A. More Spacetime and Geometry

I. Like Time or Distance? Timelike and Spacelike Vectors

The square of the interval (or modulus, as we shall call it for 4-vectors not connecting two events) associated with a 4-vector \vec{u} is, as we have seen,

$$s^2 = u_t^2 - u_x^2 - u_y^2 - u_z^2 = u_t^2 - \mathbf{u}^2,$$

where **u** is an ordinary 3-vector, the space part of \vec{u} . We should emphasize here that this separation is not a real geometric notion—it is associated with a particular reference frame (coordinate system), and is no more 'natural' than artificially separating a 3-vector into a z-component and a 2-vector in the x - y plane. It is nevertheless often useful to do, just as similar artificial constructs in 3-space are. It is clear that s^2 can have either sign, depending on the relative sizes of the time component and the length of the space part, and can, of course, also be zero. In the last case the 4-vector is called null, and is, as we have seen, parallel to the path of a photon in spacetime. If the sign of s^2 is positive, the vector connecting two events occurring at the same place in some reference frame. If, on the other hand, s^2 is negative, the vector is called spacelike; again the simplest example is a 3-vector with no time component—this is a vector connecting two simultaneous events in some frame. In this case the length of the 3-vector is $(-s^2)^{1/2}$.

The path of a particle in spacetime is called its *world line*. Now in a frame in which it is instantaneously at rest, however complex its motion, its velocity is zero, and so the 4-vector tangent to its world line has only a time component, and is hence *timelike*. Furthermore, if we let τ be the time kept by a clock carried by this particle (the particle's *proper time*, then the 4-vector

$$\vec{u} = \frac{d\vec{x}}{d\tau} = \left(\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau}\right)$$

clearly has modulus unity, because again in the frame in which the particle is instantaneously at rest the time component is 1 and the space part vanishes. \vec{u} is called the 4-velocity of the particle; for very slowly moving particles it is simply $(1, \mathbf{v})$. Newtonian mechanics is that regime in which all the particles satisfy the approximation that the velocity is much, much less than unity (c), so their 4-velocities have time components negligibly different from 1.

Notice that if a particle starts from rest in some frame, the 4-velocity has modulus unity and always does so, which means that it remains timelike, which means that $|\mathbf{dx}| < dt$, or $|\mathbf{dx}/dt| < 1$ — Particles accelerating from rest in any frame can NEVER exceed the velocity of light. We will see physically what the reason for this is shortly.

Let us return for a moment to the notion of proper time; clearly

$$d\tau^2 = dt^2 - d\mathbf{x}^2,$$

which allows one to calculate the time measured by any particle along any path in spacetime. It is important to understand this notion for cosmology, because this is the time kept by an individual galaxy in the expansion.

Notice that we can write

 $d\tau^2 = dt^2(1 - v^2),$

the well-known result that *time passes more slowly as measured by moving objects than by ones at rest.* For very rapidly moving objects (v near unity) the effect can be enormous, and is seen easily in particle accelerators, where particles which decay very rapidly at rest can be stored for long periods when they are moving very nearly at the velocity of light. (For the awake among you, this appears to violate the principle of relativity, does it not? *WHO*, after all, is moving? Can you make sense of this?)

To harangue about the obvious, please note that $d\tau$ is an *invariant*, *i. e.* a number which everyone calculates the same way, because $d\tau$ is just the modulus of the 4-vector $d\tau \vec{u}$, which connects two events in spacetime.

II. Momentum, Energy, Acceleration, and Forces

Let us first look at acceleration. The natural definition of acceleration in special relativity is

$$\vec{a} = d\vec{u}/d\tau.$$

Note that in order for \vec{a} to be a proper 4-vector, it must have invariant modulus, which is why we take the derivative with respect to τ , which is itself an invariant; besides, whose time is important when measuring acceleration? That of the body being accelerated, no? Because \vec{u} is a unit vector, it must be that

$$\vec{a} \cdot \vec{u} = 0$$

Notice that since \vec{a} is orthogonal to \vec{u} , it must be that in a frame in which the particle is at rest and \vec{u} has *only* a time component, that \vec{a} must have only a space part, so the acceleration is a 3-vector in the rest frame of the body.

