LINEAR SERIES OF STELLAR MODELS AND THERMAL STABILITY

Let us consider a stellar model in a hydrostatic and thermal equilibria, with the total mass M, and the profile of chemical composition $X(M_r)$, specified. The four differential equations describing stellar structure may be written in a form

$$\frac{dx_i}{dx_o} = y_i, \qquad i = 1, 2, 3, 4, \tag{ls.1}$$

where x_o is the independent space-like variable, usually M_r, and x_i are the four dependent variables, like T, ρ, r, and L_r. The boundary conditions have two adjustable parameters at the center, z_1 and z_2, and two parameters at the surface, z_3 and z_4. We may have for example: $z_1 = \rho_c$, $z_2 = T_c$, $z_3 = R$, $z_4 = L$.

The equations of stellar structure are integrated from the surface inwards, down to the fitting point at $M_r = M_f$, and from the center outwards to the same fitting point. The results of the envelope integrations at the fitting point may be written as

$$x_{i,e} = x_{i,e}\left(z_3, z_4\right), \qquad i = 1, 2, 3, 4, \tag{ls.2}$$

and the results of the core integrations as

$$x_{i,c} = x_{i,c}\left(z_1, z_2\right), \qquad i = 1, 2, 3, 4. \tag{ls.3}$$

At the fitting point the differences between the core and envelope integrations are calculated

$$\Delta x_i \equiv x_{i,c} - x_{i,e}, \qquad i = 1, 2, 3, 4. \tag{ls.4}$$

The model is found when $\Delta x_i = 0$. In general, this is not so when wrong values of the boundary parameters are used. The iterative process of finding the correct values is based on the linearized equation

$$\Delta x_i + \sum_{j=1}^{4} c_{ij} \delta z_j = 0, \qquad i = 1, 2, 3, 4, \tag{ls.5a}$$

where

$$c_{ij} \equiv \frac{\partial \Delta x_i}{\partial z_j}, \qquad i, j = 1, 2, 3, 4, \tag{ls.5b}$$

These equations are solved to find the corrections to the boundary parameters, δz_j, and the new values of the boundary parameters, $z_j + \delta z_j$. In order to solve the equations (ls.5a) the determinant of the matrix $|c_{ij}|$ has to be calculated.

Let us suppose that a stellar model satisfying hydrostatic and thermal equilibria has been found, i.e. we know the boundary parameters z_j, and the variation of all functions inside the model, $x_i(x_o)$. Now, we shall make a small perturbation of the equilibrium model, with the exponential time variability, i.e. we shall have

$$z_j = z_{j,o} + z_{j,1} e^{\sigma t}, \qquad j = 1, 2, 3, 4, \tag{ls.6a}$$

$$x_i = x_{i,o} + x_{i,1} e^{\sigma t}, \qquad i = 1, 2, 3, 4, \tag{ls.6b}$$

$$|z_{j,1}/z_{j,o}| \ll 1, \qquad |x_{i,1}/x_{i,o}| \ll 1. \tag{ls.6c}$$

The perturbed functions must satisfy stellar structure equations with the time dependence taken into account. The exponential time dependence implies that we may replace $\partial x_i/\partial t$ with $\sigma x_{i,1}$. In a close analogy with the equilibrium stellar model we may calculate the differences between the core and envelope perturbations, $\Delta x_{i,1}$, at the fitting point, as a function of the perturbations to the boundary parameters, $z_{i,1}$, and σ. We require:

$$\Delta x_{i,1}\left(z_{1,1}, z_{2,1}, z_{3,1}, z_{4,1}, \sigma\right) = 0. \tag{ls.7}$$

We may also calculate (numerically) the partial derivatives at the fitting point:

$$c_{ij,1} \equiv \frac{\partial \Delta x_{i,1}}{\partial z_{j,1}}, \qquad i, j = 1, 2, 3, 4, \tag{ls.8}$$

All the derivatives $c_{ij,1}$, and the determinant $|c_i j{,}1|$ are functions of σ. Notice, that if $\sigma = 0$ then $c_{ij,1} = c_{ij}$, and $|c_{ij,1}| = |c_{ij}|$. Of course, if $\sigma \neq 0$ then the corresponding partial derivatives and determinants are different.

