Magnetohydrodynamic Turbulence: solar wind and numerical simulations

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Magnetic turbulence in nature energy spectra



[Goldstein, Roberts, Matthaeus (1995)] [Armstrong, Rickett, Spangler (1995)]

Nature of Magnetohydrodynamic (MHD) turbulence

HD turbulence: interaction of eddies

MHD turbulence: interaction of wave packets moving with Alfven velocities



Guide field in MHD turbulence



B₀ imposed by external sources



B₀ created by large-scale eddies

Magnetohydrodynamic (MHD) equations

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho_0} \nabla p + \frac{1}{4\pi\rho_0} (\mathbf{B} \cdot \nabla) \mathbf{B} + \nu \nabla^2 \mathbf{v}$$
$$\partial_t \mathbf{B} = \nabla \times [\mathbf{v} \times \mathbf{B}] + \eta \nabla^2 \mathbf{B}$$

Separate the uniform magnetic field: $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$

Introduce the Elsasser variables:

$$\mathbf{z}^{\pm} = \mathbf{v} \pm \frac{1}{\sqrt{4\pi
ho_0}} \mathbf{b}$$

Then the equations take a symmetric form:

$$\partial_t \mathbf{z}^+ - (\mathbf{v}_A \cdot \nabla) \mathbf{z}^+ + (\mathbf{z}^- \cdot \nabla) \mathbf{z}^+ = -\nabla P$$

$$\partial_t \mathbf{z}^- + (\mathbf{v}_A \cdot \nabla) \mathbf{z}^- + (\mathbf{z}^+ \cdot \nabla) \mathbf{z}^- = -\nabla P$$

With the Alfven velocity $\mathbf{v}_A = \mathbf{B}_0 / \sqrt{4\pi\rho_0}$

The uniform magnetic field mediates small-scale turbulence

MHD turbulence: Alfvenic cascade $\partial \mathbf{z}^{\pm} \mp (\mathbf{v}_A \cdot \nabla) \mathbf{z}^{\pm} + (\mathbf{z}^{\mp} \cdot \nabla) \mathbf{z}^{\pm} = -\nabla P + \frac{1}{R_e} \nabla^2 \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$ Ideal system conserves the Elsasser energies $E = \frac{1}{2} \int (v^2 + b^2) d^3x$ $= \frac{E}{2} \int (v^2 + v^2) d^3 x$ $H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3 x$ $E^+ = \int (\mathbf{z}^+)^2 d^3 x$ $E^- = \int (\mathbf{z}^-)^2 \, d^3 x$ \mathbf{B}_0 V۸ \mathbf{V}_A Z+ \mathbf{B}_{0} V_A \mathbf{V}_A $E^+ \gg E^-$: imbalanced case $E^+ \sim E^-$: balanced case.

$$H^{C} = \int (\mathbf{v} \cdot \mathbf{b}) d^{3}x = \frac{1}{4} (E^{+} - E^{-}) \neq 0$$
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Strength of interaction in MHD turbulence

$$\partial \mathbf{z}^{\pm} \mp (\mathbf{v}_{A} \cdot \nabla) \mathbf{z}^{\pm} + (\mathbf{z}^{\mp} \cdot \nabla) \mathbf{z}^{\pm} = -\nabla P + \frac{1}{Re} \nabla^{2} \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$$

$$\underbrace{\langle k_{\parallel} v_{A} \rangle z^{\pm}}_{(k_{\perp} z^{\mp}) z^{\pm}} \underbrace{\langle k_{\perp} z^{\mp} \rangle z^{\pm}}_{(k_{\perp} z^{\mp}) z^{\pm}}$$

When
$$~~k_\parallel v_A \gg k_\perp z^\mp~$$
 turbulence is weak

When $~~k_\parallel v_A \sim k_\perp z^\mp$ turbulence is strong

MHD turbulence: collision of Alfven waves



user: jcperez Sun Feb 28 21:29:59 2010

Strong MHD turbulence: collision of eddies



Wave MHD turbulence: Phenomenology

Three-wave interaction of shear-Alfven waves

$$\omega(k) = |k_z| v_A$$

$$\begin{cases} \omega(k) = \omega(k_1) + \omega(k_2) \\ \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \end{cases}$$

Only counter-propagating waves interact, therefore, k_{1z} and k_{2z} should have opposite signs.

