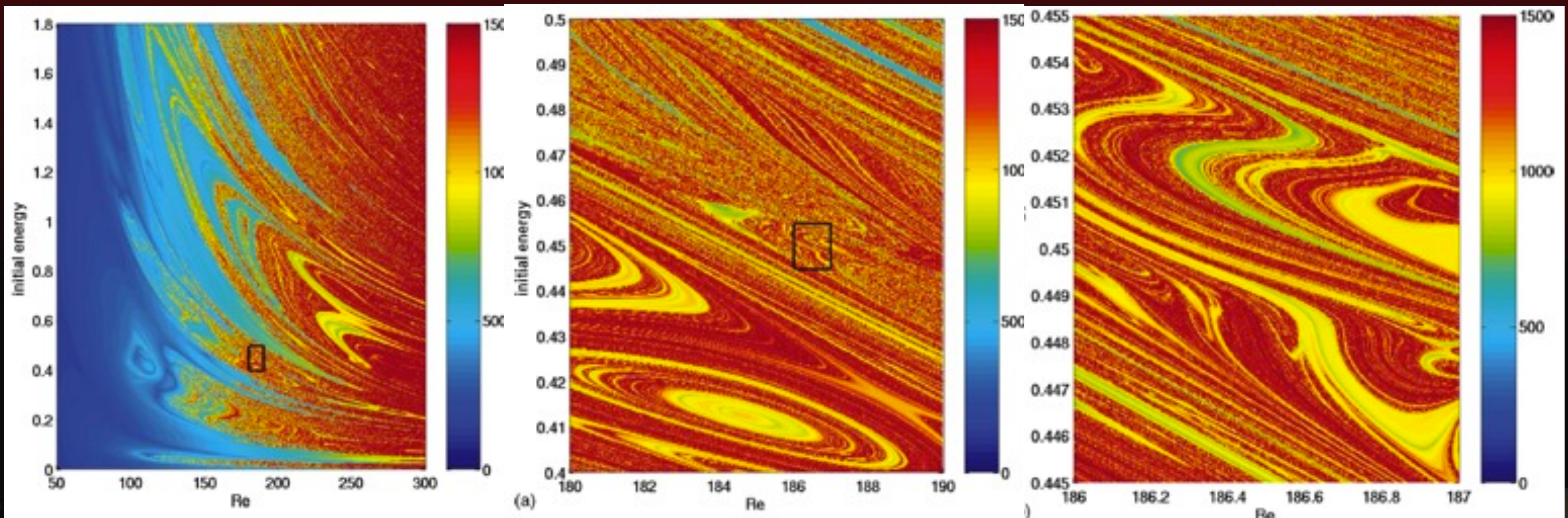



[Schneider et al., PRL 99, 034502 (2007)]

# A chaotic saddle

- Transitional dynamics structured around a chaotic repeller
  [Schmiegel & Eckhardt, PRL, 79, 5250 (1997)]

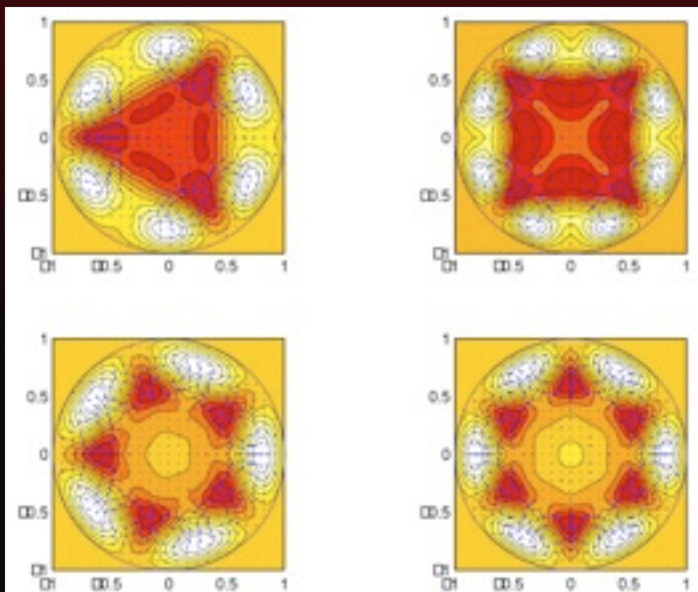
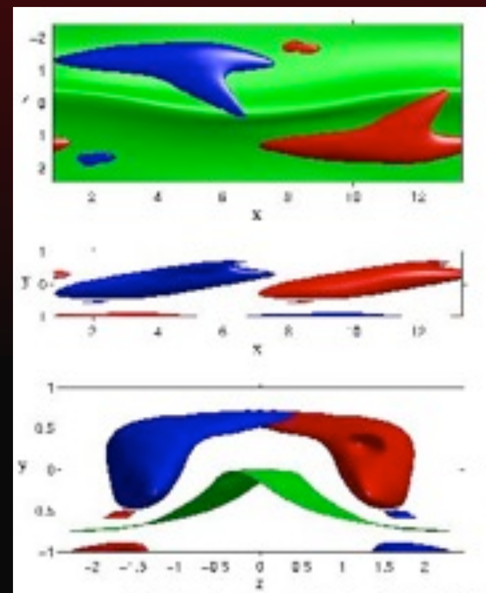
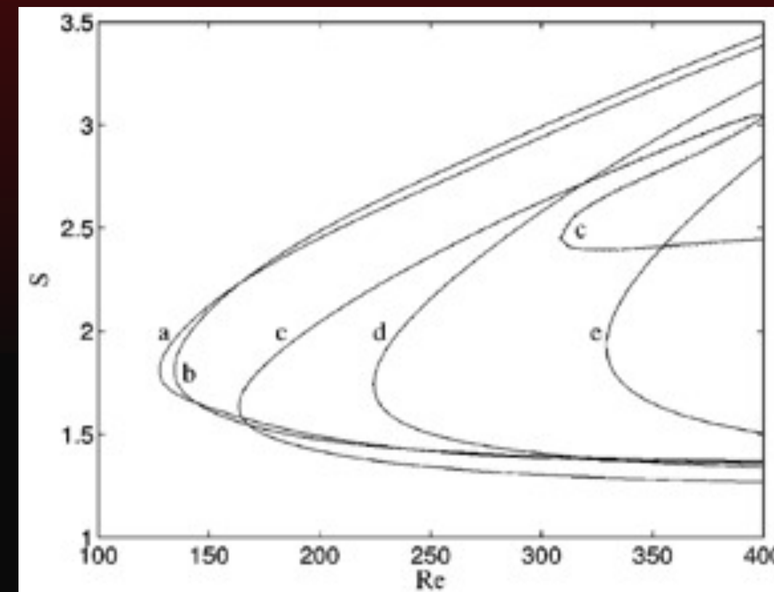



[Moehlis et al., Chaos, 14, S11 (2004)]

- (Mostly) open questions
  - What are the bifurcations mechanisms at work ?
  - Can we make a rigorous assessment of typical transition Re ?
  - What are the statistical properties of turbulence (transport etc.) ?

# A zoo of nonlinear invariant solutions

- Fixed points, nonlinear travelling waves, periodic orbits

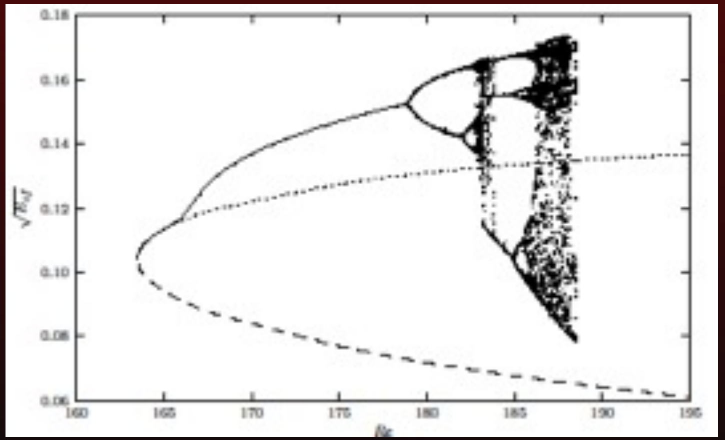





[Nagata, JFM 217, 519 (1990); Waleffe, PRL 81, 4140 (1998); Faisst & Eckhardt, PRL 91, 224502 (2003); Waleffe, Phys. Fluids 15, 1517 (2003); Wedin & Kerswell, JFM 508, 333 (2004); Viswanath, JFM 580, 339 (2007); Gibson et al., JFM 638, 243 (2009)]

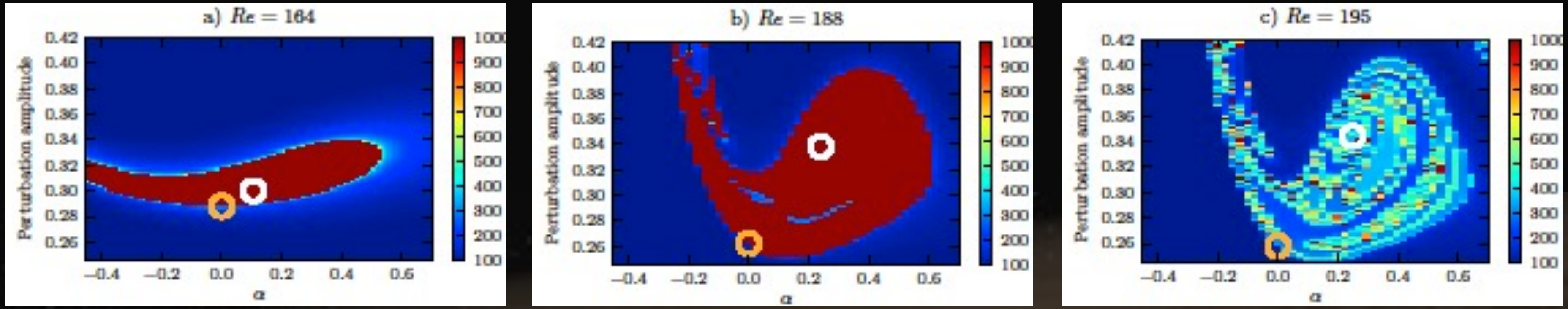
- Physically supported by the self-sustaining process
- Born out of saddle node bifurcations
  - Not connected to the laminar state

# Possible bifurcation mechanisms

- Bifurcations of invariant solutions
  - Local period doubling cascades
  - Global bifurcations
    - Homoclinic explosions, crises



[Kreilos & Eckhardt, Chaos 22, 047505 (2012)]



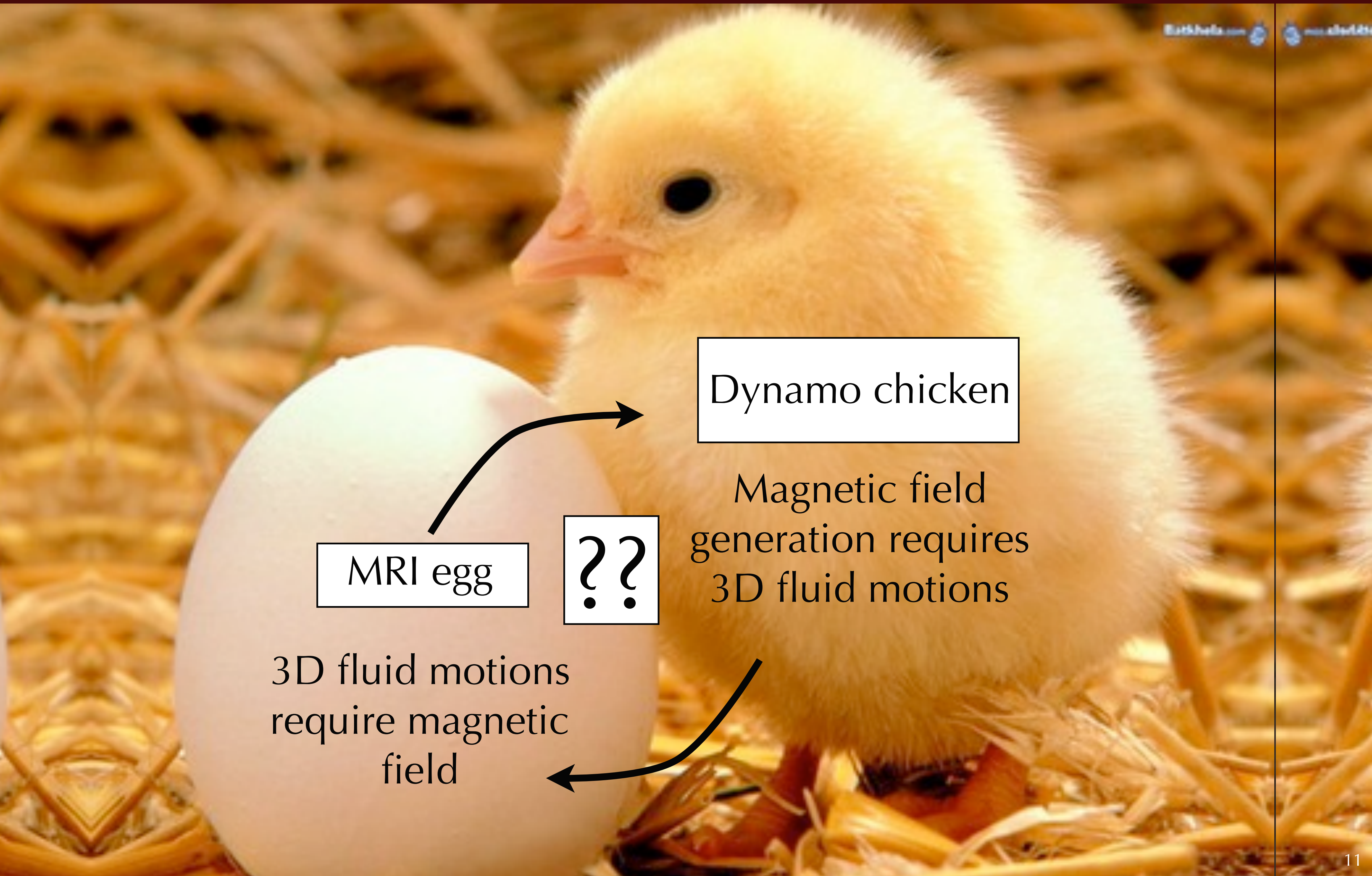

- Currently, lots of activity in this area...

