The physics of fusion in stars

Most stars derive their luminosity from the conversion of hydrogen to helium. The rest mass of one ^{4}He atom is about 0.71% less than the combined rest masses of four hydrogen atoms (note that the electrons are included in the atomic masses here). The difference, or about 26.7 MeV/c^2, is released as heat, except for ≈ 0.6 MeV worth of neutrinos (in the *pp* chain). There are two paths from $4\,^1H$ to ^4He: the *pp* cycle, which predominates in the Sun and cooler stars, and the CNO cycle, which predominates in stars with slightly higher central temperatures.

pp	**CNO**
$p + p \rightarrow {}^2\text{H} + e^+ + \nu_e$	${}^{12}\text{C} + p \rightarrow {}^{13}\text{N} + \gamma$
${}^2\text{H} + p \rightarrow {}^3\text{He}$	${}^{13}\text{N} \rightarrow {}^{13}\text{C} + e^+ + \nu_e$
${}^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + 2p$	${}^{13}\text{C} + p \rightarrow {}^{14}\text{N} + \gamma$
	${}^{14}\text{N} + p \rightarrow {}^{15}\text{O} + \gamma$
	${}^{15}\text{O} \rightarrow {}^{15}\text{N} + e^+ + \nu_e$
	${}^{15}\text{N} + p \rightarrow {}^{12}\text{C} + {}^4\text{He}$

Table 1: *The main channels of the pp and CNO cyles [2].*

In Table 1, the isotopic designations refer to nuclei rather than whole atoms, so that ^{1}H would be equivalent to a proton, p. In some books, the helium nucleus is denoted by α instead of ^{4}He, and the deuterium nucleus by d instead of ^{2}H.

About 0.4% of *pp* reactions in the Sun start with $2p + e^- \rightarrow {}^2\text{H} + \nu_e$ instead of the first reaction shown in the Table. About 15% involve

$$\begin{aligned} {}^3\text{He} + {}^4\text{He} &\rightarrow {}^7\text{Be} + \gamma \\ {}^7\text{Be} + e^- &\rightarrow {}^7\text{Li} + \nu_e \\ {}^7\text{Li} + p &\rightarrow {}^4\text{He} + {}^4\text{He} \end{aligned} \tag{1}$$

instead of the third reaction shown. Even more rarely (0.02%), the second and third reactions of (1) are replaced by

$$\begin{aligned} {}^7\text{Be} + p &\rightarrow {}^8\text{B} + \gamma \\ {}^8\text{B} + p &\rightarrow {}^8\text{Be}^* + e^+ + \nu_e \\ {}^8\text{Be}^* &\rightarrow {}^4\text{He} + {}^4\text{He} \end{aligned} \tag{2}$$

in which $^8\text{Be}^*$ is a metastable state. This last side chain is energetically negligible but experimentally important because it produces an exceptionally energetic neutrino (up to 15 MeV) which, though much rarer, is easier to detect than the paltry ≤ 0.420 MeV neutrino resulting from the first reaction in the Table.

The carbon, nitrogen, and oxygen in the CNO reactions serve as catalysts: no net production of these elements occurs. The second column of Table 1, for example, replaces the original ^{12}C nucleus. There is a side chain that goes through ^{16}O, but this also involves no net production of elements other than helium. Thus, even at high central temperatures, the CNO cycle could not have occurred in metal-free primordial high-mass stars.

On a per-proton basis, the *pp* and CNO cycles in stars proceed extremely slowly. Fusion has reduced the central hydrogen abundance of the Sun by about a factor of two in the 4.6 Gyr since its formation; thus the fusion rate per proton is $\approx 5\times10^{-18}\,\mathrm{s}^{-1}$. Let us compare this to a characteristic proton-proton collision rate, $n_p\sigma v_{\rm th}$, where $n \approx 6\times10^{25}\,\mathrm{cm}^{-3}$ is the central number density of protons and $v_{\rm th} = (3k_{\rm B}T_c/m_p)^{1/2} \approx 600\,\mathrm{km\,s^{-1}}$ is their thermal velocity. The choice of the collision cross section, σ, depends upon what one considers a collision. As will be seen later, a natural scale for cross sections is $\pi\lambda\!\!\!^{-}{}^2_{\rm dB}$ where $\lambda\!\!\!^{-}_{\rm dB} \equiv \hbar/mv$ is the reduced de Broglie wavelength. If $v = v_{\rm th}$ then $\lambda\!\!\!^{-}_{\rm dB} \approx 10^{-11}$ cm, and the collision rate $n_p\pi\lambda\!\!\!^{-}{}^2_{\rm dB}v \approx 10^{12}\,\mathrm{s}^{-1}$. Comparing this with the fusion rate estimated above, one sees that the probability of fusion per collision is $\sim 2\times10^{-31}$.