Either $k_{1z}=0$ or $k_{2z}=0$

Wave interactions change k_{\perp} but not k_z

At large k_1: $E(k_z,k_\perp) \propto g(k_z) k_\perp^{-eta}$

Montgomery & Turner 1981, Shebalin et al 1983

Analytic framework [Galtier, Nazarenko, Newell, Pouquet, 2000]

In the zeroth approximation, waves are not interacting. and z^+ and z^- are independent:

$$\langle \mathbf{z}^{+}(\mathbf{k}) \cdot \mathbf{z}^{+}(\mathbf{k}') \rangle = e^{+}(k_{z}, k_{\perp}) \delta(\mathbf{k} + \mathbf{k}')$$

$$\langle \mathbf{z}^{-}(\mathbf{k}) \cdot \mathbf{z}^{-}(\mathbf{k}') \rangle = e^{-}(k_{z}, k_{\perp}) \delta(\mathbf{k} + \mathbf{k}')$$

$$\langle \mathbf{z}^{+}(\mathbf{k}) \cdot \mathbf{z}^{-}(\mathbf{k}') \rangle = 0$$

When the interaction is switched on, the energies slowly change with time: $e^{\pm}(k_z, k_{\perp}, t)$

$$\partial_t \mathbf{z}^{\pm} - (\mathbf{v}_A \cdot \nabla) \mathbf{z}^{\pm} + (\mathbf{z}^{\mp} \cdot \nabla) \mathbf{z}^{\pm} = -\nabla P$$

$$\partial_t \langle z^+ z^+ \rangle = \dots \langle z^- z^+ z^+ \rangle + \langle z^+ z^- z^+ \rangle \dots$$

$$\partial_t \langle z^- z^+ z^+ \rangle = \dots \langle z^+ z^- z^+ z^+ \rangle + \langle z^- z^- z^+ z^+ \rangle + \langle z^- z^+ z^- z^+ \rangle \dots$$

split into pair-wise correlators using Gaussian rule

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Weak turbulence: Analytic framework [Galtier, Nazarenko, Newell, Pouquet, 2000]

$$\partial_t \langle z^+ z^+ \rangle = \dots \langle z^- z^+ z^+ \rangle + \langle z^+ z^- z^+ \rangle \dots$$

$$\partial_t \langle z^- z^+ z^+ \rangle = \dots \langle z^+ z^- z^+ z^+ \rangle + \langle z^- z^- z^+ z^+ \rangle + \langle z^- z^+ z^- z^+ \rangle \dots$$

split into pair-wise correlators using Gaussian rule

$$\partial_t e^{\pm}(k_z, k_{\perp}) = \int M_{k,pq} e^{\mp}(0, q_{\perp}) \left[e^{\pm}(k_z, k_{\perp}) - e^{\pm}(k_z, p_{\perp}) \right] \delta(\mathbf{k}_{\perp} - \mathbf{p}_{\perp} - \mathbf{q}_{\perp}) d^2 p d^2 q$$
$$M_{k,pq} = \frac{\pi}{v_A} \frac{(\mathbf{k}_{\perp} \times \mathbf{q}_{\perp})^2 (\mathbf{k}_{\perp} \cdot \mathbf{p}_{\perp})^2}{k_{\perp}^2 p_{\perp}^2 q_{\perp}^2}$$

This kinetic equation has all the properties discussed in the phenomenology: it is scale invariant, z^+ interacts only with z^- , k_z does not change during interactions. Weak turbulence: Analytic framework [Galtier, Nazarenko, Newell, Pouquet, 2000]

$$\partial_t e^{\pm}(k_z, k_{\perp}) = \int M_{k,pq} e^{\mp}(0, q_{\perp}) \left[e^{\pm}(k_z, k_{\perp}) - e^{\pm}(k_z, p_{\perp}) \right] \delta(\mathbf{k}_{\perp} - \mathbf{p}_{\perp} - \mathbf{q}_{\perp}) d^2 p d^2 q$$

Statistically balanced case: $e^+ = e^-$

$$e^+(k_z, k_\perp) = e^-(k_z, k_\perp) = g(k_z)k_\perp^{-3}$$

where $g(k_z)$ is an arbitrary function.