Matter, of course, is characterized by having mass. In relativity, mass is a basically Newtonian concept—it is defined as the mass measured by Newton's second law in a frame in which the object being measured is moving arbitrarily slowly. A *force* is also pretty Newtonian; it is a vector with purely space parts in the frame in which the body is instantaneously at rest. By analogy with the Newtonian definition, the 4-momentum is

$$\vec{p} = m\vec{u},$$

where \vec{u} is the 4-velocity, and Newton's second law becomes

$$\frac{d\vec{p}}{d\tau} = \vec{f}.$$

The statement that the force is a pure space 3-vector in the frame in which the particle is at rest means in 4-space language that $\vec{f} \cdot \vec{u} = 0$. 4-forces must be orthogonal to the 4-velocity of the particle on which they act. Thus the 4-force is parallel to the 4-acceleration. Then the derivative of the modulus of the 4-momentum is

$$\frac{d(\vec{p}\cdot\vec{p})}{d\tau} = \vec{p}\cdot\frac{d\vec{p}}{d\tau} = \vec{p}\cdot\vec{f} = 0.$$

But the 4-momentum is just the mass times the 4-velocity; both have constant modulus, so the mass is constant, which is reassuring, and we recover a familiar-looking result

$$\vec{f} = m\vec{a}.$$

Let us look briefly at the components of the 4-momentum in some coordinate system. Remembering that $d\tau = dt\sqrt{1-v^2}$, we get

$$\vec{p} = \left(\frac{m}{\sqrt{1-v^2}}, \frac{m\mathbf{v}}{\sqrt{1-v^2}}\right).$$

To lowest order in the velocity, the time component is

$$\frac{m}{\sqrt{1-v^2}} \simeq m + mv^2/2$$

or $mc^2 + 1/2mv^2$ in ordinary units, a constant plus the kinetic energy. The constant is the rest mass energy made famous by Einstein and infamous by the Bomb. It is most definitely not an arbitrary constant; when a positron meets an electron, there is no rest mass in the aftermath, but an amount of energy $2m_ec^2$ is released in two or three gammaray photons. The time component of the 4-momentum is the total energy of the particle. For small velocities, the space part of the 4-momentum is just the ordinary momentum, but for velocities near the velocity of light, even though the velocity is bounded by c, the momentum goes to infinity as $v \to c$, as does the energy. The factor $1/\sqrt{1-v^2}$ which multiplies m for the energy and $m\mathbf{v}$ for the momentum is often called the ' γ factor' or just ' γ '; its deviation from unity is a direct measure of how relativistic a particle is. For example, if γ is 2, the velocity is $\sqrt{3}/2 = 0.866$ of the velocity of light, and the kinetic energy (conveniently defined as the energy minus the rest energy) is equal to the rest mass energy. Note that γ is the energy per unit mass of a moving particle:

$$\vec{u} = (\gamma, \gamma \mathbf{v}),$$

$$\vec{p} = (\gamma m, \gamma m \mathbf{v}) = (E, \mathbf{p})$$

Note also that internal energy in an object shows up in its rest mass. If you build a very strong spherical shell and fill it with helium gas, say, the mass of the shell depends on how hot it is, because the thermal energy 3NkT/2 of the helium gas is *energy*, and the

mass of the object is the mass of the shell plus the mass of the helium plus the thermal kinetic energy of the helium (over c^2 in ordinary units, of course.) The situation shows up differently in the case of nuclei, which are always *less* massive than the sum of the masses of the free neutrons and protons which make them up, by typically of the order of an MeV per nucleon. This is because they are *bound* by attractive *negative* potentials; binding energy is negative, so the result. The fact that very heavy nuclei are less bound than lighter ones because of Coulomb repulsion is why fission bombs and reactors work, and the fact that a helium nucleus is much more tightly bound than two deuterium nuclei or two protons and two neutrons is why fusion bombs (and stars) work.

If we now take a particle of very small mass and some energy and let its mass go to zero, then clearly γ must go to infinity and the magnitude of v to unity (c). The four-velocity is not well behaved in this limit; both the space and time parts become infinite, but the 4-momentum vector is perfectly well behaved. The vector in the limit is *null*, clearly, and the magnitude of the momentum is equal to the energy. We have just described a photon, which is a particle of zero mass. Thus always

$$E^2 - p^2 = m^2$$

but m can be zero.