The rest of this lecture is devoted to explaining why the latter probability is so small. Actually, there are two principal reasons: the electrostatic repulsion between nuclei, and the weakness of the weak interactions. As a byproduct, we will see why the CNO cycle is so much more sensitive to temperature than the *pp* cycle.

Barrier penetration

The strong force binds nucleons (protons and neutrons) in nuclei but has a limited range, of order one fermi: $1\,\mathrm{fm} \equiv 10^{-13}\,\mathrm{cm} = 10^{-15}\,\mathrm{m}$, so a fermi is also a femtometer. At separation r, the electrostatic energy between nuclei of charges Z_1e and Z_2e is $\approx 1.5Z_1Z_2\,\mathrm{MeV\,fm}/r$, whereas thermal energies are $\sim k_{\rm B}T = 1.4(T/10^7\,\mathrm{K})\,\mathrm{keV}$. Since the Boltzmann distribution falls off exponentially at $E \gg k_{\rm B}T$, and $T \approx 1.58\times10^7\,\mathrm{K}$ at the center of the Sun, the probability that two colliding protons could approach within 1 fm would be $\sim e^{-670} \sim 10^{-290}$ if classical physics applied. But quantum-mechanical tunneling allows the protons to go “under” the Coulomb barrier with a probability that is much larger than this, though still exponentially suppressed.

Partial waves

If you learned about partial waves and scattering from spherical potentials, you may skip this subsection. If you need more detail, see a QM text such as [3].

Since the probability of fusion per collision is so tiny, it is a good approximation to say that the incoming nuclei are described by a stationary state of definite energy $E = mv_\infty^2/2$, where v_∞ is their relative velocity when they are widely separated, and $M = M_1M_2/(M_1 + M_2)$ is their reduced mass. The wavefunction $\Psi(\boldsymbol{r})$ of this state obeys

$$\frac{\hbar^2}{2M}\nabla^2\Psi(\boldsymbol{r}) + [E - U(r)]\,\Psi(\boldsymbol{r}) = 0. \tag{3}$$

where $\boldsymbol{r}$ is the relative coordinate and $U(r) = Z_1Z_2e^2/r$ is the electrostatic potential. When they are widely separated, $\Psi(\boldsymbol{r}) \approx N\exp(i\boldsymbol{k}_\infty\cdot\boldsymbol{r})$, a plane wave, where N is a normalization factor and $\boldsymbol{k}_\infty = Mv_\infty/\hbar$. It is convenient to expand Ψ in “partial waves”

$$\Psi(\boldsymbol{r}) = N\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} C_{\ell m}\psi_\ell(r)Y_{\ell m}(\theta,\phi), \tag{4}$$

$$C_{\ell m}\psi_\ell(r) = \int Y^*_{\ell m}\Psi\,d\Omega \equiv \int_{-1}^{1} d\cos\theta\int_0^{2\pi} d\phi\;Y^*_{\ell m}(\theta,\phi)\Psi(r,\theta,\phi),$$

where $Y_{\ell m}(\theta, \phi)$ is a spherical harmonic,

$$\frac{\hbar^2}{2Mr^2}\frac{d}{dr}\left(r^2\frac{d\psi_\ell}{dr}\right) + \left[E - U(r) - \frac{\hbar^2\ell(\ell+1)}{2Mr^2}\right]\psi_\ell(r) = 0, \tag{5}$$

and the $C_{\ell m}$ are constants.[1] The partial waves are states of definite angular momentum $L = \hbar\ell$, $L_z = \hbar m$, whereas the incoming plane wave is a state of definite *linear* momentum $\boldsymbol{p}_\infty = M\boldsymbol{v}_\infty$. $\boldsymbol{L}$ is perfectly conserved by $U(r)$ but $\boldsymbol{p}$ is not. Because of the "centrifugal barrier" term $\ell(\ell+1)/r^2$ in (5), only the $\ell = 0$ partial wave has an appreciable amplitude at $r \ll ƛ_{\mathrm{dB}}$ where the strong force comes into play. On the other hand, as $r \to \infty$ and $U(r) \to 0$,