[Gibson et al., JFM 611, 107 (2008); Halcrow et al., 621, 365 (2009); Vollmer et al., NJP 11, 013040 (2011); van Veen & Kawahara, PRL 107, 114501 (2011); Kreilos & Eckhardt, Chaos 22, 047505 (2012); Mellibovsky & Eckhardt, JFM 670, 96 (2011); JFM 709, 149 (2012); Willis et al., JFM (2013)]

# Turbulence and dynamo in accretion disks

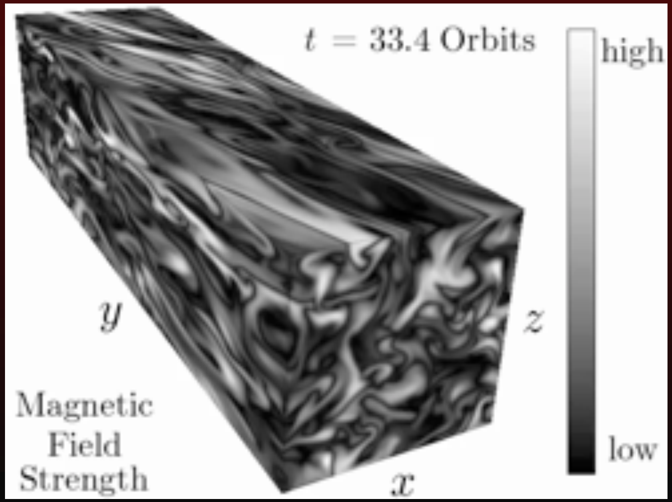
- The magnetorotational instability
  [Velikhov, JETP 36, 1098 (1959), Chandrasekhar, PNAS 46, 253 (1960)]
  - Instability of differentially rotating flows with $\dfrac{d\,\Omega^2}{dr} < 0$
  - Requires a weak magnetic field: magnetic tension is essential
  - Linear, exponentially growing in the presence of a uniform field
- Astrophysical relevance [Balbus & Hawley, ApJ 376, 214 (1991)]
  - Keplerian accretion disks $\Omega(r) \propto 1/r^{3/2}$ are thought to be MRI-unstable
  - Subsequent turbulence transports angular momentum (hence accretion)
- The magnetorotational (disk) dynamo
  - How do you get the magnetic field in the first place ?
    - Requires some form of turbulent induction
      - But turbulence in disks apparently requires a magnetic field...

# The MRI dynamo problem



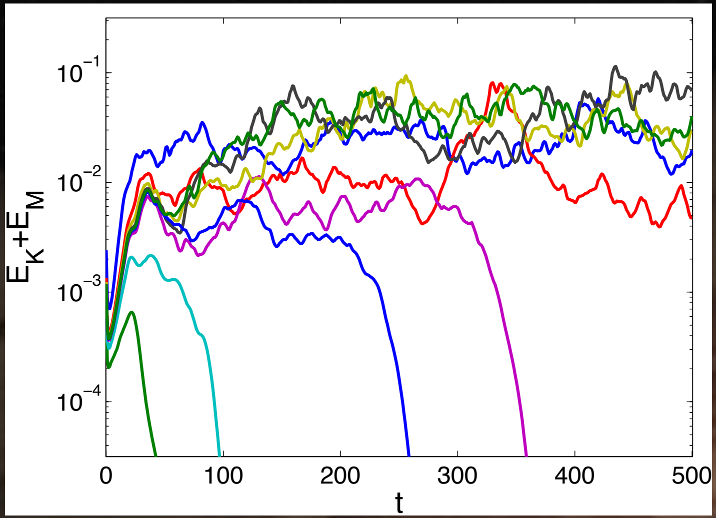
# Zero net-flux MRI is a subcritical dynamo

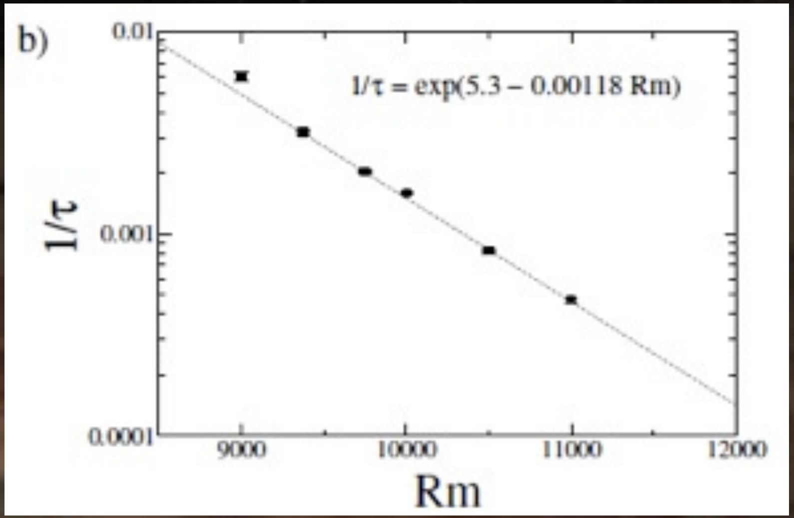
- Only 3D case is sustained
  [Brandenburg et al., ApJ 446, 741 (1995); Hawley et al., ApJ 464, 690 (1996)]



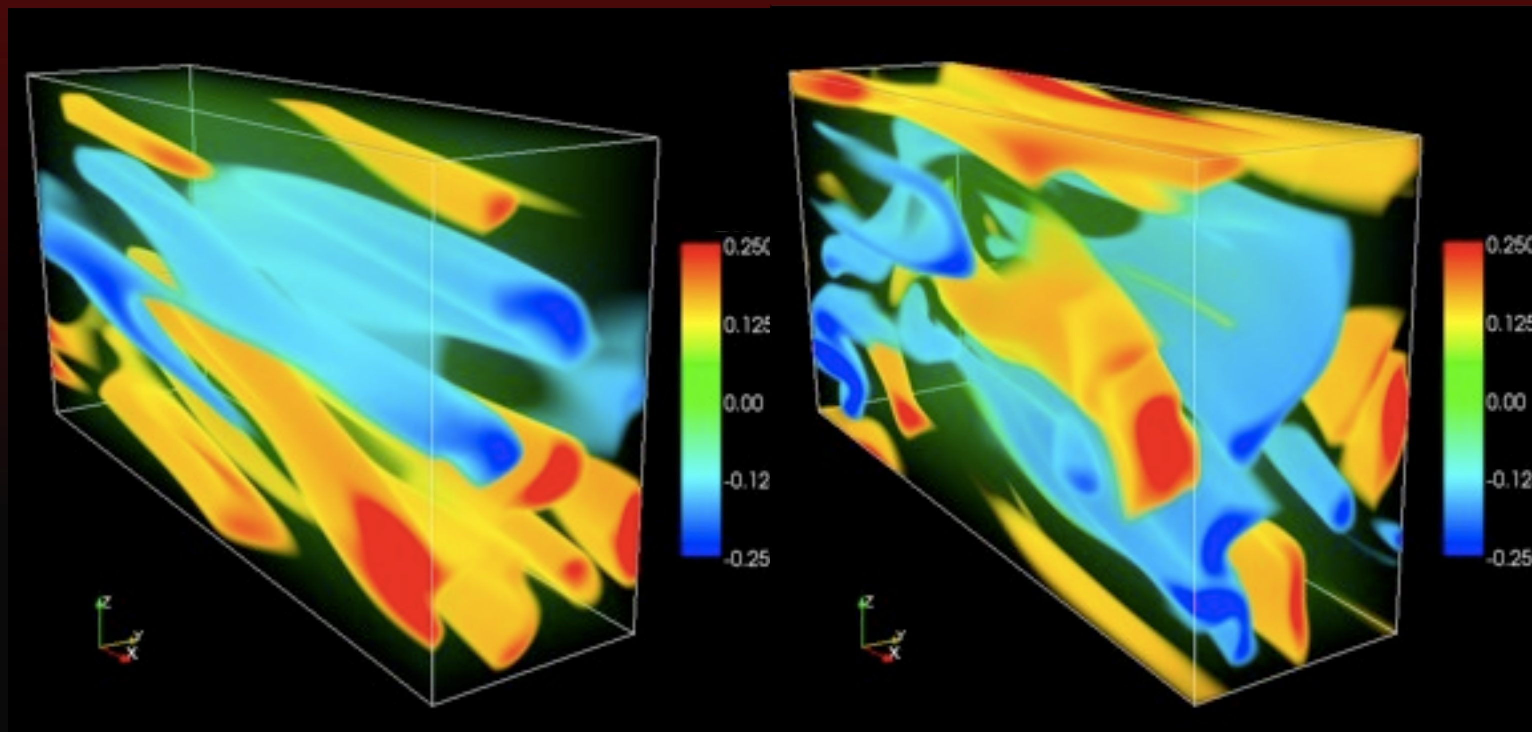

[courtesy T. Heinemann]

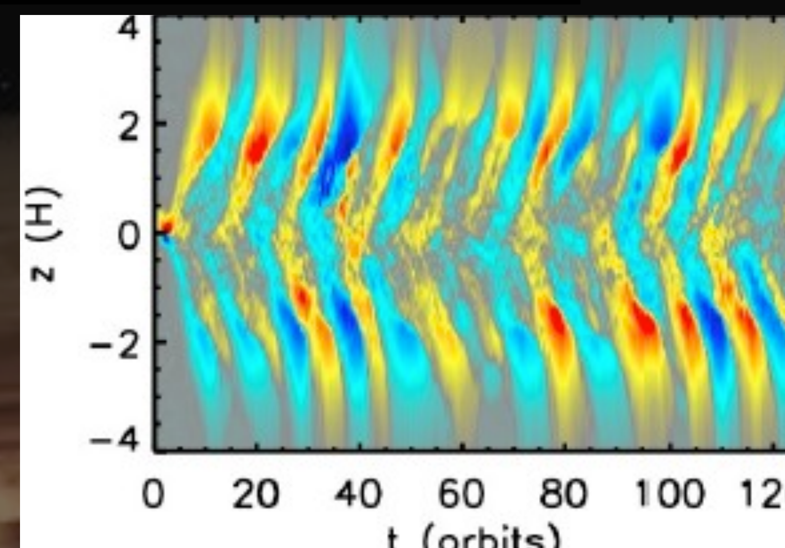
- Not a kinematic dynamo
  - Decays with no Lorentz force
  - Requires finite-amplitude perturbations
- "Supertransient [Rempel et al., PRL 105, 044501 (2010)]
  - Finite lifetime growing exponentially with Rm



# Pseudo-cyclic MRI dynamo action



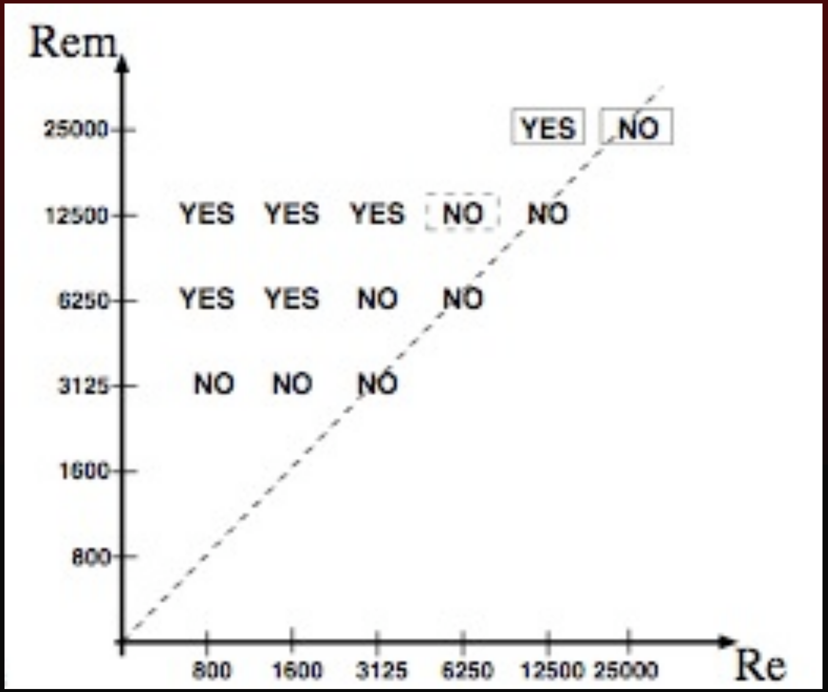




- Oscillations of the large-scale toroidal/azimuthal field

[[Brandenburg et al., ApJ 446, 741 (1995); Lesur & Ogilvie, A&A 488, 451 (2008); Davis et al., ApJ 713, 52 (2010); Simon et al., ApJ 730, 94 (2011); MNRAS 422, 2685 (2012)]

# Interesting questions

- What happens ?
  - How is the dynamics excited ?
  - How is it sustained ?
  - Why do we see recurrent dynamics ?