We are looking for the no-trivial perturbations, i.e. we would like to have $z_{j,1} \neq 0$, while the perturbations match at the fitting point, i.e. $\Delta x_{i,1} = 0$:

$$\Delta x_{i,1} = \sum_{j=1}^{4} c_{ij,1} \delta z_{j,1} = 0, \qquad i = 1, 2, 3, 4, \tag{ls.9}$$

These equations for small perturbations $\delta z_{j,1}$ are linear, and we are free to choose their amplitude. This means that only three out of four variables $\delta z_{j,1}$ are independent, while they have to satisfy four linear algebraic equations (ls.9) . This is possible only when the determinant $|c_{ij,1}|$ vanishes. This may be possible for some values of the time constant $\sigma = \sigma_k$, which are called eigen-values, while the corresponding solutions for the perturbations are the eigen-functions. In general σ_k may be complex, but we shall consider here only real eigen-values. $\sigma_k > 0$ corresponds to the unstable mode, while negative eigen-values correspond to stable modes.

In general, there is the largest eigen-value, σ_o, which may be either positive or negative, and there is an infinite number of smaller, negative eigen-values. The determinant $|c_{ij,1}|$ is an oscillatory function of σ for $\sigma < \sigma_o$, and it either rises to $+\infty$, or falls down to $-\infty$ for $\sigma \gg \sigma_o$. If the sign of determinant for $\sigma = 0$ is opposite than for $\sigma \rightarrow +\infty$, then there must be at least one $\sigma > 0$ for which the determinant vanishes, i.e. there must be at least one positive eigen-value, $\sigma_o > 0$, and the stellar model must be unstable. If the sign of determinant for $\sigma = 0$ is the same as for $\sigma \rightarrow +\infty$, then there may be no positive eigen-value, and the model may be stable. Of course, there is also a possibility of having 2, or 4 positive eigen-values in the last case, and then the model is unstable. Therefore, the sign of determinant for $\sigma = 0$ is indicative of the stability of the model. Notice, that this determinant is identical with the one calculated while finding the equilibrium model. Therefore, even if no perturbations are made, we obtain some information about the model stability. In practice it may not be easy to apply this method to an isolated model, as we may not know the sign of determinant for $\sigma \rightarrow +\infty$.

Imagine a family of models that may be described with a single parameter, z_o. For example, this may be the total stellar mass M, the hydrogen content of the chemically homogeneous star X, or the mass of the helium core M_c in a star with a fixed total mass, and so on. As long as the single parameter describes completely the variation of chemical composition throughout the star, and the mass of the star, the family is called a linear series of stellar models. Many different topologies are possible. The sequence may end at some value of the parameter, $z_{o,min}$ or $z_{o,max}$. Those points are called terminating points. The parameter z_o, while varying continuously along the sequence of models, may reach a local minimum and/or a local maximum - these are called the turning points. The sections of the linear series located between the successive turning points are called branches. A linear series that has only two terminating points and no turning points has only one branch. It is possible to have a loop-like linear series of models, with no terminating points, just an even number of turning points, and equal number of branches. Finally, it may be possible to have a series that self-intersects at the so called bifurcation point. The concept of a linear series is at least 100 years old. In particular, it was used to study the properties of rotating fluid bodies.

Imagine now a curve that shows the variations of determinant $|c_{ij,1}|$ as a function of σ for a single stellar model. This curve varies from one stellar model to another, while we vary the parameter z_o along the linear series. The curve varies continuously, the eigen-values vary continuously, and the structure of the models varies continuously too. The differences Δx_i at the fitting point must also vary continuously with z_o along the linear series, and we may calculate (numerically)

$$c_{io} \equiv \frac{\partial \Delta x_i}{\partial z_o}, \qquad i = 1, 2, 3, 4. \tag{ls.10}$$

For all equilibrium models $\Delta x_i = 0$. Therefore, along the linear series we have, for the equilibrium models

$$d(\Delta x_i) = \sum_{j=0}^{4} c_{ij} dz_j = 0, \qquad i = 1, 2, 3, 4. \tag{ls.11}$$

This is a set of 4 ordinary, first order differential equations. It has a formal solution

$$\frac{dz_o}{C_o} = \frac{dz_j}{C_j}, \qquad j = 1, 2, 3, 4, \tag{ls.12}$$

where C_j are the determinants of the 4×4 matrices obtained from the 4×5 matrix of elements c_{ij}, with the "j" column removed.

