# Pre-Algebra

## Lecture 5: Fractions III

Today we will cover the following topics:

1. Multiplying Rational Numbers
2. Dividing Rational Numbers
3. Solving Equations: Using Multiplication and Division

### 1. Multiplying Rational Numbers

A rational number can be written as

$$\frac{a}{b}$$

where $a$ and $b$ are integers, and $b \neq 0$. The general rule for multiplying rational numbers can be written as:

$$\frac{\mathbf{a}}{\mathbf{b}} \times \frac{\mathbf{c}}{\mathbf{d}} = \frac{\mathbf{a} \times \mathbf{c}}{\mathbf{b} \times \mathbf{d}}$$

In other words, the resulting fraction has the product of two numerators as the numerator and the product of two denominators as the denominator. Multiplying rational numbers is much simpler than adding them, since

$$\frac{a}{b} + \frac{c}{d} \neq \frac{a+c}{b+d}$$

**Notation:** There are two equivalent multiplication symbols for numbers:

$$7 \times 2 = 7 \cdot 2$$

When using variables such as $a$ or $x$, multiplication symbols are not necessary:

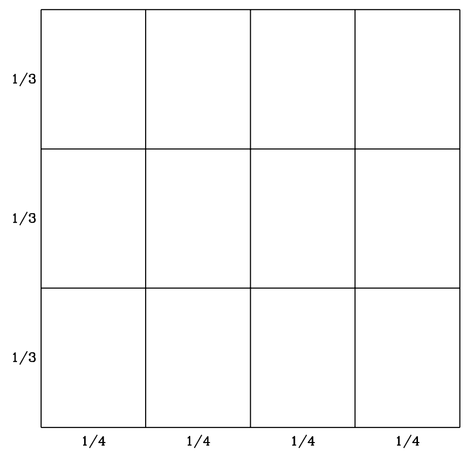
$$a \times x = a \cdot x = ax$$

(on board)
Multiplication of rational numbers can be illustrated by drawing a unit square and subdividing it as shown below. There are 12 basic rectangles of the same area that equals $\frac{1}{12}$. The area of such a rectangle can also be computed by multiplying the lengths of its sides:

$$\text{Area } = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$$

So we get the same answer by counting rectangles and by multiplication.

Students should practice finding the areas of various rectangles in the figure, by counting the number of basic rectangles and by multiplication, and checking whether the answers are the same.




**Examples:**
**(a)** positive proper fractions

$$\frac{2}{7} \times \frac{3}{4} = \frac{2 \times 3}{7 \times 4} = \frac{3}{7 \times 2} = \frac{3}{14}$$

**(b)** one positive and one negative proper fraction

$$\frac{3}{5} \times \left(-\frac{7}{9}\right) = (-1) \times \frac{3 \times 7}{5 \times 9} = -\frac{21}{45}$$

**(c)** negative proper fractions

$$\left(-\frac{1}{3}\right) \times \left(-\frac{6}{7}\right) = (-1) \times (-1) \times \frac{6}{3 \times 7} = \frac{2}{7}$$

**(d)** one positive proper fraction and one positive mixed number

$$1\frac{1}{3} \times \frac{6}{7} = \frac{4}{3} \times \frac{6}{7} = \frac{4 \times 6}{3 \times 7} = \frac{8}{7} = 1\frac{1}{7}$$

**(e)** two positive mixed numbers

$$2\frac{1}{3} \times 3\frac{6}{7} = \frac{7}{3} \times \frac{27}{7} = \frac{7 \times 27}{3 \times 7} = 9$$

**(f)** one positive and one negative mixed number

$$3\frac{2}{5} \times \left(-1\frac{2}{3}\right) = (-1) \times \frac{17}{5} \times \frac{5}{3} = -\frac{17}{3} = -5\frac{2}{3}$$

**2. Dividing Rational Numbers**

First we need to learn about **multiplicative inverses** or **reciprocals** of numbers. Consider a number $x$ and let $y$ be its multiplicative inverse. Then

$$x \times y = 1$$

**Examples:**

**(a)** $x = 5$ and $y = \frac{1}{5}$, since $x \times y = 5 \times \frac{1}{5} = 1$

**(b)** $x = \frac{1}{5}$ and $y = 5$, since $x \times y = \frac{1}{5} \times 5 = 1$

**(c)** $x = -7$ and $y = -\frac{1}{7}$, since $x \times y = (-1) \times (-1) \times 7 \times \frac{1}{7} = 1$

**(d)** $x = 2\frac{1}{7} = \frac{15}{7}$ and $y = \frac{7}{15}$, since $x \times y = \frac{15}{7} \times \frac{7}{15} = 1$

These examples show that for every rational number $\frac{a}{b}(a \neq 0,\ b \neq 0)$ its multiplicative inverse is $\frac{b}{a}$, that is

$$\mathbf{\frac{a}{b} \times \frac{b}{a} = 1}$$

Now we are ready to divide rational numbers. Let $b \neq 0,\ c \neq 0,$ and $d \neq 0$. We would like to divide a rational number $\frac{a}{b}$ by another rational number $\frac{c}{d}$. The result $R$ can be written as

$$R = \frac{\frac{a}{b}}{\frac{c}{d}}$$

Multiplying the numerator **and** the denominator by the multiplicative inverse of the denominator we get

$$R = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}}$$

$$= \frac{\frac{a}{b} \times \frac{d}{c}}{1}$$

$$= \frac{a}{b} \times \frac{d}{c}$$

So we have shown that to divide by a rational number, we should multiply by its multiplicative inverse.