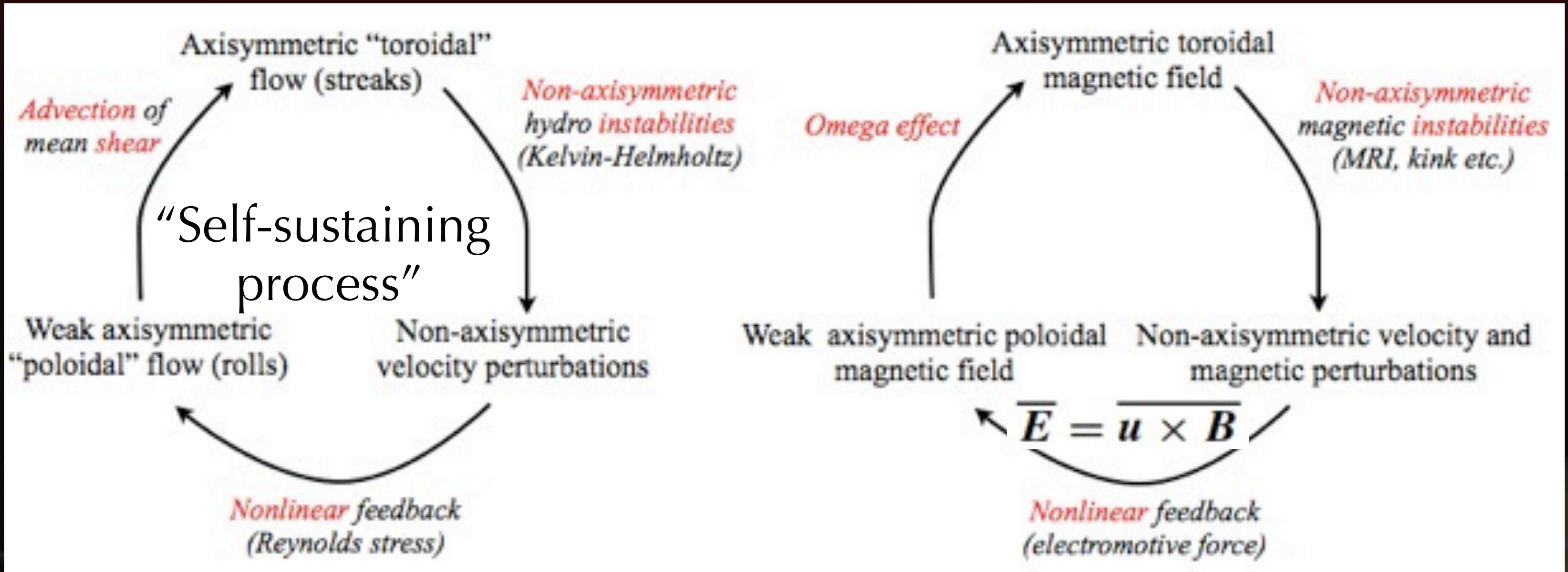



[Fromang et al., A&A 476, 1123 (2007)]

- Astrophysical relevance
  - The magnetic Prandtl number issue
  - Transport efficiency
- Dynamos in general
  - Theoretical framework for such a dynamo ?
  - Is the MRI dynamo a good candidate for experiments ?
  - Is this problem unique in dynamo theory ?

# Connexion between hydro & MRI dynamo

- Basic requirements for instability-driven dynamos
  - A background shear flow
  - A configuration prone to non-axisymmetric MHD instability



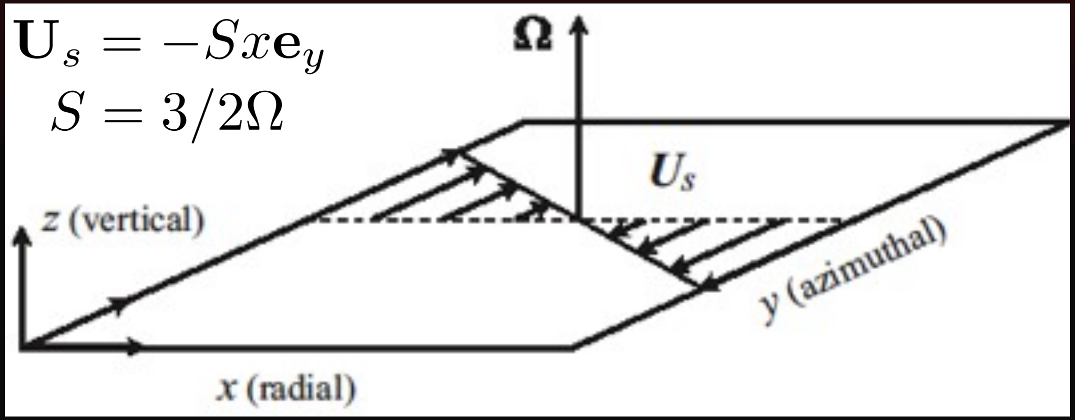

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = -S\,\overline{B}_x\,\boldsymbol{e}_y + \overline{\nabla \times \boldsymbol{E}} + \eta \Delta \overline{\boldsymbol{B}}.$$

[Rincon et al., PRL 98, 254502 (2007); Astron. Nachr. 329, 750 (2008)]

- The MRI dynamo is an archetype of this class of dynamos
  - Also: magnetic buoyancy, Spruit-Pitts-Tayler dynamo, magnetoshear

# Proposed approach

- Proceed along the lines of the hydro transition problem
  - Self-sustaining process, nonlinear invariant solutions, bifurcations ?
- Simplest possible physical configuration
  - Local approximation of Keplerian flow
  - 3D dissipative MHD
  - Incompressible
  - Zero net-flux conserved

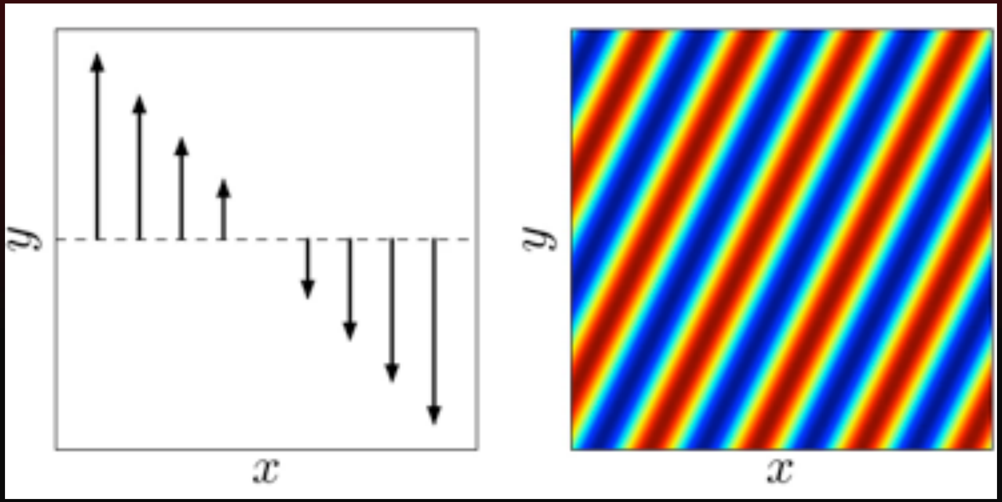


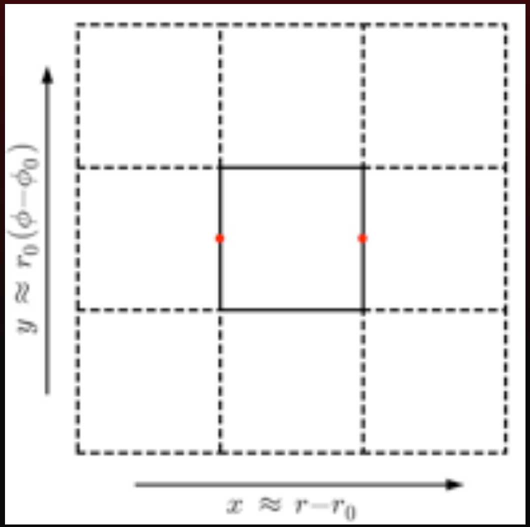

[Goldreich & Lynden-Bell, MNRAS 130, 125 (1965)]

- Reduce the dynamical complexity
  - Constrain the dynamics by using a “minimal” box
  - Reduce the dynamics to symmetric MHD subspaces
  - Focus on transitional regimes: low Re & Rm ~ a few 100

# The shearing box

- Shearing-periodic boundary conditions




[Courtesy G. Lesur & T. Heinemann]

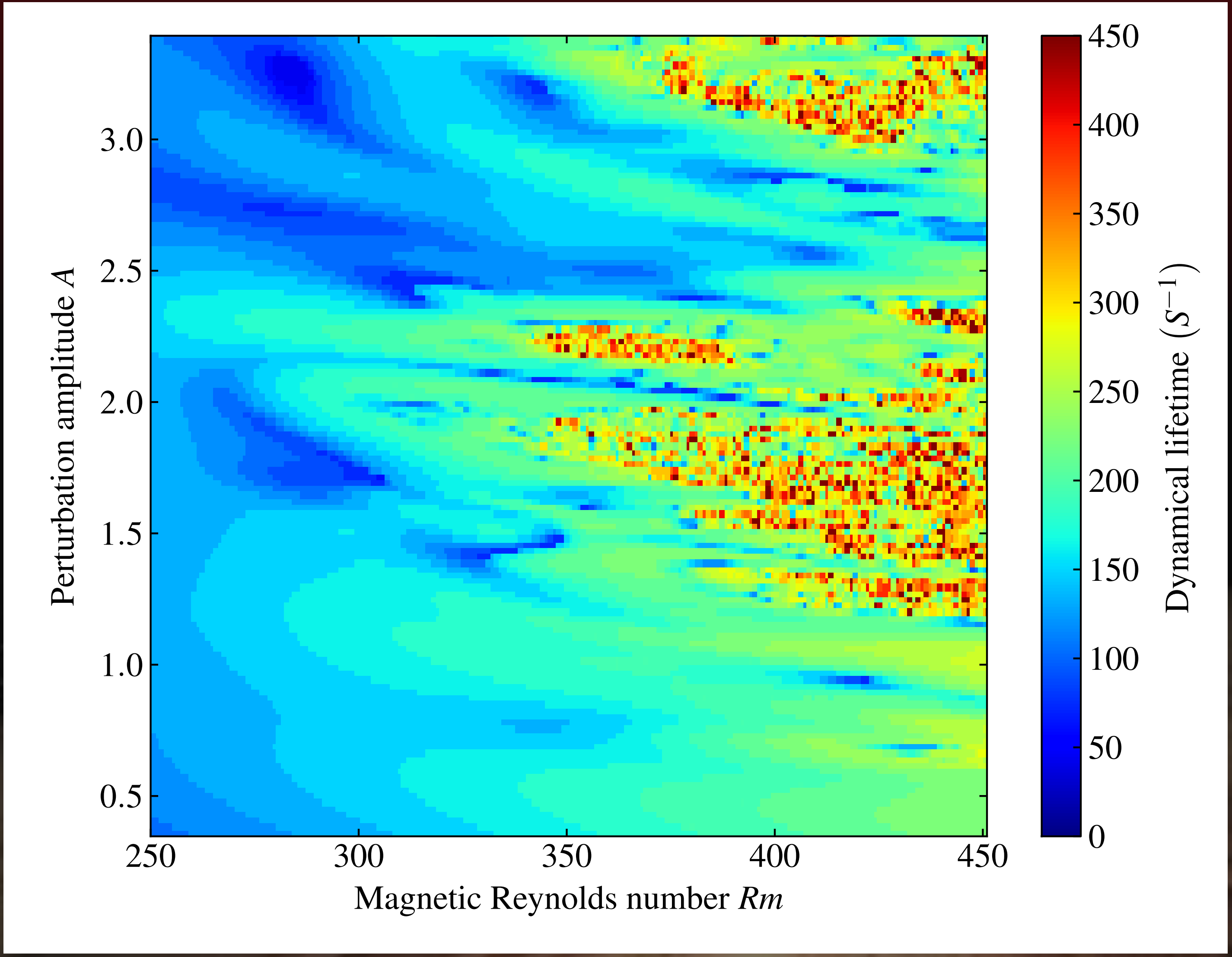
- SNOOPY: a spectral implementation
  - Shearing waves basis

$$\mathbf{Q}(x, y, z, t) = \sum_{\mathbf{k}} \widehat{\mathbf{Q}}_{\mathbf{k}}(t) \exp\left[\exp(i\mathbf{k}(t)\cdot\mathbf{x})\right]$$

$$\mathbf{k}(t) = (k_x^0 + Stk_y)\mathbf{e_x} + k_y\mathbf{e_y} + k_z\mathbf{e_z}$$

# The edge of MRI dynamo chaos

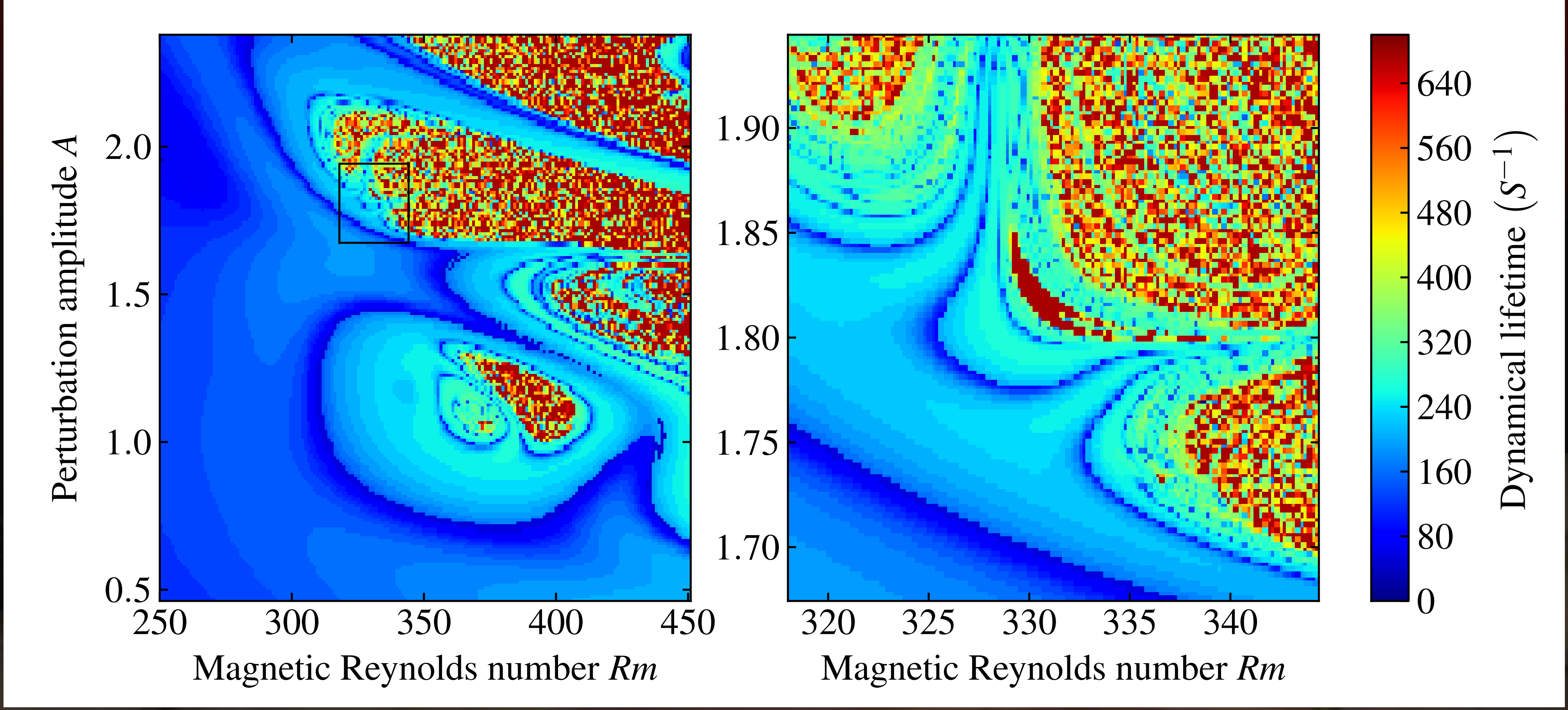
Re=70; Pm~4-8; $(L_x,L_y,L_z)=(0.7,20,2)$




[Riols, Rincon, Cossu et al., submitted to J. Fluid Mech.]