The spectrum of weak balanced MHD turbulence is therefore:

$$E^{\pm}(k_z,k_{\perp}) = e^{\pm}(k_z,k_{\perp})2\pi k_{\perp} \propto k_{\perp}^{-2}$$

Ng & Bhattacharjee 1996, Goldreich & Sridhar 1997

Weak MHD turbulence



 $\partial_t e^{\pm}(k_z, k_{\perp}) = \int M_{k,pq} e^{\mp}(0, q_{\perp}) \left[e^{\pm}(k_z, k_{\perp}) - e^{\pm}(k_z, p_{\perp}) \right] \delta(\mathbf{k}_{\perp} - \mathbf{p}_{\perp} - \mathbf{q}_{\perp}) d^2 p d^2 q$

The kinetic equation has a one-parameter family of solutions:

$$\begin{aligned} e^+(k_z, k_\perp) &= g^+(k_z) k_\perp^{-3-\alpha} \\ e^-(k_z, k_\perp) &= g^-(k_z) k_\perp^{-3+\alpha} \end{aligned} \quad \text{with -1 < \alpha < 1} \end{aligned}$$

What do these solutions mean? Hint: calculate energy fluxes.

Assume that e+ has the steeper spectrum and denote the energy fluxes ϵ^+ and ϵ^- . Then $\epsilon^+ > \epsilon^-$

and:
$$\alpha = f(\epsilon^+/\epsilon^-)$$

Imbalanced weak MHD turbulence

(where problems begin)

$$\partial_t e^{\pm}(k_z, k_{\perp}) = \int M_{k,pq} e^{\mp}(0, q_{\perp}) \left[e^{\pm}(k_z, k_{\perp}) - e^{\pm}(k_z, p_{\perp}) \right] \delta(\mathbf{k}_{\perp} - \mathbf{p}_{\perp} - \mathbf{q}_{\perp}) d^2 p d^2 q$$

The kinetic equation has a one-parameter family of solutions:



$$e^{+}(k_{z}, k_{\perp}) = g^{+}(k_{z})k_{\perp}^{-3-\alpha} -1 < \alpha < 1$$

$$e^{-}(k_{z}, k_{\perp}) = g^{-}(k_{z})k_{\perp}^{-3+\alpha} \qquad \alpha = f(\epsilon^{+}/\epsilon^{-})$$

The spectra are "pinned" at the dissipation scale.

• If the ratio of the energy fluxes is specified, then the slopes are specified, but the amplitudes depend on the dissipation scale, or on the Re number.



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Imbalanced weak MHD turbulence: Numerical results



[Boldyrev & Perez (2009)] 24

Residual energy in weak MHD turbulence

$$\begin{aligned} \langle \mathbf{z}^{+}(\mathbf{k}) \cdot \mathbf{z}^{+}(\mathbf{k}') \rangle &= e^{+}(k_{\parallel}, k_{\perp}) \delta(\mathbf{k} + \mathbf{k}') & \checkmark \\ \langle \mathbf{z}^{-}(\mathbf{k}) \cdot \mathbf{z}^{-}(\mathbf{k}') \rangle &= e^{-}(k_{\parallel}, k_{\perp}) \delta(\mathbf{k} + \mathbf{k}') & \checkmark \\ \langle \mathbf{z}^{+}(\mathbf{k}) \cdot \mathbf{z}^{-}(\mathbf{k}') \rangle &= q^{r}(k_{\parallel}, k_{\perp}) \delta(\mathbf{k} + \mathbf{k}') & \neq \mathbf{0} \\ \langle \mathbf{z}^{+} \cdot \mathbf{z}^{-} \rangle &= \langle v^{2} - b^{2} \rangle & \text{since the waves are not independent!} \end{aligned}$$

What is the equation for the residual energy?

SB & Perez PRL 2009

Residual energy in weak MHD turbulence

- Waves are almost independent one would not expect any residual energy!
- Analytically tractable:

$$\partial_t q^r = 2ik_{\parallel} v_A q^r - \gamma_k q^r +$$

 $+\int R_{k,pq} \{e^{+}(\mathbf{q}) \left[e^{-}(\mathbf{p}) - e^{-}(\mathbf{k})\right] + e^{-}(\mathbf{q}) \left[e^{+}(\mathbf{p}) - e^{+}(\mathbf{k})\right] \} \delta(q_{\parallel}) \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) d^{3}p d^{3}q d$

where:
$$R_{k,pq} = (\pi v_A/2)(\mathbf{k}_\perp \times \mathbf{q}_\perp)^2 (\mathbf{k}_\perp \cdot \mathbf{p}_\perp) (\mathbf{k}_\perp \cdot \mathbf{q}_\perp) / (k_\perp^2 p_\perp^2 q_\perp^2)$$

Conclusions:

- Residual energy is always generated by interacting waves!
- ∫ ... < 0, so the residual energy is negative: magnetic energy dominates!

