Multiscale Gyrokinetics: Fluctuations, Transport and Energy Flows

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Outline

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Scale Separation

Time Scales

- Cyclotron Frequency Ω_i
- Turbulence ω
- Profile Evolution τ_E

- $\Omega_i \gg \omega \gg T^{-1} \gg \tau_E^{-1}$

Spatial Scales

- Gyroradius ρ_i
- Fluctuations $k_\perp$, $k_\|$
- Profiles a

- $k_\perp \sim \rho_i^{-1} \ll \lambda^{-1} \ll k_\| \sim a^{-1}$

- We use a single small parameter ϵ and order

$$\epsilon \sim \frac{\omega}{\Omega_i} \sim \frac{\rho_i}{a} \sim \frac{k_\|}{k_\perp} \sim \frac{\delta f}{f} \tag{1}$$

Small-scale Averaging

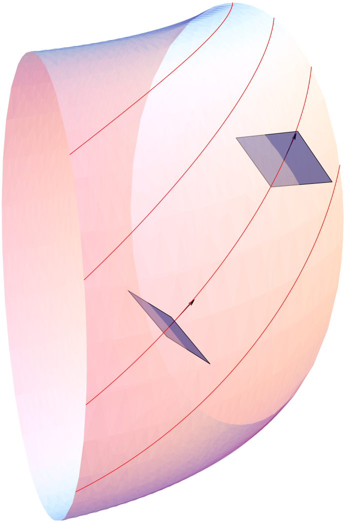
- Introduce the patch average $\langle \cdot \rangle_{\mathrm{turb}}$

$$\langle \cdot \rangle_{\mathrm{turb}} = \frac{1}{T} \int_{t-T/2}^{t+T/2} dt' \frac{1}{\lambda_\perp^2} \int_{\lambda_\perp^2} d^2 \mathbf{r}'_\perp \tag{2}$$

This is an average over a time T and a perpendicular area λ^2.

- Using this, all quantities separate into mean and fluctuating parts,

$$f_s = F_s + \delta f \qquad F_s = \langle f_s \rangle_{\mathrm{turb}} \tag{3}$$



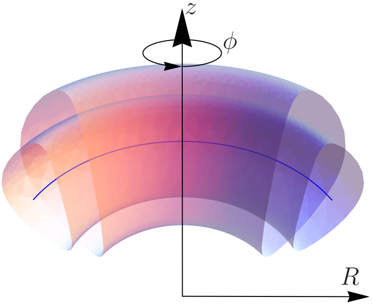
Axisymmetry

- All quantities are axisymmetric

$$\frac{\partial}{\partial \phi} \langle g \rangle_{\mathrm{turb}} = 0 \tag{4}$$

- Magnetic field has the usual form

$$\boldsymbol{B} = I \nabla \psi + \nabla \psi \times \nabla \phi \tag{5}$$



- $F_s = F_{0s} + F_{1s} + \cdots$
- F_{0s} is Maxwellian (as $\nu_{ii} \sim \omega$) with density n_s and temperatures T_s and velocity $\boldsymbol{u}$.
- $\boldsymbol{u} = \omega(\psi) R^2 \nabla\phi$ and is *species independent*
- n_s is related to the flux functions $N_s(\psi)$ and $T(\psi)$ via

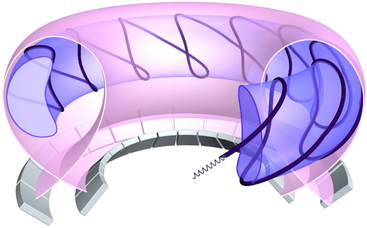
$$n_s = N_s\left(\psi\right) \exp\left[\frac{m_s \omega^2(\psi) R^2}{2T_s} - \frac{Z_s e \varphi_0}{T_s}\right] + \mathcal{O}(\epsilon n_s), \tag{6}$$

Magnetic Equilibrium and Neoclassical Theory

- $\psi(R, z)$ is given by the Grad-Shafranov equation
- $I(\psi, t)$ is evolved via

$$\left.\frac{\partial}{\partial t}\right|_{\psi} q = \frac{c}{4\pi^2} \frac{\partial}{\partial \psi} V' \left\langle \boldsymbol{E} \cdot \boldsymbol{B} \right\rangle_{\psi} . \tag{7}$$

- F_{1s} is given by neoclassical theory



The Gyrokinetic Equation

δf splits into h_s and the Boltzmann response,

$$\delta f_s = -\frac{Z_s e \delta\varphi'}{T_s} F_{0s} + h_s(\boldsymbol{R}_s, \varepsilon_s, \mu_s, t), \tag{8}$$