$$\psi_\ell(r) \to A_\ell^{(+)}\frac{\exp(+ik_\infty r)}{r} + A_\ell^{(-)}\frac{\exp(-ik_\infty r)}{r} + O\left(\frac{1}{r^2}\right), \tag{6}$$

for some constants $A_\ell^{(\pm)}$. The term in e^{-ikr}/r can be identified as an incoming wave, meaning that its radial group velocity < 0, while the other term is an outgoing wave. If the colliding particles are conserved, then what goes in must come out, $|A_\ell^{(+)}| = |A_\ell^{(-)}|$; in this case, the effect of $U(r)$ is expressed by the relative phase of these coefficients. In our case $1 - |A_\ell^{(+)}|^2/|A_\ell^{(-)}|^2$ is positive but tiny—it is the probability of fusion.

Since the $Y_{\ell m}$ are orthogonal and $Y_{00} = 1/\sqrt{4\pi}$, one finds by evaluating the second line in (4) at large r with (6) for $\psi_0(r)$ and the incoming plane wave for $\Psi(\boldsymbol{r})$ that $A_0^{(-)}C_{\ell m} = i\sqrt{\pi ƛ_{\mathrm{dB}}^2}$. Furthermore, by integrating the radial probability flux $\Psi^*(-i\hbar\partial/\partial r)\Psi$ over a sphere at large r, one can show that the radial probability "current" crossing this sphere per unit time in the $\ell = 0$ partial wave is $|\Psi|^2 v_\infty \pi ƛ_{\mathrm{dB}}^2$. **Therefore $\ell_0 \equiv \pi ƛ_{\mathrm{dB}}^2$ can be interpreted as the cross section for the incoming plane wave to be a zero-angular-momentum state.** This makes sense because the classical angular momentum would be $L = p_\infty b$, where b is the impact parameter, so that $L < \hbar$ if $b < \hbar/p_\infty$; the classical cross-section for this is $\pi(\hbar/p_\infty)^2$.

WKB estimate of the penetration factor

The region where the coefficient of ψ_ℓ in (5) is negative is classically forbidden. In particular, if $U(r)$ is the Coulomb potential and $\ell = 0$, this occurs at $r < R_E \equiv Z_1Z_2e^2/E$. We would like to calculate the radial probability current deep within the forbidden region where $r \sim 1\,\mathrm{fm} \sim 10^{-2}R_E$. While this can be done exactly for $U(r) = Z_1Z_2e^2/r$ in terms of special functions, a good approximation and a much more enlightening result can be found by WKB. We set (the incoming or outgoing part of) $\psi_0(r)$ equal to $\exp[\chi(r)]/r$, which would satisfy (5) exactly if

$$\frac{d^2\chi}{dr^2} + \left(\frac{d\chi}{dr}\right)^2 = \frac{2M}{\hbar^2}[U(r) - E]. \tag{7}$$

The WKB approximation assumes that $d\chi/dr$ is large but slowly varying (at large r, $d\chi/dr \to \pm ik$, a constant), so that $|d^2\chi/dr^2| \ll |d\chi/dr|^2$. Then to leading order,

$$\chi(r) \approx \pm\frac{\sqrt{2M}}{\hbar}\int^r d\bar{r}\,\sqrt{U(\bar{r}) - E}\,. \tag{8}$$

[1]Conventionally, a factor of $i^\ell\sqrt{4\pi(2\ell+1)}$ is included in each term of (4), with a compensating change in $C_{\ell m}$.

The lower limit has been deliberately left unspecified, which is equivalent to allowing an arbitrary constant of integration. Plugging (8) into the previously neglected second-derivative term of (7), one can obtain a more accurate approximation for χ, though we will not need it here. The two choices for the sign in (8) yield two independent approximate solutions for $\psi_0(r)$. In the permitted region where the integrand of (8) is imaginary, the solution whose phase decreases (increases) with increasing r can be interpreted as the incoming (outgoing) wave.