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Y. Wang, S. B. & J. C. Perez (2011)
S.B, J. C. Perez & V. Zhdankin (2011)
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Residual energy in weak MHD turbulence $e^{r}(k) = Re\langle z^{+}(k) \cdot z^{-}(k) \rangle \propto -\epsilon^{2}k_{\perp}^{-2}\Delta(k_{\parallel})$



Y. Wang, S. B. & J. C. Perez (2011) S.B, J. C. Perez & V. Zhdankin (2011)

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 $E^{\pm}(k_{\perp}) \propto k_{\perp}^{-2}$

Residual energy in MHD turbulence



Wang et al 2011 ¹⁸

Universal picture of MHD turbulence



Energy spectra in the solar wind and in numerical simulations

Solar wind observations: spectral indices in 15,472 independent measurements. (From 1998 to 2008, fit from 1.8×10^{-4} to 3.9×10^{-3} Hz)

Numerical simulations: spectral indices in 80 independent snapshots, separated by a turnover time.

S.B., J. Perez, J Borovsky & J. Podesta (2011)



Spectrum of strong MHD turbulence: balanced case



Computational resources: DoE 2010 INCITE, Machine: Intrepid, IBM BG/P at Argonne Leadership Computing Facility

Perez et al, Phys Rev X (2012)

Spectrum of strong MHD turbulence: imbalanced case



Perez et al, Phys Rev X (2012)

Possible explanation of the -3/2 spectrum Dynamic Alignment theory

Fluctuations δv_{λ} and δb_{λ} become spontaneously aligned in the field-perpendicular plane within angle θ_{λ}



SB (2005, 2006)

Numerical verification of dynamic alignment

$$S_{cross}(r) = \langle |\delta \tilde{\mathbf{v}}_r \times \delta \tilde{\mathbf{b}}_r| \rangle \qquad S_2(r) = \langle |\delta \tilde{\mathbf{v}}_r| |\delta \tilde{\mathbf{b}}_r| \rangle$$

Alignment angle: $\theta_r \approx \sin(\theta_r) \equiv S_{cross}(r)/S_2(r)$



Magnetic and velocity fluctuations build progressively stronger correlation at smaller scales.

Form sheet-like structures

Mason et al 2011, Perez et al 2012

Physics of the dynamic alignment

Hydrodynamics:
$$\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \nu\nabla^2\mathbf{v}$$

 $E = \frac{1}{2}\int \mathbf{v}^2(\mathbf{x}) d^3x$

MHD:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho_0} \nabla p + \frac{1}{4\pi\rho_0} (\mathbf{B} \cdot \nabla) \mathbf{B} + \nu \nabla^2 \mathbf{v}$$

$$\partial_t \mathbf{B} = \nabla \times [\mathbf{v} \times \mathbf{B}] + \eta \nabla^2 \mathbf{B}$$

$$E = \frac{1}{2} \int (v^2 + b^2) d^3 x \quad H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3 x$$

Energy E is dissipated faster than cross-helicity H^C

$$\frac{\delta}{\delta \mathbf{v}} \left[\int (v^2 + b^2) d^3 x - \lambda \int (\mathbf{v} \cdot \mathbf{b}) d^3 x \right] = 0$$

$$\frac{\delta}{\delta \mathbf{b}} \left[\int (v^2 + b^2) d^3 x - \lambda \int (\mathbf{v} \cdot \mathbf{b}) d^3 x \right] = 0$$

$$\mathbf{v}(\mathbf{x}) = \pm \mathbf{b}(\mathbf{x})$$



Summary

- Weak MHD turbulence spontaneously generates a condensate of the residual energy $E_r = E_b E_v$ at small $k_{||}$. Condensate broadens with k_{\perp} .
- The universal spectrum or weak turbulence (balances or imbalanced) is:

 $E^{\pm}(k_{\perp}) \propto k_{\perp}^{-2}$.

At large k turbulence becomes strong, with the spectrum $E^{\pm}(k_{\perp}) \propto k^{-3/2}$

Residual energy is nonzero for strong turbulence at all k, so
 E_b(k) can look steeper, while E_v(k) shallower in a limited interval (consistent with the solar wind!)