h_s is given by

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s)\cdot\frac{\partial}{\partial \boldsymbol{R}_s}\right] h_s + \left(w_\parallel \boldsymbol{b} + \boldsymbol{V}_D + \langle \boldsymbol{V}_\chi\rangle_{\boldsymbol{R}}\right)\cdot\frac{\partial h_s}{\partial \boldsymbol{R}_s} - \langle C[h_s]\rangle_{\boldsymbol{R}} \\
&= \frac{Z_s e F_{0s}}{T_s}\left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s)\cdot\frac{\partial}{\partial \boldsymbol{R}_s}\right]\langle\chi\rangle_{\boldsymbol{R}} \\
&- \left\{\frac{\partial F_{0s}}{\partial \psi} + \frac{m_s F_{0s}}{T_s}\left[\frac{I(\psi) w_\parallel}{B} + \omega(\psi) R^2\right]\frac{d\omega}{d\psi}\right\}\langle \boldsymbol{V}_\chi\rangle_{\boldsymbol{R}}\cdot\nabla\psi
\end{aligned} \tag{9}$$

and the gyrokinetic potential (and associated turbulent flow) is

$$\chi = \delta\varphi' - \frac{1}{c}\delta\boldsymbol{A}\cdot\boldsymbol{w} \qquad \boldsymbol{V}_\chi = \frac{c}{B}\boldsymbol{b}\times\nabla\chi \tag{10}$$

Transport Equations

- Particle Transport

$$\left.\frac{\partial}{\partial t}\right|_{\psi} n_s + \frac{1}{V'}\frac{\partial}{\partial \psi} V' \left\langle \Gamma_s \right\rangle_{\psi} = 0 \tag{11}$$

- Heat Transport

$$\begin{aligned} \frac{1}{V'}\frac{3}{2}\left.\frac{\partial}{\partial t}\right|_{\psi} V' \left\langle n \right\rangle_{\psi\ s} T_s + \frac{1}{V'}\frac{\partial}{\partial \psi} V' \left\langle q_s \right\rangle_{\psi} = P_s^{\mathrm{visc}} + P_s^{\mathrm{turb}} + P_s^{\mathrm{Ohm}} \\ - \left\langle Z_s e \varphi_0 \frac{\partial n_s}{\partial t} \right\rangle_{\psi} + \frac{\omega^2(\psi)}{2V'} \left.\frac{\partial}{\partial t}\right|_{\psi} V' m_s \left\langle R^2 n_s \right\rangle_{\psi} + \left\langle C_s^{(E)} \right\rangle_{\psi}, \end{aligned} \tag{12}$$

- Momentum Transport

$$\frac{1}{V'}\left.\frac{\partial}{\partial t}\right|_{\psi} V' J \omega(\psi) + \frac{1}{V'}\frac{\partial}{\partial \psi} V' \left\langle \Pi^{\psi\phi} \right\rangle_{\psi} = 0, \tag{13}$$

Focusing on the heating,

$$P_s^{\mathrm{turb}} = Z_s e \left\langle \left\langle \int d^3\boldsymbol{w} \left\langle h_s \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \right) \chi \right\rangle_{\boldsymbol{r}} \right\rangle_{\mathrm{turb}} \right\rangle_{\psi} + \omega(\psi) \frac{Z_s e}{c} \left\langle \left\langle \int d^3\boldsymbol{w} \langle h_s \boldsymbol{w} \rangle_{\boldsymbol{r}} \cdot (\delta \boldsymbol{A} \times \nabla z) \right\rangle_{\mathrm{turb}} \right\rangle_{\psi}, \tag{14}$$

and the viscosity $\Pi^{\psi\phi} = \pi_s^{(\psi\phi)} + \pi_{\mathrm{EM}}^{(\psi\phi)}$, gives rise to viscous heating

$$P_s^{\mathrm{visc}} = - \left[\left\langle \pi_s^{(\psi\phi)} \right\rangle_{\psi} + m_s \omega(\psi) \left\langle R^2 \Gamma_s \right\rangle_{\psi} \right] \frac{d\omega}{d\psi}, \tag{15}$$

Energy Conservation

The total energy,

$$\sum_s \int d^3\mathbf{v} \frac{1}{2} m v^2 f \approx U = \sum_s \frac{3}{2} \langle n_s \rangle_\psi T_s + \frac{1}{2} J \omega^2(\psi), \tag{16}$$

is evolved via

$$\frac{1}{V'} \left.\frac{\partial}{\partial t}\right|_\psi V' U + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left\langle J^{(\mathrm{U})} \right\rangle_\psi = \langle \mathbf{E} \cdot \mathbf{j} \rangle_\psi + \sum_s P_s^{\mathrm{turb}} - \left\langle \pi_{\mathrm{EM}}^{(\psi\phi)} \right\rangle_\psi \frac{d\omega}{d\psi} \tag{17}$$

Poynting's Theorem and Turbulent Heating

But the fluctuations cannot be a source of energy!

$$\frac{\partial}{\partial t}\left(\frac{\delta B^2}{8\pi}\right) - c\nabla\cdot(\delta \boldsymbol{E}\times\delta\boldsymbol{B}) = -\delta\boldsymbol{E}\cdot\delta\boldsymbol{j}, \tag{18}$$

Averaging this over the fluctuations

$$-\langle\delta\boldsymbol{j}\cdot\delta\boldsymbol{E}\rangle_{\mathrm{turb}} = \frac{c}{4\pi}\nabla\cdot\langle\delta\boldsymbol{E}\times\delta\boldsymbol{B}\rangle_{\mathrm{turb}}, \tag{19}$$

So any “heating” must in fact be a flux of energy, not net heating.