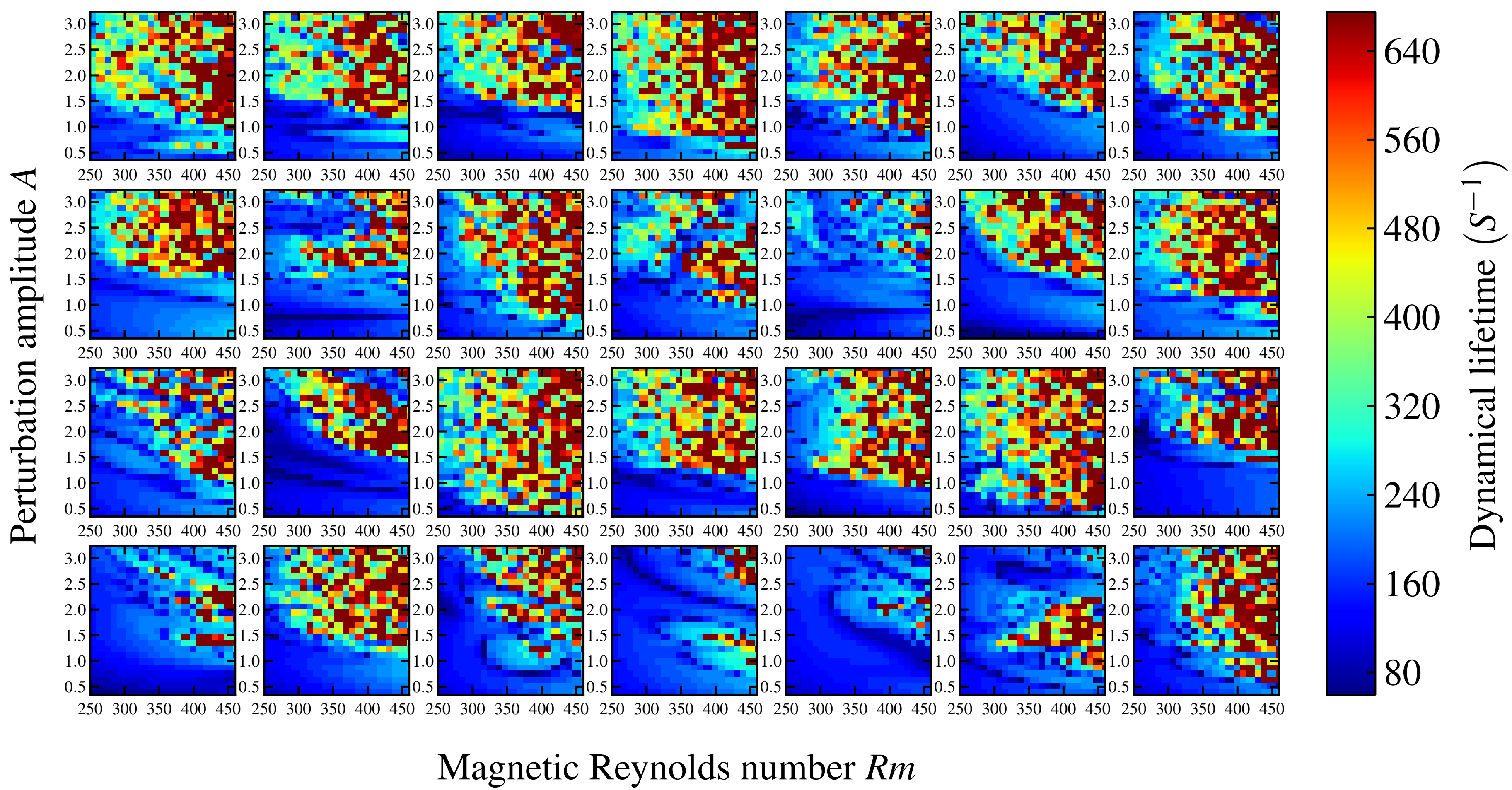
# Same in a symmetric subspace

Re=70; Pm ~4-8; $(L_x,L_y,L_z)=(0.7,20,2)$



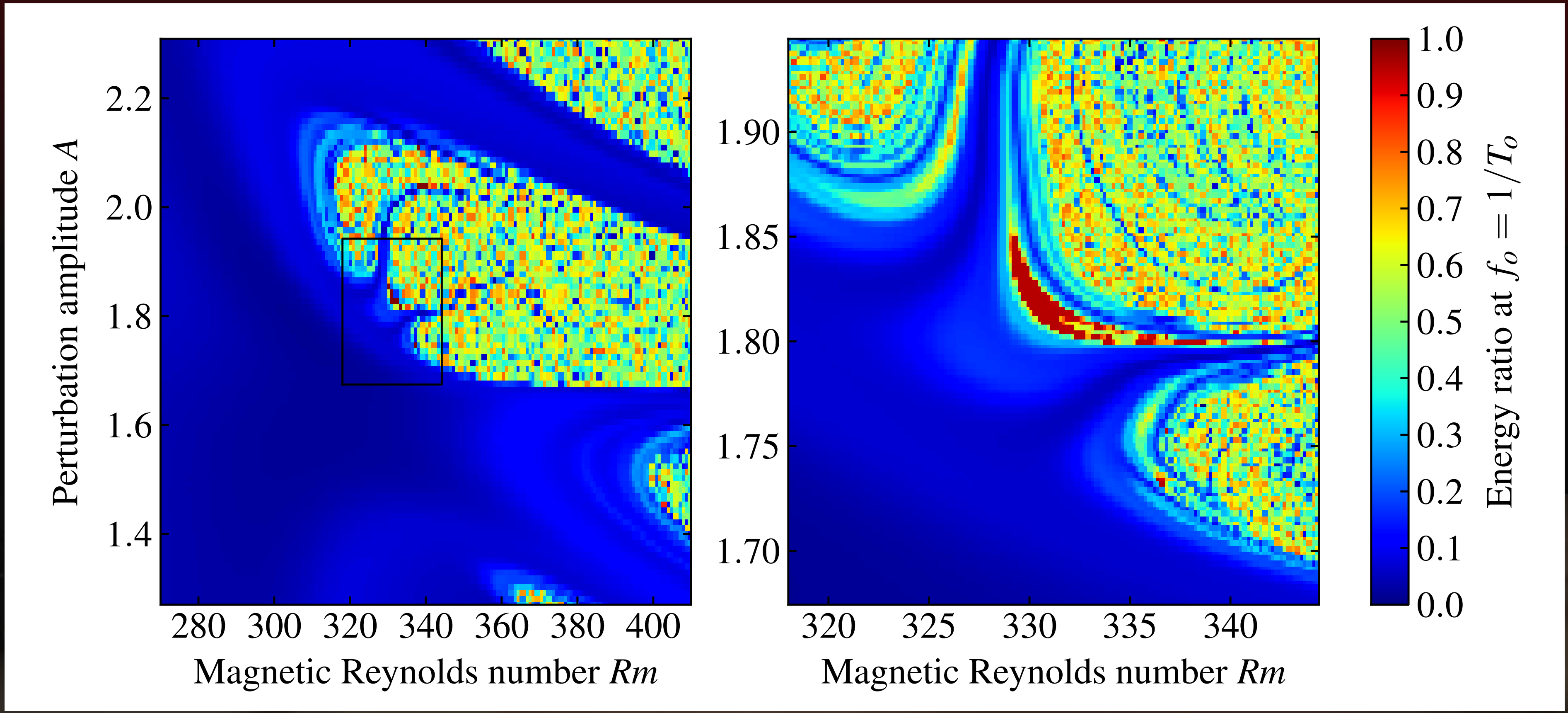
# Exciting the system in different ways

Re=70; Pm ~4-8; $(L_x,L_y,L_z)=(0.7,20,2)$



# Transitional "fingers"...and periodic orbits

Re=70; Pm ~4-8; $(L_x,L_y,L_z)=(0.7,20,2)$




- Nonlinear periodic orbits: skeleton of the transition ?

# Dynamical systems approach

- Physical systems are commonly described as flows in time of physical fields X (velocity, B...) depending on space

$$\frac{\partial \vec{X}}{\partial t} = \mathcal{NL}(\vec{X}) \qquad \text{A vector flow}$$

- Interesting behaviours of these systems include
  - Fixed points (stationary solutions): $\mathcal{NL}(\vec{X}) = \vec{0}$
  - Periodic orbits: $\vec{X}(t = T, \vec{x}) = \vec{X}(t = 0, \vec{x})$

    $$\vec{X}(T, \vec{X}(0)) \equiv \phi^T(\vec{X}(0)) = \vec{X}(0)$$
  - Bifurcations of these solutions to more complex dynamical behaviour

- How do we compute all these things ?

# Newton's method

- Fixed point: look for $\vec{X}^*$ such that $\mathcal{NL}(\vec{X}^*) = \vec{0}$
  - Predict: start from a guess $\vec{X}_0$ "not too far" from the solution
  - Compute the residual $\mathcal{NL}(\vec{X}_o) = \vec{R}$
  - Now, $\vec{X}^* = \vec{X}_o + \delta\vec{X}$

Jacobian operator $\mathbf{J}_0$

$$\begin{aligned} \mathcal{NL}(\vec{X}^*) &= \mathcal{NL}(\vec{X}_0 + \delta\vec{X}) \\ &\simeq \mathcal{NL}(\vec{X}_0) + \left.\frac{\partial \mathcal{NL}}{\partial \vec{X}}\right|_{\vec{X}_0} \delta\vec{X} = \vec{0} \end{aligned}$$

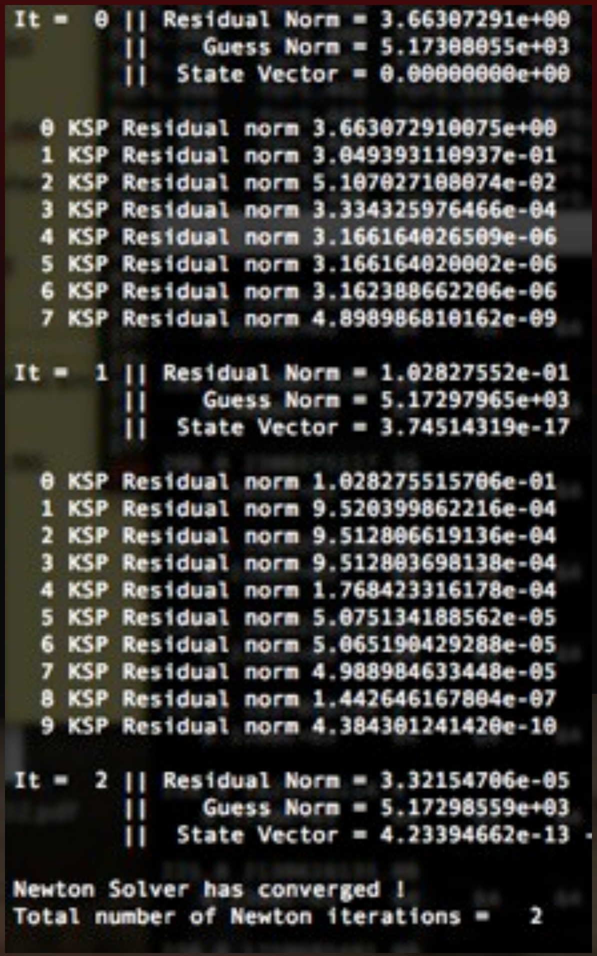
  - Correct: find $\delta\vec{X}$ by solving $\left.\frac{\partial \mathcal{NL}}{\partial \vec{X}}\right|_{\vec{X}_0} \delta\vec{X} = -\vec{R}$
  - Iterate as many times as needed until residual is small enough
- Periodic orbit: same for functional $\Psi(\vec{X}, T) = \phi^T(\vec{X}) - \vec{X}$
  - A state vector $\vec{X}^*$ is on a T-periodic orbit if $\Psi(X^*, T) = \vec{0}$
  - Note: $\phi^T(\vec{X})$ is the result of integration of PDE for time T, i.e. of a DNS !