Notice, that the determinant C_o is identical with the determinant of the 4×4 matrix of elements c_{ij} as defined with equations (ls.5b) for a single equilibrium model. According to the solution (ls.12) the variations of this determinant are proportional to dz_o. In particular, whenever dz_o changes sign, i.e. at the turning points of a linear series, the determinant of the matrix calculated at the fitting point changes its sign as well. At the turning point $dz_o = 0$, hence the determinant $C_o = 0$, and $\sigma = 0$ is an eigen-value.

The value of the determinant at the fitting point calculated for $\sigma \gg 0$ is either very large or very small, depending on the particular series of models, but it has the same sign along the whole sequence. This sign cannot change, as that would require some model to have the largest eigen-value $\rightarrow +\infty$, and this is physically unacceptable. As the sign of the determinant for $\sigma \rightarrow +\infty$ remains constant, while the determinant for $\sigma = 0$ changes sign at every turning point of the linear series, we expect that models located on some branches of the linear series **must be unstable**, i.e. they must have $\sigma_o > 0$, while models on the alternate branches **may be stable**.

Examples of linear series of stellar models

Chemically homogeneous stars. Zero age main sequence stars (ZAMS) are chemically homogeneous hydrogen rich stars. They form a linear series of models with the stellar mass M being the parameter that varies along the sequence. There is no terminating point known, and there is only one turning point known: it corresponds to the minimum mass for hydrogen burning, about $0.1 M_\odot$. The "normal" branch of the main sequence has stars like our Sun, known to be stable. Along this branch stellar radius, luminosity, and central temperature decrease with the decreasing stellar mass, all the way down to the minimum mass. The main sequence may be extended beyond that point, with the stellar radius, luminosity and central temperature all decreasing even more, while the stellar mass is increasing. Following this "high density" branch we have models supported more and more by the pressure of degenerate electron gas. As the "normal" branch is stable, the "high density" branch must be unstable, and indeed it is unstable.

Stars with a constant mass M, with helium cores and hydrogen rich envelopes form a linear series of stellar models. The helium core mass M_c is the parameter that varies along the series. Various phases of stellar evolution may be reasonably well approximated with various stable branches of this series. In a real star the helium core mass may only increase with time. For example, the evolution within the main sequence corresponds to the core mass growing from zero on ZAMS, to the Schonberg-Chandrasekhar limit (approximately $0.1M$) at the end of the main sequence life. The Schonberg-Chandrasekhar limit is the first turning point of that series. There are no more stable nearby models beyond this point, and a star must evolve on a thermal, i.e. relatively rapid time scale to the red giant phase, on which the helium core is degenerate. Evolution up the giant branch corresponds to another stable branch of the linear series. If the stellar mass is above approximately $0.5M_{\odot}$ then the next turning point corresponds to helium ignition in the degenerate core. This is known as a helium flash. The core helium burning phase of stellar evolution may be crudely described with another branch of the same linear series. Finally, evolution from the red giant phase, through the phase when the star is a nucleus of a planetary nebula, and evolves towards the white dwarf stage, may be represented with a linear series, with ever decreasing envelope mass. This branch of the series reaches a turning point at which the envelope mass has the minimum value necessary to sustain hydrogen burning in the shell source. By this time stellar radius is only slightly larger than radius of a white dwarf of the same mass, while the luminosity is almost as high as that of a red giant. As a consequence the star is very hot, just as nuclei of planetary nebulae are. Beyond that turning point there is the next branch of stellar models with increasing envelope mass. Real stars terminate their nuclear burning upon reaching that point, and gradually cool off and become white dwarfs.

Notice, that the existence of many branches on a linear series of stellar models shows that the Vogt-Russell "theorem" is not globally valid: there may be a number of different stellar models with exactly the same mass and chemical composition. However, everywhere along the series, with an exception of the turning points, the Voigt-Russell "theorem" is valid locally.

The concept of a linear series with its implications for the stability of models on various branches is very general, and applicable to many objects, not only stars. For example, interstellar medium under constant pressure: the heating rate is a parameter that varies along the sequence of equilibrium models, in which heating is balanced by cooling. There are at least three branches of models, separated by two turning points. The two extreme branches are stable, while the middle branch is unstable. There is another example in the theory of accretion disks, with the "S-shape" relation between the surface mass density and the surface brightness. There is a general theory of such phenomena known as a catastrophe theory.

Some references

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