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So any "heating" must in fact be a flux of energy, not net heating.

We can in fact show that

$$\sum_s P_s^{\mathrm{turb}} = \sum_s Z_s e \int d^3\boldsymbol{w} \left\langle \left\langle h_s \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \right) \chi \right\rangle_\perp \right\rangle_\psi + \sum_s \frac{Z_s e}{c} \omega(\psi) \left\langle \int d^3\boldsymbol{w} \left\langle h_s \delta \boldsymbol{A} \cdot \nabla z \times \boldsymbol{w} \right\rangle_\perp \right\rangle_\psi = \left\langle \pi_{\mathrm{EM}}^{(\psi\phi)} \right\rangle_\psi \frac{d\omega}{d\psi}, \quad (20)$$

so this is consistent.

But yet there is still dissipation of fluctuations, so where does the energy go!

We can in fact show that

$$\begin{aligned}\sum_s P_s^{\rm turb} &= \sum_s Z_s e \int d^3\boldsymbol{w} \left\langle \left\langle h_s \left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla \right) \chi \right\rangle_\perp \right\rangle_\psi \\ &\quad + \sum_s \frac{Z_s e}{c} \omega(\psi) \left\langle \int d^3\boldsymbol{w} \left\langle h_s \delta\boldsymbol{A}\cdot\nabla z \times \boldsymbol{w} \right\rangle_\perp \right\rangle_\psi \qquad (20) \\ &= \left\langle \pi_{\rm EM}^{(\psi\phi)} \right\rangle_\psi \frac{d\omega}{d\psi},\end{aligned}$$

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But yet there is still dissipation of fluctuations, so where does the energy go!

The free energy conservation law for our system is (multiplying the gyrokinetic equation by h_s/F_{0s} and integrating)

$$\frac{\partial W}{\partial t} = \sum_s T_s P_s^{\mathrm{turb}} - Q + D \tag{21}$$

- The Free Energy

$$W = \left\langle \left\langle \int d^3\boldsymbol{w} \frac{T_s \delta f_s}{2F_{0s}} \right\rangle_{\mathrm{turb}} \right\rangle_{\psi} + \left\langle \left\langle \frac{\delta \boldsymbol{B}^2}{8\pi} \right\rangle_{\mathrm{turb}} \right\rangle_{\psi} \tag{22}$$

- Energy Injection

$$Q = \sum_s \Gamma_s \left(T_s \frac{d \ln N_s}{d\psi} - \frac{3}{2} \frac{dT_s}{d\psi} \right) + q_s \frac{dT_s}{d\psi} + \pi_s^{\psi\phi} \frac{d\omega}{d\psi} \tag{23}$$

- Collisional Dissipation

$$D = \sum_s T_s \left\langle \int d^3\boldsymbol{w} \left\langle \frac{h_s}{F_{0s}} C[h_s] \right\rangle_{\mathrm{turb}} \right\rangle_{\psi} \tag{24}$$

Entropy Balance

The average entropy,

$$\widetilde{H} = -\sum_s \left\langle \int d^3\boldsymbol{w} f_s \ln f_s \right\rangle_{\mathrm{turb}} \tag{25}$$

evolves via

$$\frac{1}{V'} \left.\frac{\partial}{\partial t}\right|_{\psi} V' \left\langle \widetilde{H} \right\rangle_{\psi} + \frac{1}{V'} \frac{\partial}{\partial \psi} \left(V' \left\langle \Gamma^{(H)} \right\rangle_{\mathrm{turb}} \right) = \left\langle \langle \sigma \rangle_{\mathrm{turb}} \right\rangle_{\psi}, \tag{26}$$

with entropy produced by

$$\left\langle \langle \sigma \rangle_{\mathrm{turb}} \right\rangle_{\psi} = \sum_s - \left\langle \int d^3\boldsymbol{w} \ln F_s C[F_s] \right\rangle_{\psi} - \left\langle \left\langle \int d^3\boldsymbol{w} \frac{h_s}{F_{0s}} C[h_s] \right\rangle_{\mathrm{turb}} \right\rangle_{\psi} \tag{27}$$

Energy and Entropy Revisited

In steady state we have,

$$\sum_s T_s P_s^{\mathrm{turb}} = Q - D \tag{28}$$

So all power extracted to excite fluctuations is returned as heat. We also have

$$D = Q + \left\langle \pi_{\mathrm{EM}}^{(\psi\phi)} \right\rangle_\psi \frac{d\omega}{d\psi} \tag{29}$$

and thus the entropy generated by the dissipation of the fluctuations is caused by relaxation of gradients.

Summary

- Fluctuations contribute no net bulk heating
- Fluctuations increase entropy by relaxing gradients, not by heating
- In order to conserve energy in the mean, the fluctuations must conserve free energy
- Free energy is conserved locally – no turbulence spreading

References

- http://arxiv.org/abs/1209.4782