# Important practical concerns

- How do you solve $\mathbf{J}_n\, \delta\vec{X} = -\vec{R}_n$ numerically ?
  - 1D equation (Ginzburg-Landau, Kuramoto-Sivashinsky): $N = N_x = 64$
  - 3D, incompressible MHD problem at $32^3$
    - $N \sim N_x \times N_y \times N_z \times$ (2 V components + 2 B components) ~ 130 000 !
- Two major problems
  - The Jacobian operator is usually dense and fills your computer memory very easily if you form it explicitly
  - Solving the system by Gaussian elimination is very expensive for large systems and difficult to parallelize -- impractical for N > a few thousands
- Use an iterative, matrix-free approach
  - Krylov methods (GMRES)
  - The Jacobian is not formed in the process

# The PEANUTS code

- General features
  - Written in C, parallel if DNS code is parallel
- Philosophy
  - Separate problem-specific code (DNS) from nonlinear solver layer
  - Avoid reinventing the wheel…use libraries !
- Based on object-oriented toolkits
  - PETSc (ANL): matrix-free Newton-Krylov solver
  - SLEPc (Valencia): matrix-free stability solver

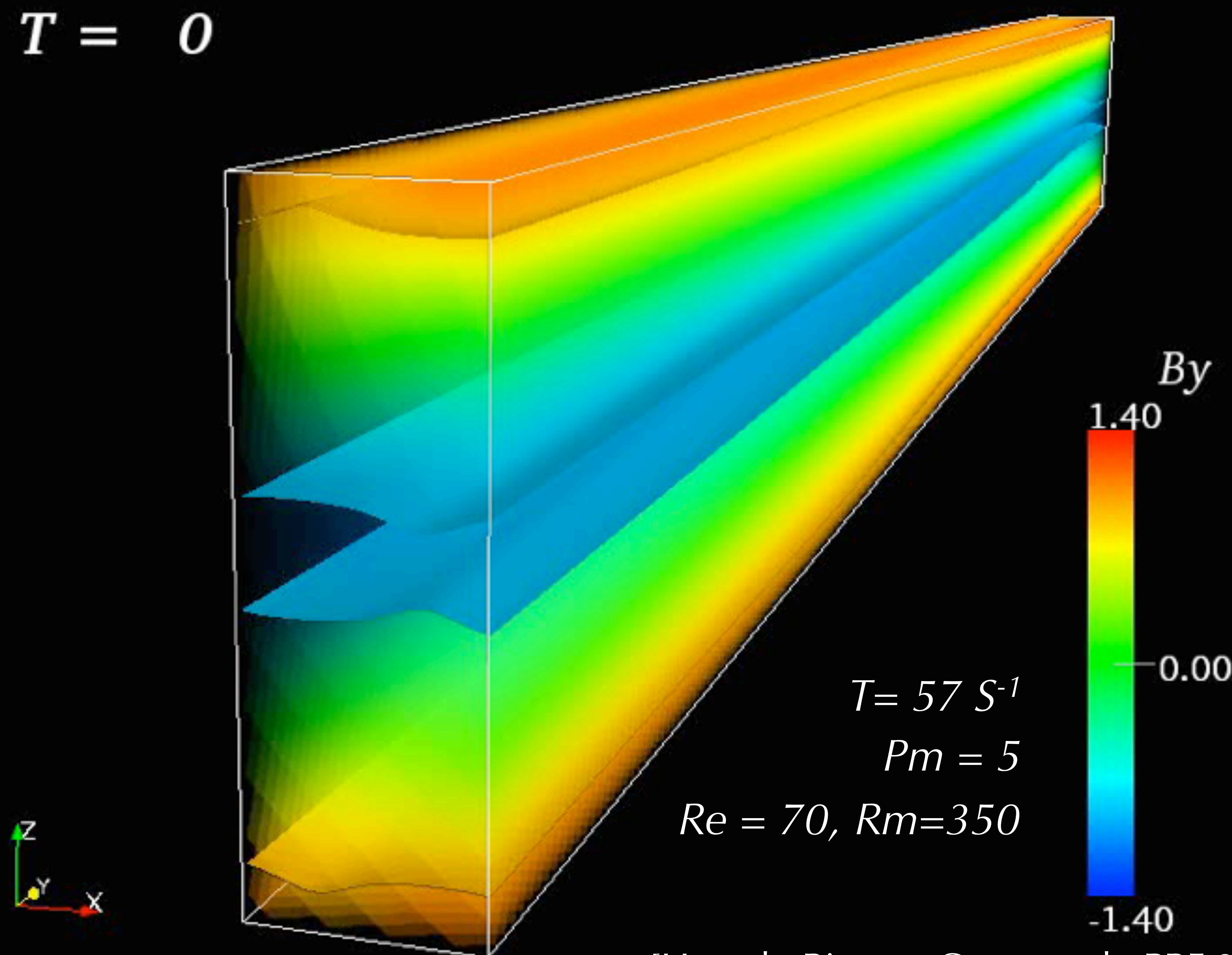


```
It =   0 || Residual Norm = 3.66307291e+00
         ||    Guess Norm = 5.17308055e+03
         ||  State Vector = 0.00000000e+00

   0 KSP Residual norm 3.663072910075e+00
   1 KSP Residual norm 3.049393110937e-01
   2 KSP Residual norm 5.107027108074e-02
   3 KSP Residual norm 3.334325976466e-04
   4 KSP Residual norm 3.166164026509e-06
   5 KSP Residual norm 3.166164020002e-06
   6 KSP Residual norm 3.162388662206e-06
   7 KSP Residual norm 4.898986810162e-09

It =   1 || Residual Norm = 1.02827552e-01
         ||    Guess Norm = 5.17297965e+03
         ||  State Vector = 3.74514319e-17

   0 KSP Residual norm 1.028275515706e-01
   1 KSP Residual norm 9.520399862216e-04
   2 KSP Residual norm 9.512806619136e-04
   3 KSP Residual norm 9.512803698138e-04
   4 KSP Residual norm 1.768423316178e-04
   5 KSP Residual norm 5.075134188562e-05
   6 KSP Residual norm 5.065190429288e-05
   7 KSP Residual norm 4.988984633448e-05
   8 KSP Residual norm 1.442646167804e-07
   9 KSP Residual norm 4.384301241420e-10

It =   2 || Residual Norm = 3.32154706e-05
         ||    Guess Norm = 5.17298559e+03
         ||  State Vector = 4.23394662e-13

Newton Solver has converged !
Total number of Newton iterations =    2
```

- Two nested iterative loops (Newton, Krylov)
  - Convergence to periodic solutions typically requires 100 DNS