III. A Simple Geometric Problem

Let us do a problem which illustrates the power of geometric reasoning. Suppose we have an observer U with some 4-velocity \vec{u} and another, W, with another 4-velocity \vec{w} . Suppose that U emits light which W receives. What is the redshift that W observes? How do we do this? Well, a photon from U to W must travel along a null vector, say $\vec{\sigma}$. A little proper U-time later, say $d\tau$, which we may as well make the inverse of the frequency of the light, so that exactly one wavelength goes out, there is another null vector, $\vec{\sigma} + d\vec{\sigma}$, going from U to W. The arrivals at W are separated in that frame by some small proper W-time time $d\eta$. Now $d\tau\vec{u}$, σ , $d\eta\vec{w}$, and $\vec{\sigma} + d\vec{\sigma}$ make a closed quadrilateral in spacetime; in particular,

$$d\tau \vec{u} + \vec{\sigma} + d\vec{\sigma} = \vec{\sigma} + d\eta \vec{w}$$

or

$$d\tau \vec{u} + d\vec{\sigma} = d\eta \vec{w}.$$

But $d\eta/d\tau$ is the ratio of the times for one undulation of the wave in U's frame to that in W's frame, which is the ratio of the wavelengths, which is 1 + z, so if we can calculate this ratio, we have solved the problem.

If we take the dot product of this with the null vector $\vec{\sigma}$, we get

$$d\tau \vec{u} \cdot \vec{\sigma} = d\eta \vec{w} \cdot \vec{\sigma} - d\vec{\sigma} \cdot \vec{\sigma}.$$

But $\vec{\sigma} \cdot \vec{\sigma} = 0$; 0 is a constant, so $d\vec{\sigma} \cdot \vec{\sigma} = 0$, and

$$d\tau \vec{u} \cdot \vec{\sigma} = d\eta \vec{w} \cdot \vec{\sigma},$$

$$1 + z = \frac{d\eta}{d\tau} = \frac{\vec{u} \cdot \vec{\sigma}}{\vec{v} \cdot \vec{\sigma}}.$$

Very compact. For a specific case, let us calculate the redshift of a galaxy receding from us at some distance along the x axis at very high velocity. Then

$$\vec{w} = (1, 0, 0, 0), \quad \vec{u} = (\gamma, \gamma v, 0, 0), \quad \vec{\sigma} = (d, -d, 0, 0)$$

and

$$(1+z) = \frac{d(\gamma + \gamma v)}{d} = \gamma + \gamma v$$
$$= \frac{(1+v)}{\sqrt{(1-v^2)}}$$
$$= \sqrt{\frac{1+v}{1-v}}.$$

Thus if we lived in a special relativistic world, the z = 5.81 quasar would be receding from us at a velocity v which satisfies

$$\sqrt{\frac{1+v}{1-v}} = (1+z) = 6.81,$$

which is trivially solved to yield

$$v = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} = 0.958,$$

so it is receding from us at about 96% of the speed of light.

Now we will see that there are almost certainly effects in real cosmological models which invalidate this calculation, but as we expect from relativity, a large redshift is *not* associated with velocities in excess of c, and if the universe is nearly empty, so that gravitation is negligible, this calculation is correct. In models in which gravitation matters, as is almost certainly the case in the real universe, we will see that the question of what somebody's *velocity* is at great distance is next to meaningless.

V. Lorenz Transformations, Tensors and the Metric

I. Lorenz Transformations

We talked about orthogonal transformations which carry one Cartesian coordinate system into another and by which one transforms the components of vectors. An analogous set of transformations carry the components of 4-vectors in one special relativistic reference frame into those in another. These have the property, as they must, that they preserve the dot products of arbitrary 4-vectors. Since if (u_0, u_1, u_2, u_3) , (w_0, w_1, w_2, w_3) are the (t, x, y, z) components of a two 4-vectors \vec{u} and \vec{w} in one frame and $(u_{0'}, u_{1'}, u_{2'}, u_{3'}, (w_{0'}, w_{1'}, w_{2'}, w_{3'}, u_{3'}, w_{3'}, w$

$$\vec{u} \cdot \vec{w} = \eta_{\alpha\beta} u^{\alpha} w^{\beta} = \eta_{\mu'\nu'} u^{\mu'} w^{\nu'}$$

Here we have introduced *three* new pieces of notation. The matrix

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Is called the *Minkowski metric tensor*, and in this context merely expresses the fact that to do the dot product we take the product of the time components and *subtract* the ordinary dot product of the space components. The second notational convention is that we sum over repeated *Greek* indices from 0 to 3. This latter is simply to make things compact and to keep from writing lots of summation signs. The third is that we use *superscripts* for the components of ordinary vectors—this is unnecessary here, but we do it for consistency with proper calculations in general relativity (these are the *contravariant* components.)