The WKB approximation breaks down near the turning point $r = R_E$ because the integrand in (8) is not smooth there (its derivative is singular). There are standard prescriptions for matching the WKB solutions across the turning point. Since their derivation would take up too much space, we will just quote results.[2] Neglecting the small possibility of fusion, the outgoing and ingoing waves must have equal magnitudes. In this situation, the matching conditions say that we must use the upper sign of (8) at $r < R_E$, so that the solution decays inward into the forbidden zone. Let $R_0 \ll R_E$ be the range of the strong nuclear force. Then

$$\begin{aligned} \chi(R_E) - \chi(R_0) &\approx \frac{\sqrt{2M}}{\hbar} \int_{R_0}^{R_E} d\bar{r} \sqrt{\frac{Z_1 Z_2 e^2}{\bar{r}} - E} = \frac{\sqrt{2ME}}{\hbar} R_E \int_{R_0/R_E}^{1} dx \sqrt{\frac{1}{x} - 1} \\ &\approx \pi \sqrt{\frac{Z_1^2 Z_2^2 e^4 M}{2E\hbar^2}} = \pi Z_1 Z_2 \alpha \frac{c}{v_\infty}, \end{aligned} \tag{9}$$

where $\alpha \equiv e^2/\hbar c \approx 1/137$ is the fine-structure constant. To obtain the second line, we have replaced the lower limit of the x integral by 0: this makes only a small error because the singularity of the integrand at $x = 0$ is integrable.

The probability that the two nuclei come within R_0 is during a collision is of order

$$P_B \equiv \frac{R_0^2}{R_E^2} \frac{|\psi_0(R_0)|^2}{|\psi_0(R_E)|^2} \approx \exp\left[-2\pi \sqrt{\frac{Z_1^2 Z_2^2 e^4 M}{2E\hbar^2}}\right]. \tag{10}$$

The additional factor of two in the exponential relative to (9) occurs because the wavefunction is squared. You might think that the factor R_0/R_E should be cubed rather than squared, to reflect the relative volumes, but the barrier penetration probability was originally defined by the physicist Gamow for radioactive *decay* by fission, and in that case it is the probability flux rather than the probability density that comes in. The most sensitive dependence on energy is due to the exponential factor in any case; discrepancies in the prefactor are absorbed into the nuclear factor $S(E)$ defined below. Even for two colliding protons in the solar core, where $Z_1 = Z_2 = 2$, $M = m_p/2$, and $(\overline{v_\infty^2})^{1/2} = \sqrt{6k_B T_c/m_p} \approx c/340$, the argument of the exponential is moderately large, ≈ -16, so $P_B \ll 1$, but this is not nearly small enough to explain the low probability of fusion per collision as estimated at the beginning of this lecture ($\sim 10^{-31}$).

It is conventional to write the fusion cross section as the product of three factors:

1. The cross section $\sigma_0 = \pi \lambda\!\!\!^{-2}_{\rm dB} = \pi\hbar^2/2ME$ for the plane wave to intercept an $\ell = 0$ state.

[2] Airy functions are involved; see any standard QM text.

2. The probability of what ever nuclear transition is necessary to transform the two nuclei into one once they come into "contact."

3. The probability of barrier penetration, as approximated by (10).

Conventionally then, the energy-dependent fusion cross section is written as

$$\sigma(E) = E^{-1} S(E) \exp\left[-2\pi\sqrt{\frac{Z_1^2 Z_2^2 e^4 M}{2E\hbar^2}}\right], \tag{11}$$

in which the three factors E^{-1}, $S(E)$, and the exponential correspond to items 1,2,3 above.

By far the slowest reaction in the *pp* chain is the first one (Table 1). The strong force is unable to bind two protons, *i.e.* the isotope ^{2}He has a negligible half-life. It can bind a proton and a neutron, though not terribly strongly: the binding energy of ^{2}H (deuterium) is 1.44 MeV, or about one tenth of the binding energy per nucleon of ^{4}He. So a beta decay must occur during the brief time that the two protons in contact. Using the fact that the weak and electromagnetic interactions actually have comparable streangth at energies $\gtrsim M_W \approx 70\,\text{GeV}$, the mass of the carrier of the weak-interaction charged current, the beta-decay probability can be very crudely estimated as $\sim \alpha^2(1.44\,\text{MeV}/M_W)^4 \sim 10^{-23}$. All the other important weak decays in the *pp* and *CNO* cycles occur only after a bound (though only metastable) nucleus forms, so $S(E)$ is much larger for them.[3]