# Thanks to PEANUTS, SNOOPY now loves to cycle

# A nonlinear MRI dynamo limit cycle !



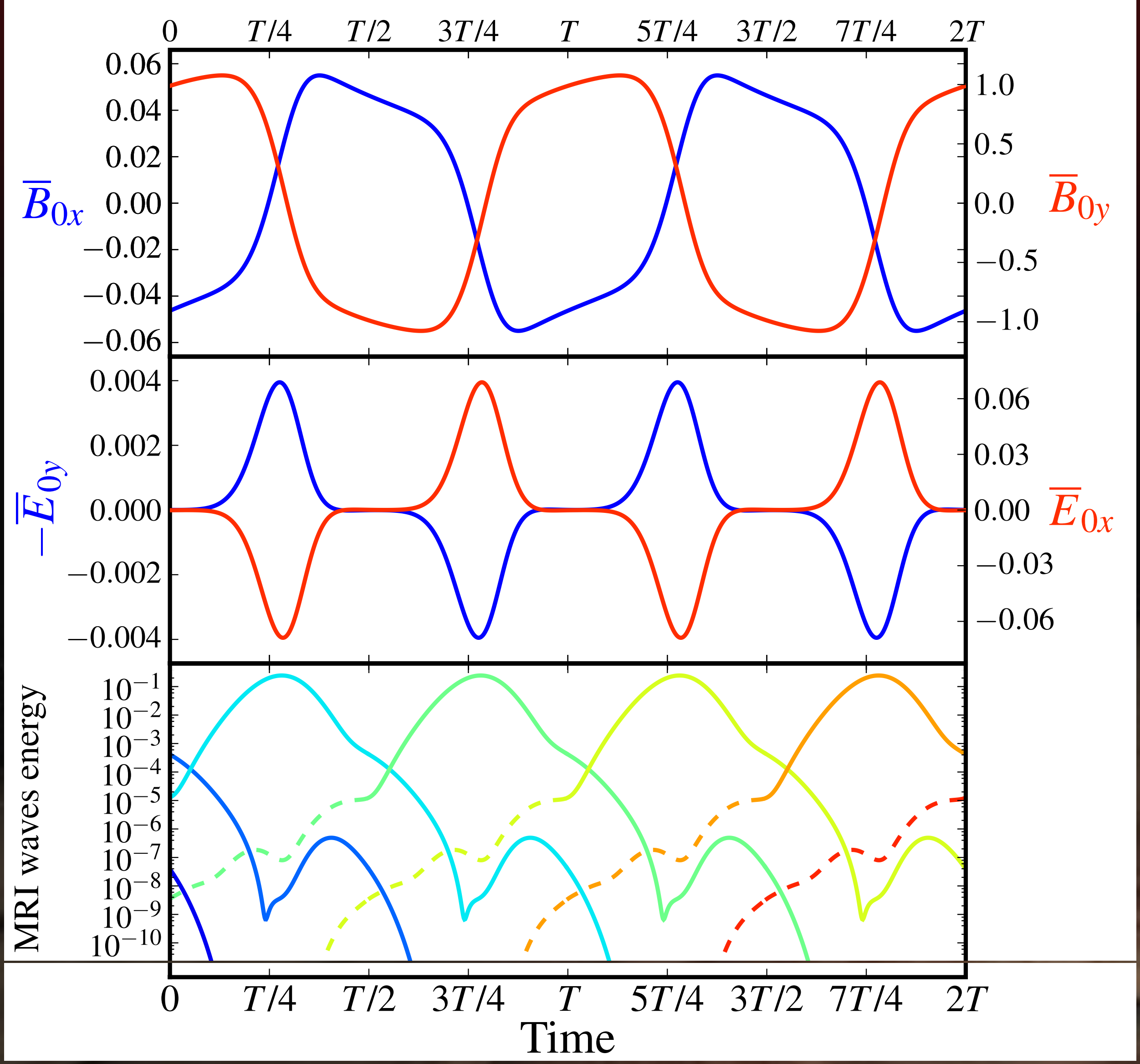

$T = 57\ S^{-1}$

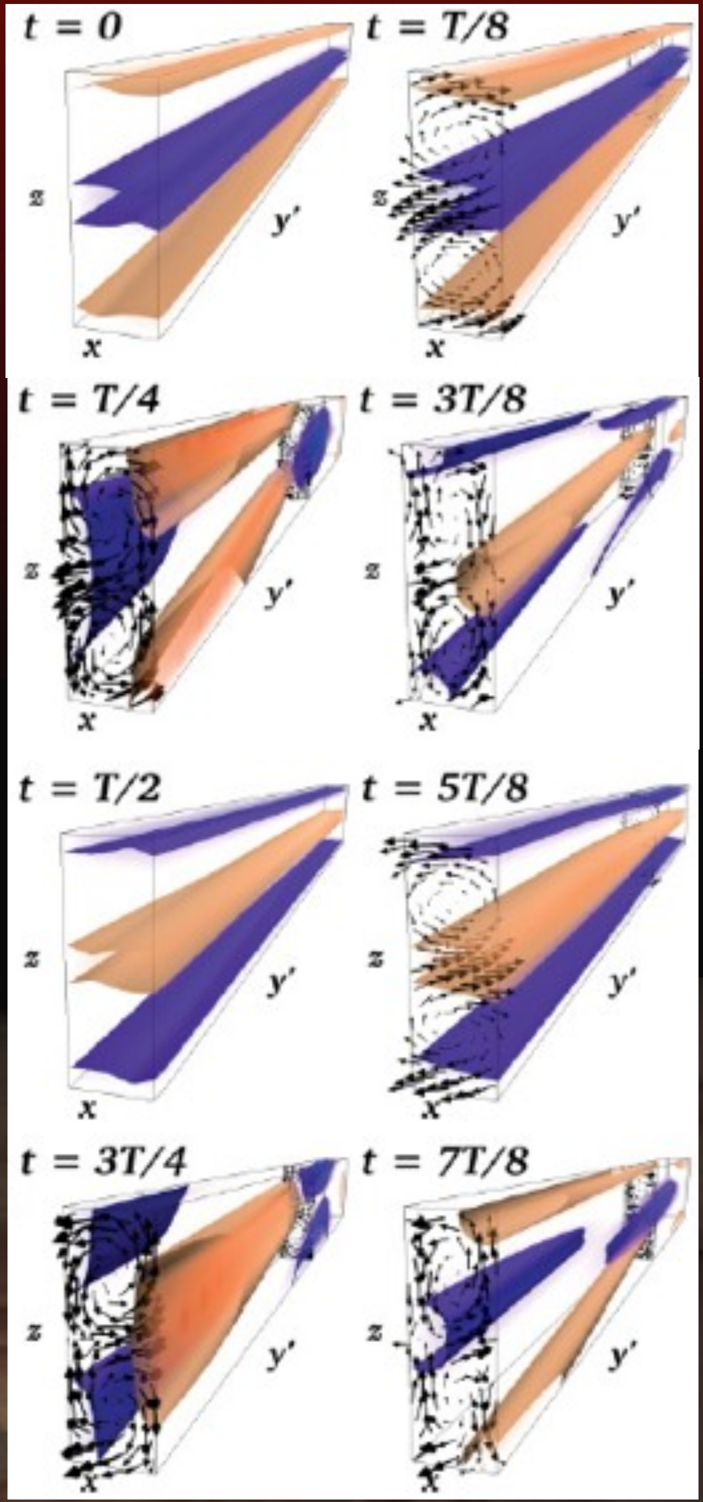
$Pm = 5$

$Re = 70$, $Rm = 350$

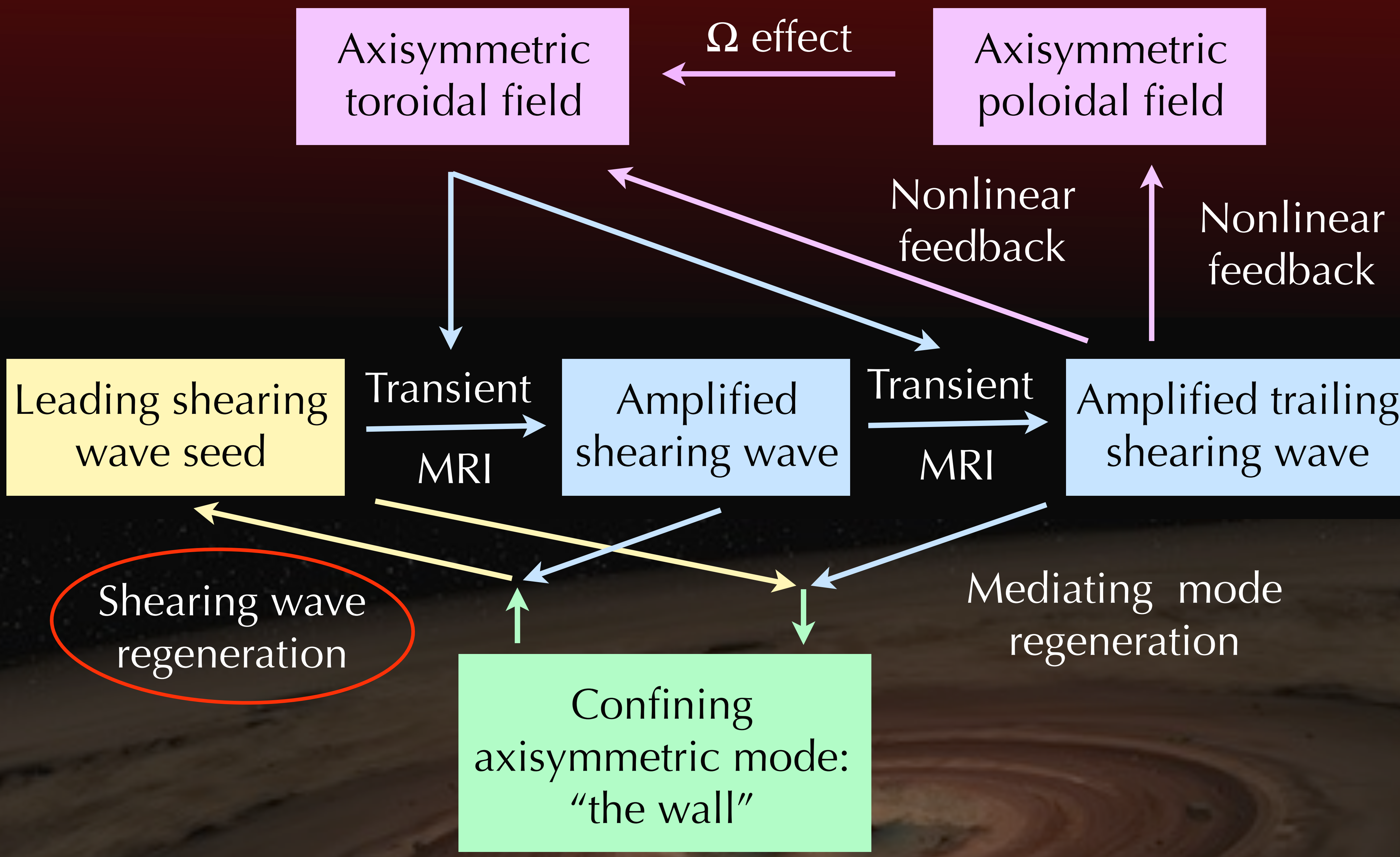
[Herault, Rincon, Cossu et al., PRE 2011]

# Non-axisymmetric MRI-driven reversals

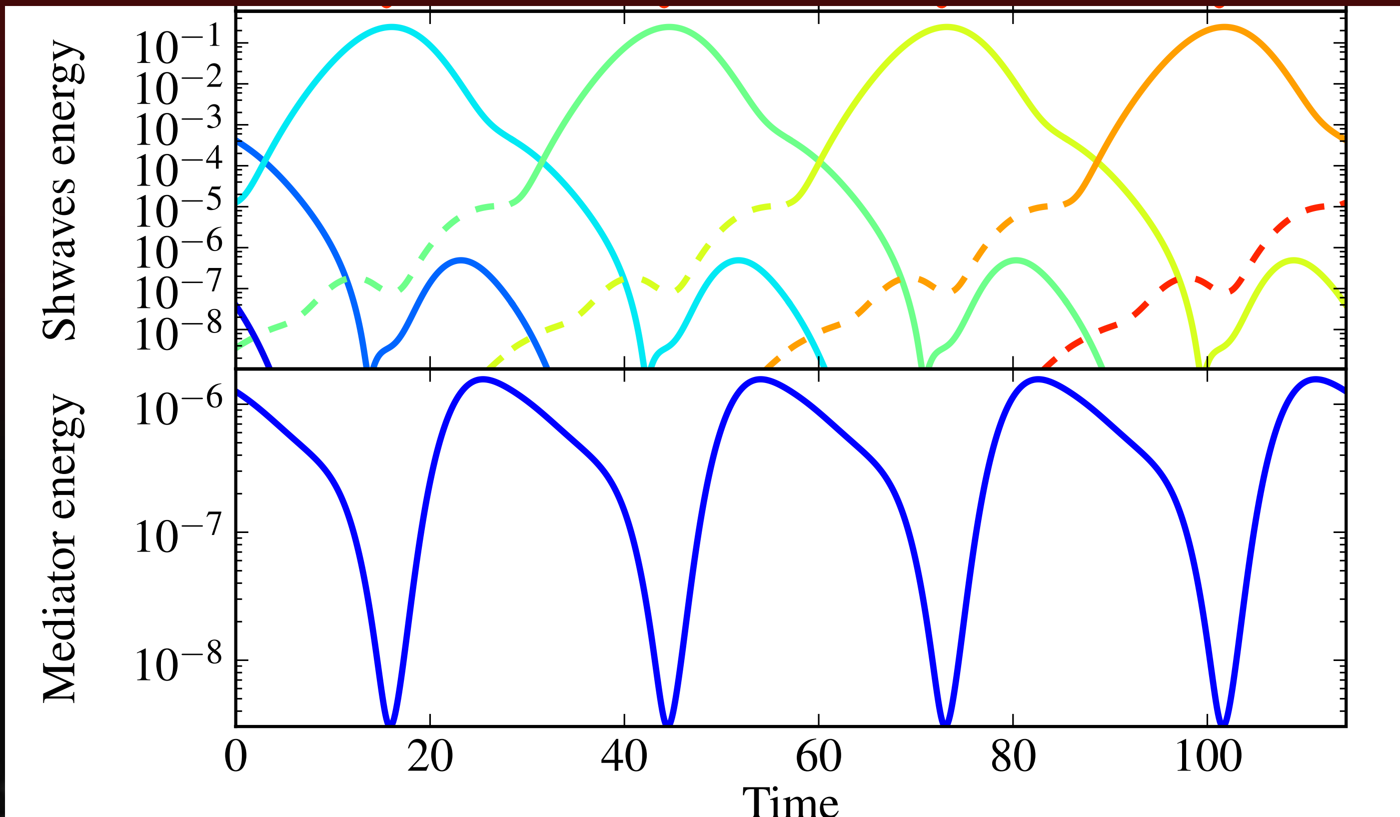


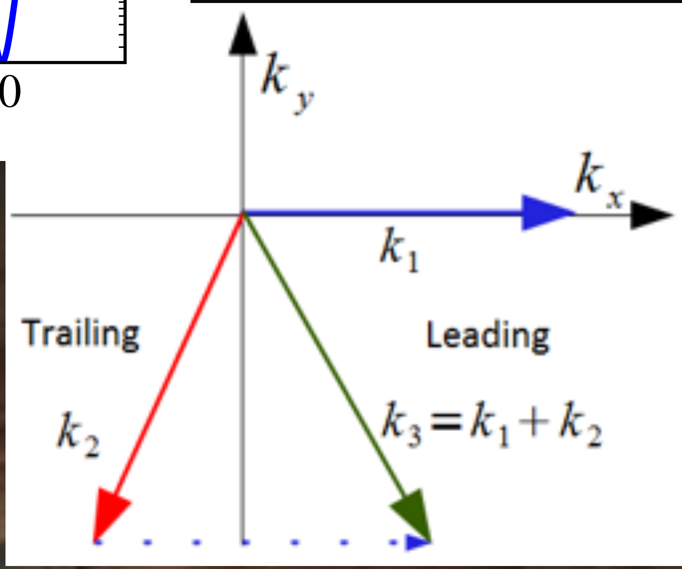



# MRI dynamo: the full cycle



# Regeneration of leading shearing waves

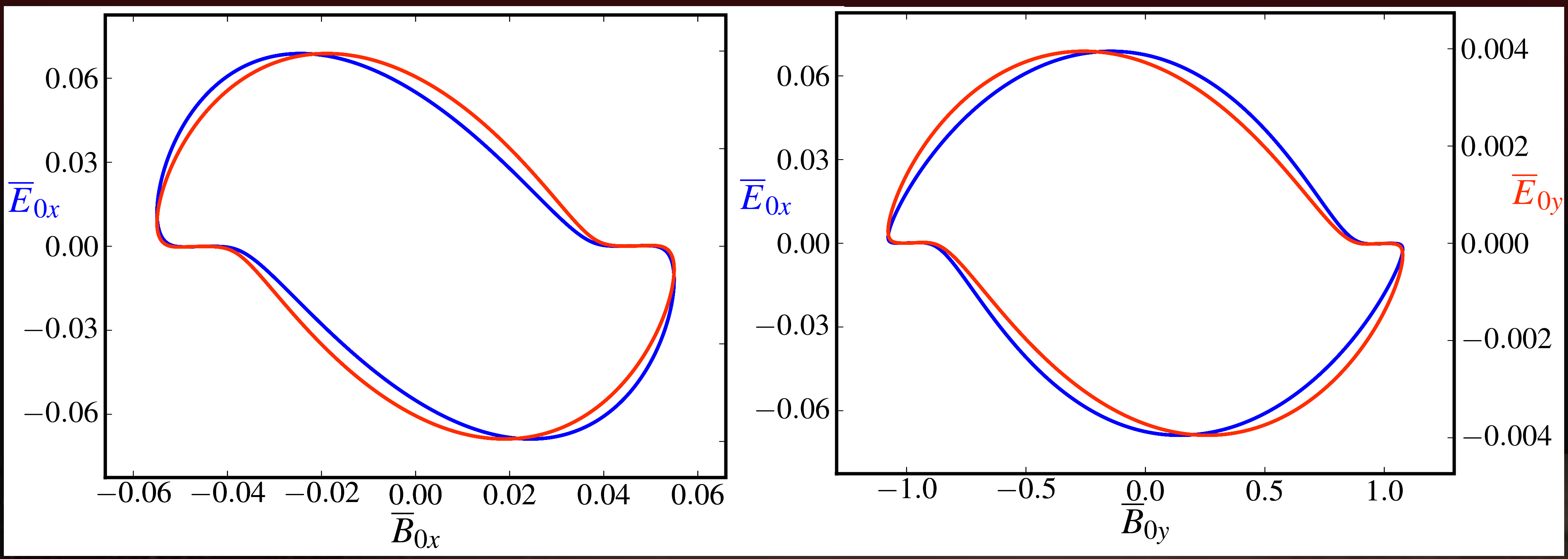


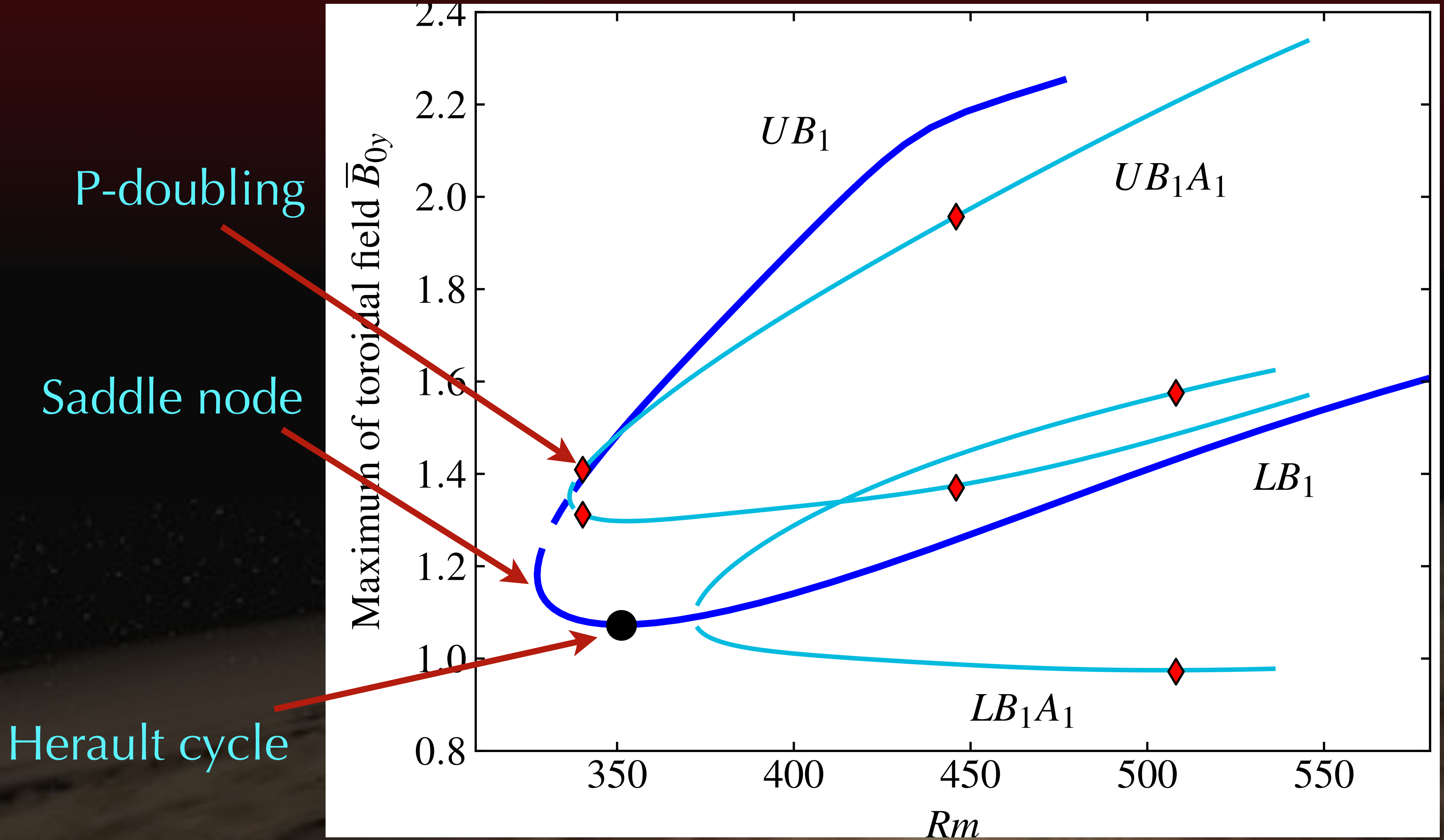

Wall-type axisymmetric confining mode with shearwise modulation is required to regenerate leading waves

[something similar in Lithwick, ApJ 670, 789 (2007)]