So what must the properties of a linear transformation

$$u^{\alpha'} = L^{\alpha'}_{\ \beta} u^{\beta}$$

be in order that the dot products are preserved? The reasoning is precisely the same as for the orthogonal transformation in 3-space, and leads to the similar result that the matrix product

$$L^T \eta L = \eta$$

such a transformation is called a *Lorenz* transformation.

Let's see how this works. If an observer is moving in our frame at velocity v along the x axis, he has 4-velocity $\vec{u} = (\gamma, \gamma v, 0, 0)$ A unit spacelike vector in the x - t plane which is orthogonal to this is clearly

$$\vec{w} = (\gamma v, \gamma);$$

 $\vec{u} \cdot \vec{w} = \gamma^2 v - \gamma^2 v = 0$, $\vec{w} \cdot \vec{w} = \gamma^2 v^2 - \gamma^2 = -1$. The spacelike vectors \vec{e}_y and \vec{e}_z along our y and z axes are also clearly orthogonal both to \vec{u} and \vec{w} as well, so the set of four vectors $\vec{u}, \vec{w}, \vec{e}_x, \vec{e}_z$ are the unit vectors pointing along the t', x', y', z' axes of the system in which the moving observer is at rest. We have the unit vectors along the axes, and the components of a vector, just as they are in 3-space, are the dot products of the vector with each of the basis vectors, though if we want the new axes to point in roughly the same directions as the old ones, we must take the *negative* of the dot product for the spacelike three. If we put the origin of both systems at one event, setting t, x, y, z and t', x', y', z' to zero simultaneously, then it must be that if we erect a 4-vector \vec{q} from the origin to some event (t, x, y, z), then the components of q in our system are just those (t, x, y, z), and the components in the system in which the moving observer is at rest are

$$t' = \vec{q} \cdot \vec{u} = \gamma t - v\gamma x$$
$$x' = -\vec{q} \cdot \vec{w} = -v\gamma t + \gamma x$$
$$y' = -\vec{q} \cdot \vec{e}_y = y$$
$$z' = -\vec{q} \cdot \vec{e}_z = z.$$

In a more standard form this is

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$
$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$
$$y' = y$$
$$z' = z$$

Thus we see, as we knew, that time and space in the moving frame are inextricably mixed.

II. The Metric and Tensors

There is a concept in relativity which we have been using all along but have not been aware of which assumes vital importance in the next step, that is, trying to understand gravity in terms of geometry. It is, in fact, quite useful in special relativity as well. We have seen that we can write

$$ds^{2} = dt^{2} - dx^{2} - dy^{2} - dz^{2} = \eta_{\mu\nu} dx^{\mu} dx^{\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski metric. There is no particular reason in relativity any more than in 3-space to use these simple Euclidean coordinates, and so one can write in general that

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$$

in any coordinate system whatever. The matrix of numbers $g_{\mu\nu}$ is not, of course, just diag(1, -1, -1, -1), but can be essentially anything for sufficiently complex coordinates. It is in general the matrix of components of a geometric entity called *metric tensor*. The quantity ds^2 must, of course, not change not only under a Lorenz transformation but under any transformation of coordinates whatsoever. Note that the matrix of g is always symmetric; the order of dot products does not matter.

A word at this point about tensors. A tensor is a sum of things made of two or more vectors, the simplest example of which is

$$T = \vec{u} \otimes \vec{v}$$

where \otimes is the symbol which represents the *tensor product*. A tensor is an object which has a dot product with a vector, and the result is another *vector*, in this simple case just a constant times \vec{u}

$$T \cdot \vec{w} = (\vec{u} \otimes \vec{v}) \cdot \vec{w} = \vec{u}(\vec{v} \cdot \vec{w}).$$

This dot product does *not* commute, since, naturally, if you take

 $\vec{w}\cdot T$

You should do the dot product with the *first* vector in the tensor, so

$$\vec{w} \cdot T = (\vec{w} \cdot \vec{u})\vec{v}$$

producing a multiple of \vec{v} . The *components* of T in some coordinate system make the obvious matrix:

$$T^{\mu\nu} = u^{\mu}v^{\nu}$$

The components of the dot product are then (remembering that the dot product involves the metric):

$$(T \cdot \vec{w})^{\mu} = T^{\mu\nu} g_{\nu\alpha} w^{\alpha}$$

You may have noticed that these implied sums always involve one superscript and one subscript, and we are now at a place where we have to insist that this is true, and define a way to change a superscript to a subscript:

We define the *covariant* components of a vector \vec{v} as

$$v_{\mu} = g_{\mu\nu}v^{\nu}$$

and likewise for any component of a multicomponent tensor. Is there an operation which retrieves the ordinary (which are called *contravariant* components)? We are looking for a matrix $h^{\mu\nu}$ for which

$$v^{\mu} = h^{\mu\nu}v_{\nu} = h^{\mu\nu}g_{\nu\alpha}v^{\alpha}$$

for any \vec{v} . There is such a matrix h, which is just the inverse of the matrix g; the matrix product of h and g is the identity = diag(1,1,1,1). The inverses of symmetric matrices are symmetric, so h is symmetric. Given this, the contravariant components of the *metric*, if this makes any sense, are

$$g^{\mu\nu} = h^{\mu\alpha}g_{\alpha\beta}h^{\beta\nu}$$
$$= h^{\mu\nu},$$

and the matrix

$$g^{\mu}_{\ \nu} = g_{\mu\alpha}h^{\alpha\nu} = \delta^{\mu}_{\ \nu}$$

i. e. the *IDENTITY*. So the metric *is* a tensor, the trivial one which when dotted with a vector gives you back the vector you started with. In the case of the Minkowski metric, the contravariant components are the same as the covariant ones, $\eta^{\mu\nu} = diag(1, -1, -1, -1)$, but the mixed components are (as for any metric) the identity, $\eta^{\mu}_{\nu} = diag(1, 1, 1, 1)$.

III. Tensors to Know and Love

We will not deal directly with very many tensors, but there are two (in addition to the metric) that are very worth knowing about at this time. The first is the Maxwell Field Tensor. We are used to thinking about the electric and magnetic fields as *vectors*, but the magnetic field involves cross products and therefore the calculation depends on whether the coordinate system you are using is right- or left- handed—it is an *axial vector*. In relativity theory, the electromagnetic field is an *antisymmetric tensor*, whose components are

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

It is accompanied by a vector, which is just the sum of all the 4-vectors of the charges making the *current*, and is the 4-current

$$\vec{j} = (\rho, \mathbf{j})$$

where ρ is the charge density and **j** the ordinary 3-vector current density. *Charge* is an invariant, as it must be–an electron must look the same to everyone, and it is a bit of work but not very hard to show that \vec{j} is a proper 4-vector. These things (the fields and the charge/current) are the ingredients of Maxwell's equations, and in this language they become $\vec{\nabla}F = 4\pi \vec{i}$

or

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = 4\pi j^{\mu}$$

Do it and check. So there is *no modification* to Maxwell's theory, as promised, except that this formulation specifies exactly how the fields and currents change as one goes from one frame to another, which was not even a part of the original formulation.

The second tensor we need to know about is the *stress-energy tensor*, which it will turn out replaces the ordinary density of matter as the source of the *gravitational field*. It is a *symmetric* tensor whose components in any frame are

$$T^{\mu\nu} = \begin{pmatrix} \rho & f_x & f_y & f_z \\ f_x & P_{xx} & P_{xy} & P_{xz} \\ f_y & P_{xy} & P_{yy} & P_{yz} \\ f_z & P_{xz} & P_{yz} & P_{zz} \end{pmatrix}$$

where here ρ is the density of mass-energy, **f** the *flux* of mass-energy, and P_{ij} is the stress, or pressure, tensor: P_{xy} , which is defined as the force transmitted in a given direction (one of the subscripts) across a unit area normal to a unit vector belonging to the other

subscript by stresses, etc, in the medium. In the simple, familiar case of an isotropic fluid at rest in a frame, this tensor takes the simple form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0\\ 0 & P & 0 & 0\\ 0 & 0 & P & 0\\ 0 & 0 & 0 & P \end{pmatrix}$$

In which P is the ordinary pressure. The stress-energy tensor of an ideal gas is, in fact, just the sum of all the tensor products of $\vec{u} \otimes \vec{p}$ of the particles contained in a unit volume in the rest frame of the gas.