The factor $S(E)$ is hard to calculate because it involves nuclear structure. It is also difficult to measure experimentally at the relevent low energies ($E \lesssim 10\,\text{keV}$), precisely because $\sigma(E)$ is terribly small. However, there is reason to believe that this factor should vary slowly with energy in the case of "nonresonant" reactions such as the first one in Table 1, so that it can be estimated by extrapolation. A reasonably recent estimate [1] is $S(E) \approx 4.00 \pm 0.03 \times 10^{-22}$ keV barn, where[4] 1 barn = 10^{-24} cm^2. For application to stars, one averages the rate coefficient $v_\infty \sigma(E)$ over a thermal distribution of kinetic energies,

$$\begin{aligned}\overline{\sigma v}(T) &= \left(\frac{8}{\pi M (k_{\mathrm{B}}T)^3}\right)^{1/2} \int_0^\infty \sigma(E) E e^{-E/k_{\mathrm{B}}T}\, dE, \\ &= \left(\frac{8}{\pi M (k_{\mathrm{B}}T)^3}\right)^{1/2} \int_0^\infty S(E) \exp\left[-\sqrt{\frac{E_C}{E}} - \frac{E}{k_{\mathrm{B}}T}\right] dE\,, \end{aligned} \tag{12}$$

where I have introduced the "Coulomb energy"

$$E_C \equiv \frac{2\pi^2 Z_1 Z_2 e^4 M}{\hbar^2} = 2\pi^2 Z_1 Z_2 \alpha^2 M c^2 \approx 0.987\, Z_1 Z_2 (M/m_p)\,\text{MeV}. \tag{13}$$

Since $S(E)$ is presumed to be slowly varying, we may estimate this by the method of steepest descent: Writing the argument of the exponential as $-f(E)$, and noting that $f \gg 1$, we expect

[3] An exception is $^3\text{He} + p \rightarrow {}^4\text{He} + e^+ + \nu_e$, but this is responsible for only $\sim 10^{-7}$ of ^{4}He production in the Sun.

[4] The name of this unit comes from the American expression, applied to a poor marksman, "He couldn't hit the broad side of a barn." The Thompson cross section $\sigma_T \approx 0.665$ barn, and larger nuclear cross sections are of similar size.

the integral to be dominated by the neighborhood of the energy E_0 that minimizes $f(E)$. Solving $f'(E_0) = 0$ yields

$$E_0 = (E_C/2)^{1/3}(k_B T)^{2/3} \approx 5.678(2Z_1 Z_2 M/m_p)^{1/3} T_7^{2/3}\,\text{keV}, \tag{14}$$

where $T_7 \equiv T/(10^7\,\text{K})$. Note $2Z_1Z_2M/m_p = 1$ for pp collisions. The energy (14) is about $4.2k_B T_c$ in the Sun, so it is out on the tail of the thermal distribution, but not terribly far. Expanding $f(E) \approx f(E_0)+\frac{1}{2}f''(E_0)(E-E_0)^2$ around the minimum and replacing $S(E) \to S(E_0)$, one finds that

$$\overline{\sigma v}(T) \approx \left(\frac{8}{\pi M(k_B T)^3}\right)^{1/2} \left[\frac{2\pi}{f''(E_0)}\right]^{1/2} e^{-f(E_0)}\, S(E_0).$$

The quantity $f(E_0) = 3E_0/k_B T = 3(E_C/2k_B T)^{1/3}$, so the temperature dependence of non-resonant thermonuclear reactions is dominated by an exponential of the form $\exp[-(T_0/T)^{1/3}]$ as a result of a compromise between the barrier-penetration probability (which increases with energy) and the thermal distribution (which decreases). The constant $T_0 \propto Z_1Z_2A_1A_2/(A_1+A_2)$, where $A_{1,2}$ are the atomic weights of the nuclei, so it is larger for the CNO reactions than for the pp ones. This is why the former are more temperature sensitive. On the other hand, they have much larger $S(E_0)$, so they dominate at higher temperatures (more massive or more evolved stars).

References

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[2] J. N. Bahcall. *Neutrino Astrophysics*. Cambridge, 1989.

[3] G. Baym. *Lectures on Quantum Mechanics*. Benjamin/Cummings, 1981.