# Note: not a simple mean field dynamo !

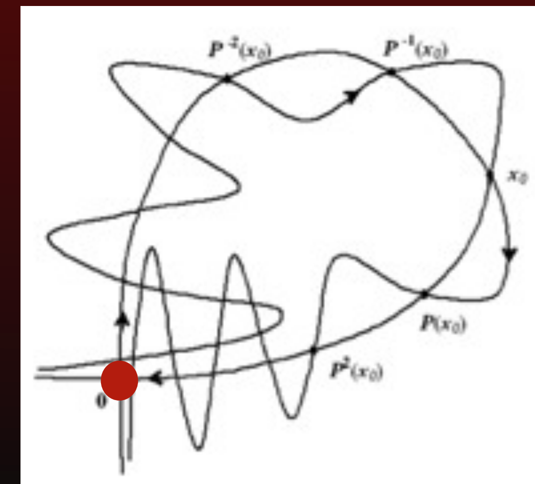


# Continuation & local bifurcations of MRI dynamo cycles

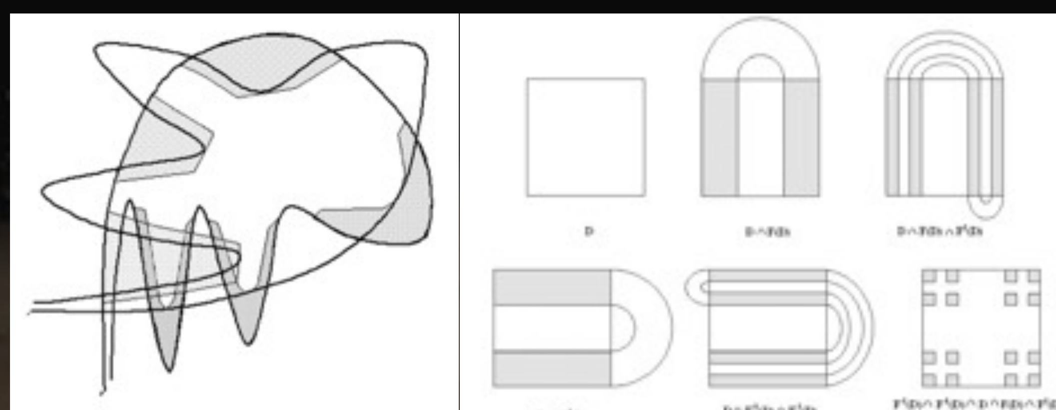



[Riols, Rincon, Cossu et al., submitted to J. Fluid Mech.]

# Global bifurcations



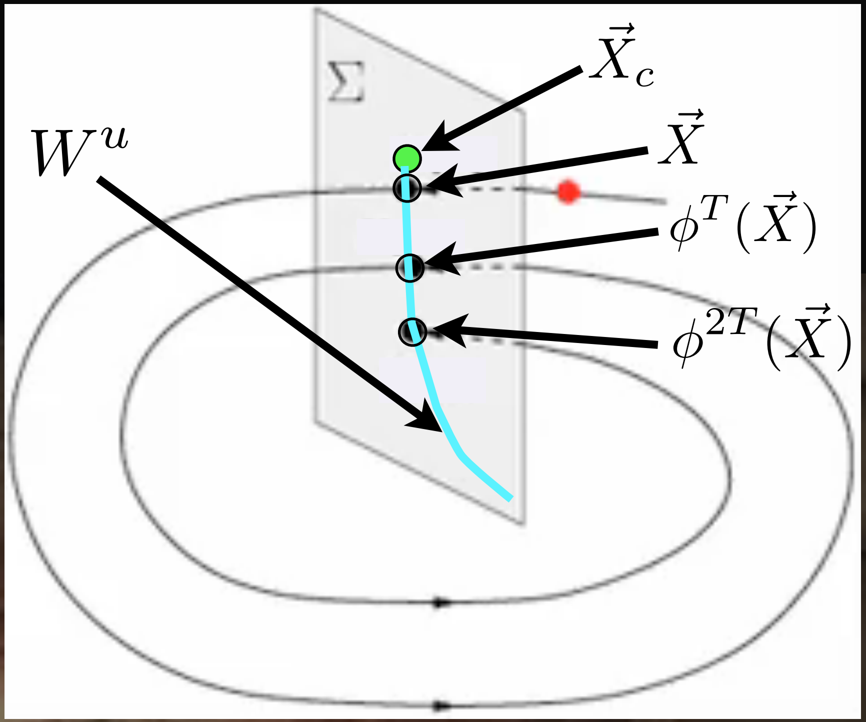

- Homoclinic bifurcations and tangles
  [Poincaré 1890]
  - Phase-space collisions between stable and unstable manifolds of nonlinear solutions
- Smale-Birkhoff theorem [1935-1967]
  (in my own poor physicist's language)
  - Hyperbolic maps with a transverse homoclinic point are topologically equivalent to a horseshoe map for a sufficiently high number of iterates



- Ergo: homoclinic bifurcations ⇒ (transient) chaos
  [Palis & takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations* (1993)]

# Global MRI dynamo bifurcations ?

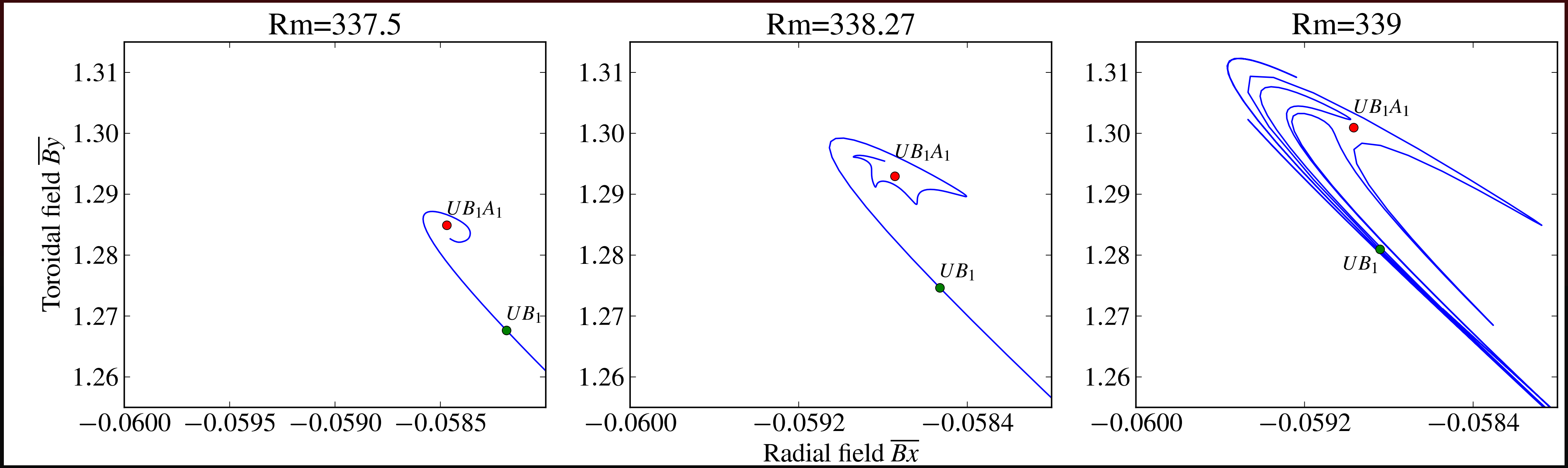
- $LB_1$ & $UB_1$ both have 1 unstable eigenvalue at Rm ~ 337
  - Track their unstable manifold $W^u$ = a 2D-surface in phase space
- Basic technique to obtain $W^u$ and visualize it
  - Construct a set of initial perturbed states consisting of increasingly larger linear perturbations of a cycle along its unstable direction
  - Integrate all these initial conditions
  - Poincaré section: "MRI dynamo map"
    - Look at system states every period T
  - Project these states on a 2D plane
- In this representation
  - A MRI dynamo cycle = a point
  - Unstable manifolds $W^u$ = 1D lines



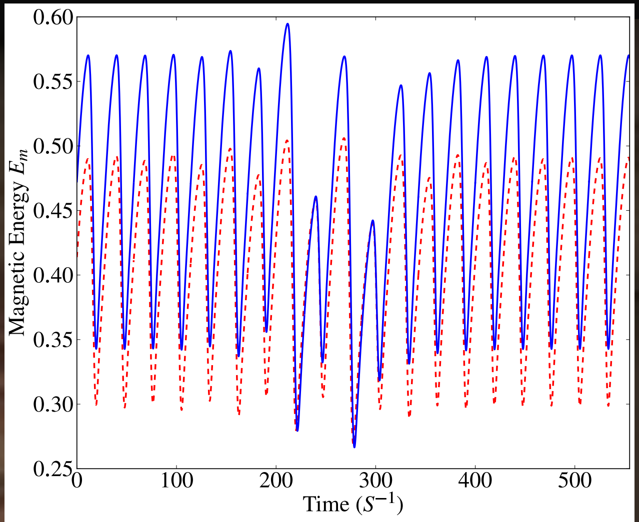

[Credits J. Moehlis, Scholarpedia]

# Of tangles and magnetic fields

- LB$_1$ & UB$_1$ both have 1 unstable eigenvalue at Rm ~ 337



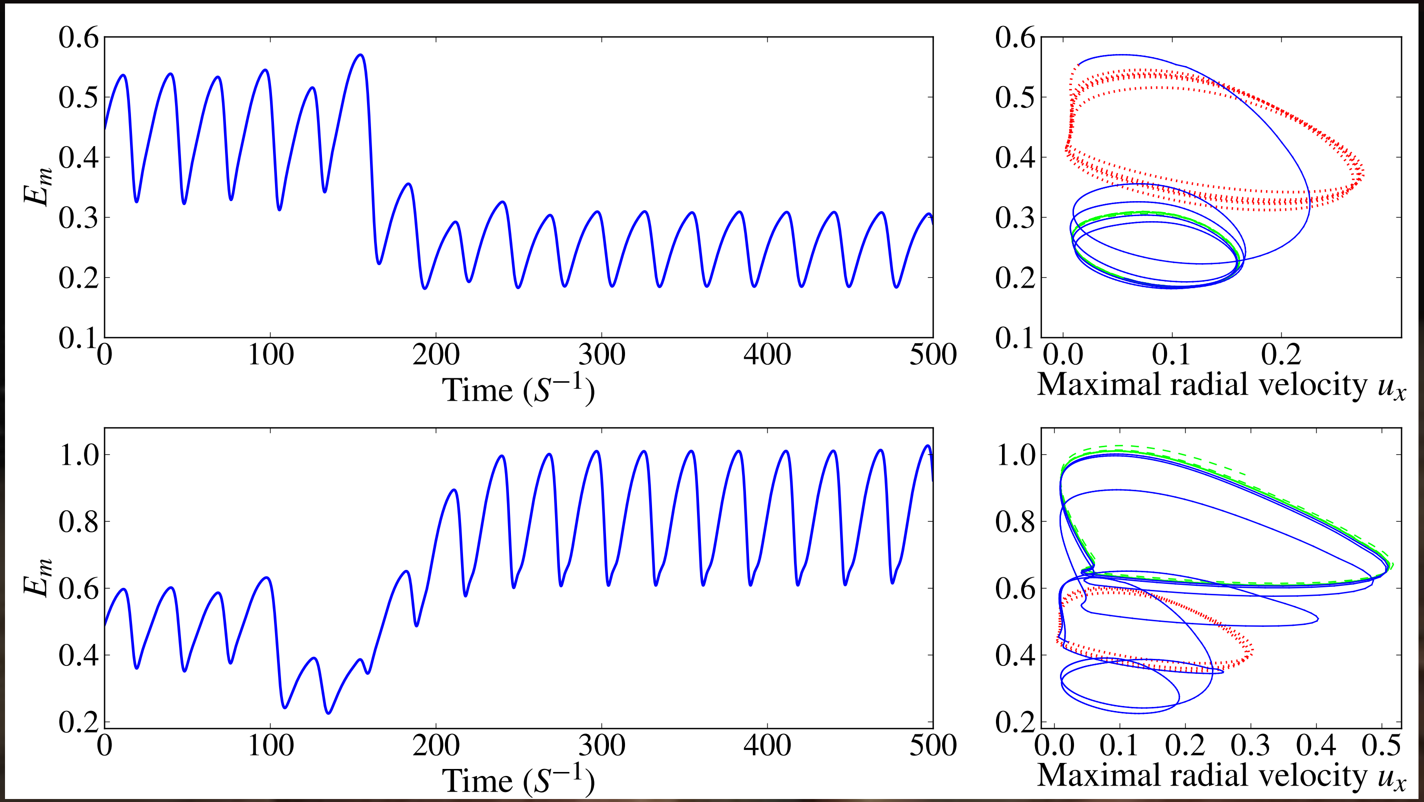

- Two homoclinic tangles
  - Homoclinic tangle in UB$_1$
  - Homoclinic tangle of LB$_1$
- Homoclinic orbits



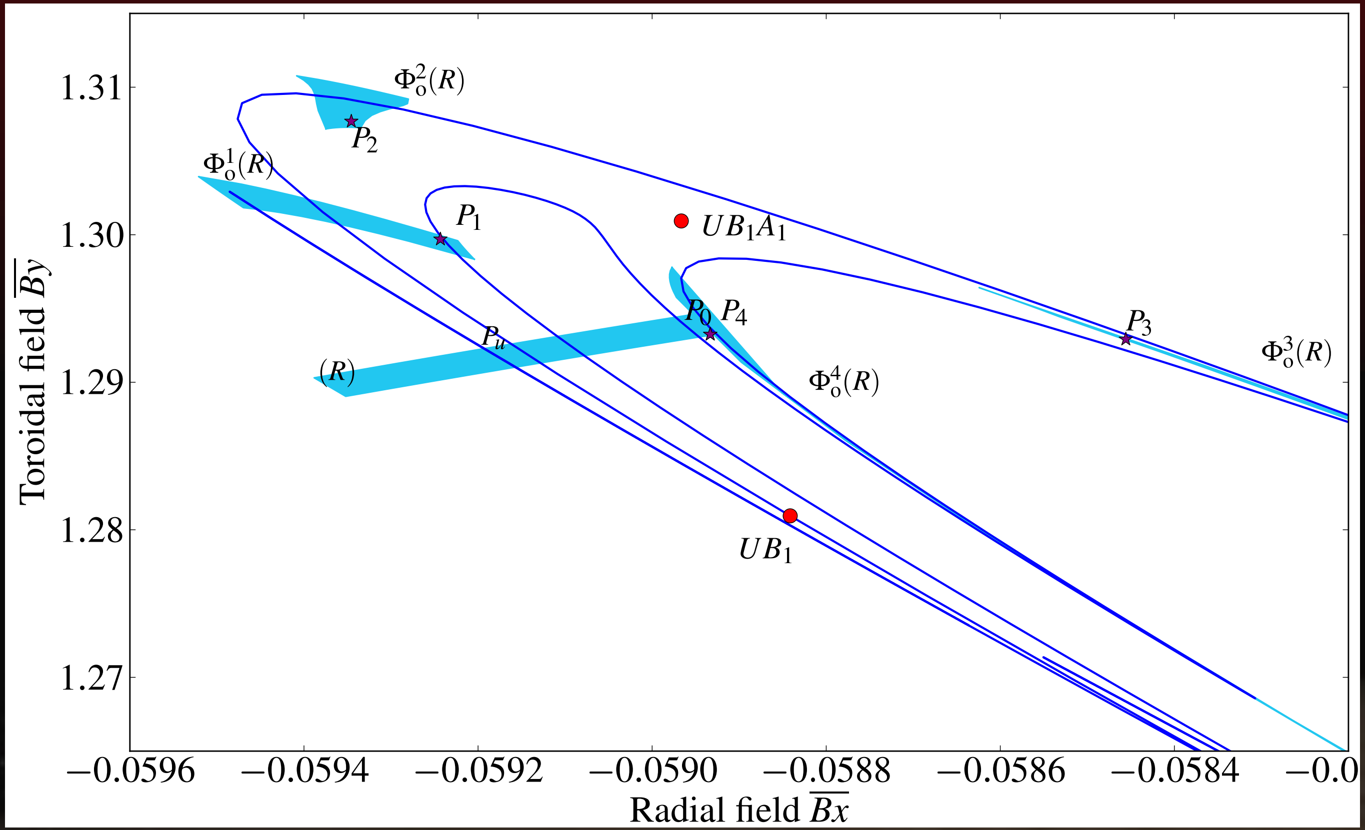

[Riols, Rincon, Cossu et al., submitted to J. Fluid Mech.]

# Heteroclinic tangles

- Other connections spotted
  - UB1 -> LB1, LB1 -> UB1, UB1 -> LB2
  - These things make the dynamics jump from one cycle to another
  - Transitions from low to high energy states (and conversely)

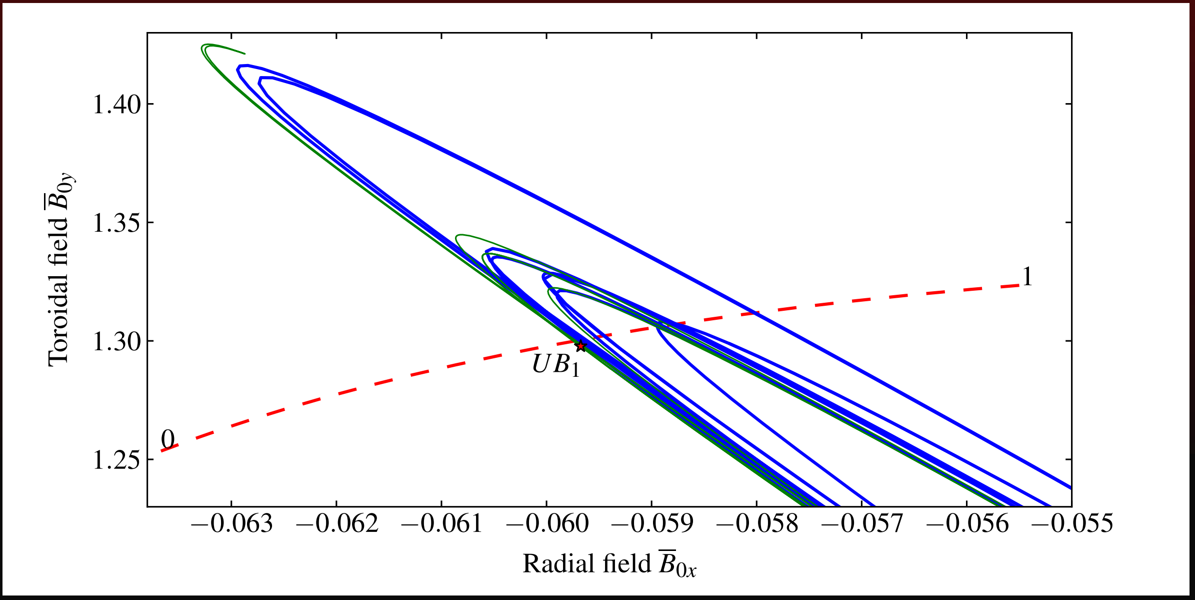


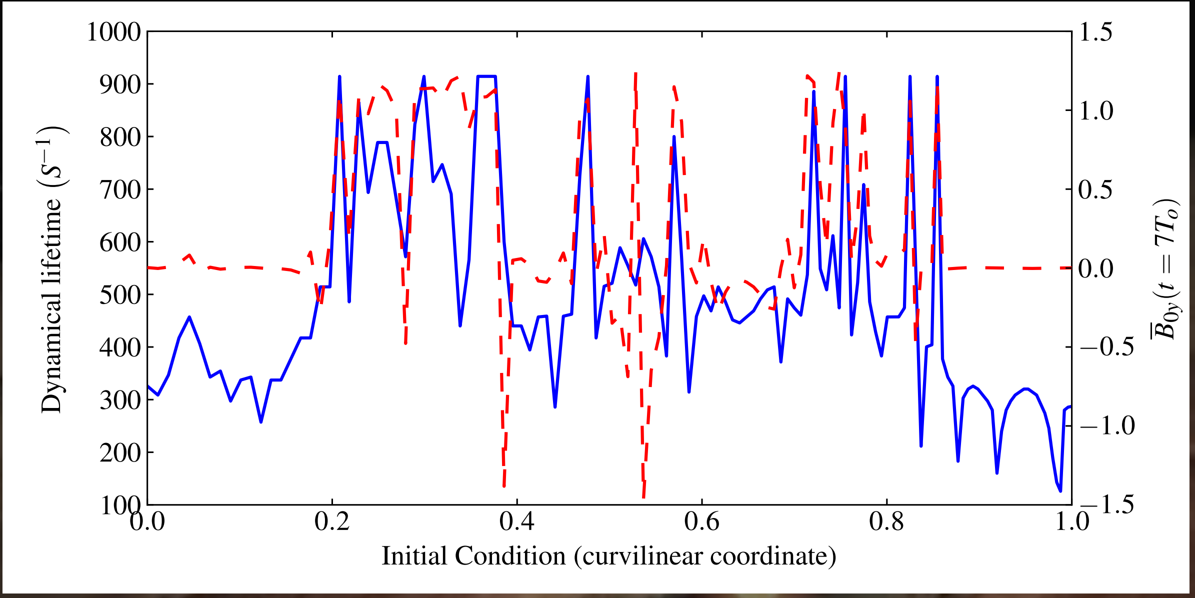
# Verification of Smale's theorem and other mathematical results on 2D maps




- Long-period orbits born out of saddle nodes
  - stable upper branches [Newhouse, Topology 12, 9 (1974)]
  - UBs lose stability and period-double [Yorke & Alligood, BAMS 9, 319 (1983)]

# Chaos in Poincaré tangles



# Summary

- The MRI dynamo is a subcritical, self-sustaining dynamo
  - Buy 1, get 1 free: both a dynamo and a turbulence activation process
  - The building blocks of the full nonlinear mechanism are well identified
  - Strong similarities with the hydro transition of non-rotating shear flows
- Dynamo cycles in elongated, incompressible shearing boxes
  - Clear precursors of subcritical MRI turbulence (homoclinic bifurcations)
  - Illustrate the MRI dynamo loop in its purest form
- Instability-driven dynamos
  - Fully non-kinematic & 3D, not a classical form of mean-field dynamo
  - A very natural way to obtain system-scale chaotic dynamo action

# Ongoing & future work

- The Pm issue
- How do we connect the results to different set-ups ?
  - Aspect ratio, stratification, boundary conditions, larger boxes
  - Can we do this in cylindrical geometry ? (PCX@Madison, PPPL Gallium)
- Can we do a closure theory for this dynamo ?
- Turbulent transport
  - Can we use cycles to characterize the statistics of turbulence ?
- Connexions with other problems
  - Other instability-driven dynamos
  - Hydro shear flows (we finally caught up !)
  - Various dynamical plasma phenomena

# Subcritical sheared plasma turbulence

[see Edmund Highcock's talk next]

MRI replaced with ~ ITG ?

Feedback on "large-scale" fields less relevant ?

Replaced with zonal flow ? (discussed by J. Krommes)

Leading shearing wave seed

Transient

MRI

Amplified shearing wave

Transient

MRI

Amplified trailing shearing wave

Shearing wave regeneration

Mediating mode regeneration

Confining axisymmetric mode "the wall"

[Highcock et al., PRL 109, 265001 (2012); Schekochihin et al., PPCF 54, 055011 (2012)]

# Directions for slab fusion people

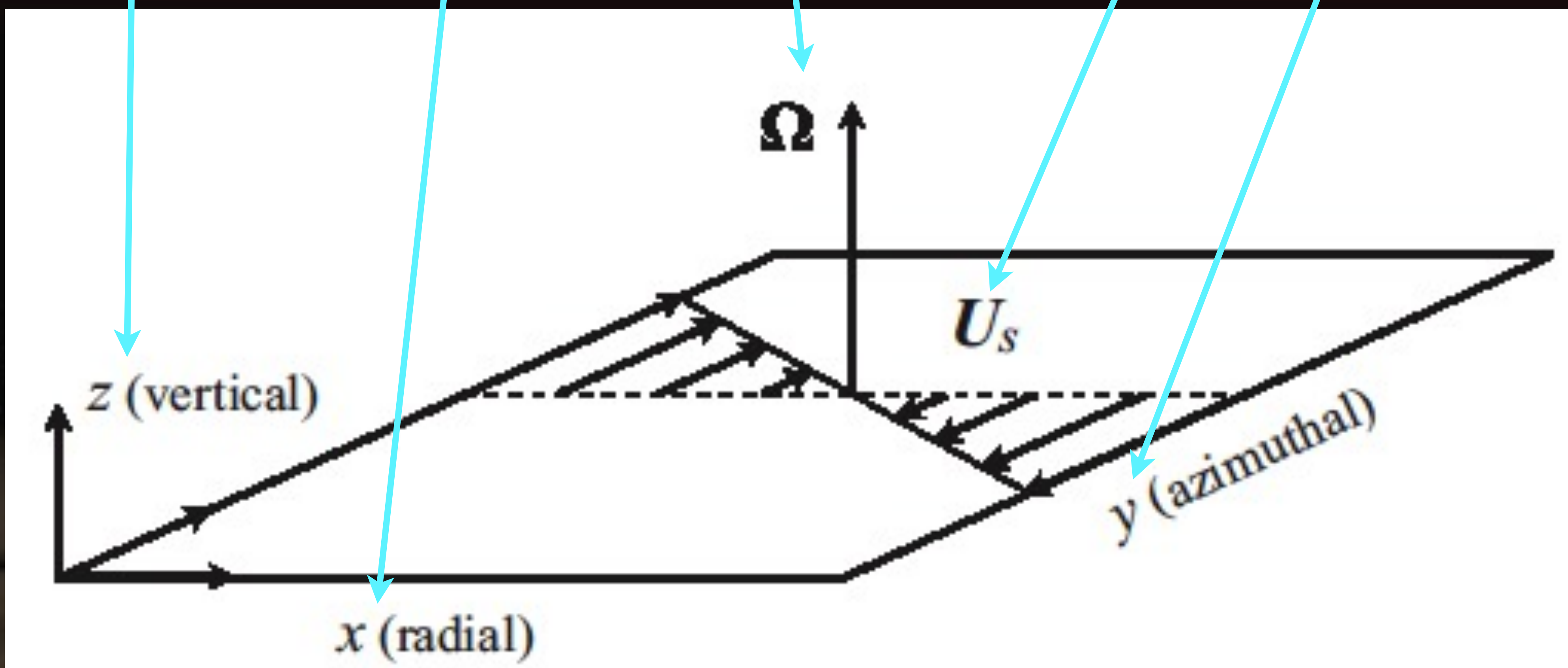
my Z is your Y

I don't think you have that

you strangely call that toroidal rotation

we have the same X

my Y is your -Z



# Another diagram with many arrows

[see also Troy Carter's talk next]

- Nonlinear instabilities galore

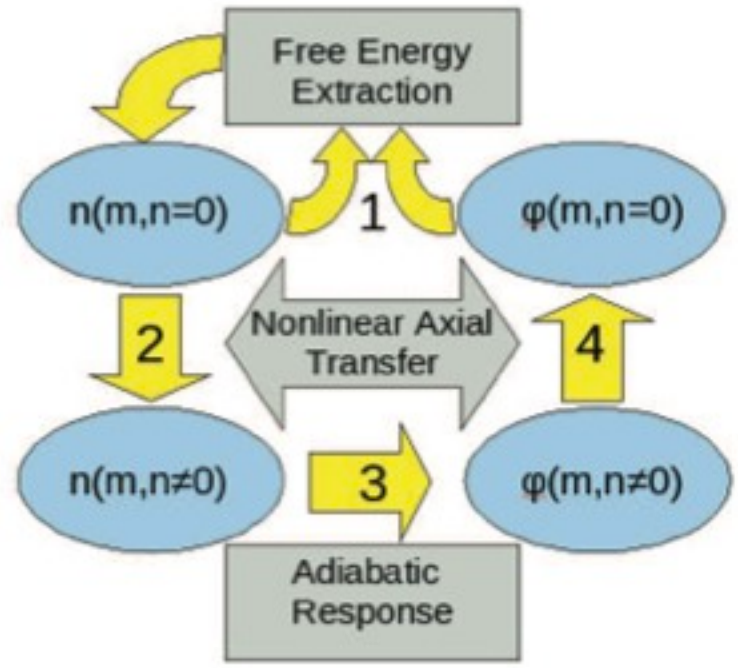



FIG. 5. Diagram of the nonlinear instability process that drives flute modes.

[Friedman et al., PoP 19, 102307 (2012)]