**Notation:** There are three equivalent division symbols for numbers and variables:

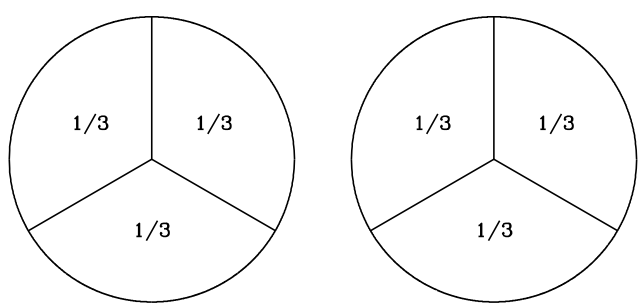
$$\frac{a}{b} = a \div b = a/b$$

(on board)
Division of rational numbers can be illustrated by drawing 2 pies as shown below. Each pie is divided into thirds. How many thirds are in two pies? By direct counting we find that there are 6. But we can also use division:

$$2 \div \frac{1}{3} = \frac{2}{1} \times \frac{3}{1} = \frac{6}{1} = 6$$

So we get the same answer by using division.

Students should practice finding answers to the following questions, using direct counting and division: How many (1/6)s are in 2/3 of a pie? How many (1/8)s are in in 1 pie, in 2 pies, in 1/2 of a pie? etc.



**Examples:**

**(a)**

$$\frac{2}{7} \div \frac{3}{4} = \frac{2}{7} \times \frac{4}{3} = \frac{2 \times 4}{7 \times 3} = \frac{8}{21}$$

**(b)**

$$\frac{2}{7} \div 6 = \frac{2}{7} \times \frac{1}{6} = \frac{2 \times 1}{7 \times 6} = \frac{1}{21}$$

**(c)**

$$5 \div \frac{3}{4} = \frac{5}{1} \times \frac{4}{3} = \frac{5 \times 4}{1 \times 3} = \frac{20}{3} = 6\frac{2}{3}$$

**(d)**

$$2\frac{1}{3} \div \frac{3}{4} = \frac{7}{3} \times \frac{4}{3} = \frac{7 \times 4}{3 \times 3} = \frac{28}{9} = 3\frac{1}{9}$$

**(e)**

$$3\frac{1}{3} \div 2\frac{3}{4} = \frac{10}{3} \div \frac{11}{4} = \frac{10}{3} \times \frac{4}{11} = \frac{10 \times 4}{3 \times 11} = \frac{40}{33} = 1\frac{7}{33}$$

**(f)**

$$\frac{7}{8} \div \left(-\frac{3}{4}\right) = (-1) \times \frac{7}{8} \times \frac{4}{3} = -\frac{7}{6} = -1\frac{1}{6}$$

## 3. Solving Equations: Using Multiplication and Division

**Examples:**

**(a)** Consider the following equation

$$\frac{5}{6}x = \frac{2}{3}$$

We can solve this equation by first finding the least common denominator (lcd) of the two fractions. It is 6. We then multiply **both** sides of this equation by 6 to obtain

$$5x = 4$$

and dividing **both** sides of this equation by 5 we find the solution

$$x = \frac{4}{5}$$

A simpler way of solving this equation is to divide **both** sides of the original equation by $\frac{5}{6}$ to obtain

$$x = \frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \times \frac{6}{5} = \frac{2 \times 6}{3 \times 5} = \frac{4}{5}$$

This is the same as multiplying **both** sides of the original equation by the reciprocal (or multiplicative inverse) of $\frac{5}{6}$.

We can check whether we got the correct answer as follows

$$\frac{5}{6}x = \frac{5}{6} \times \frac{4}{5} = \frac{5 \times 4}{6 \times 5} = \frac{2}{3}$$

So the answer is correct.

It is very important to remember that any division or multiplication must be done on both sides of the equation. If there are two or more terms on either side of the equation, then any division or multiplications must be done on every such term.

**(b)** Let's consider another example.

$$\frac{5}{6}x + \frac{1}{4} = \frac{2}{3}$$

Multiplying all terms on both sides of this equation by the multiplicative inverse of $\frac{5}{6}$ we get

$$x + \frac{1}{4} \times \frac{6}{5} = \frac{2}{3} \times \frac{6}{5}$$

$$\begin{aligned} x &= \frac{4}{5} - \frac{3}{10} \\ &= \frac{8}{10} - \frac{3}{10} = \frac{5}{10} = \frac{1}{2} \end{aligned}$$

To check the answer compute

$$\frac{5}{6}x + \frac{1}{4} = \frac{5}{6} \times \frac{1}{2} + \frac{1}{4} = \frac{5}{12} + \frac{3}{12} = \frac{8}{12} = \frac{2}{3}$$

So we have found the correct solution.

**(c)** We can apply these skills to everyday problems. For example, suppose a book store is advertising 20% off on all algebra books. The sale price for Algebra I book is \$17.20. What is the list price of this book?

Let $x$ be the list price for Algebra I in dollars. We know that the bookseller is lowering the price by 20%, that is the sale price is $80\% = \frac{4}{5}$ of the list price. So we want to solve the equation

$$\frac{4}{5}x = \$17.20$$

To solve this equation, we divide both sides by $\frac{4}{5}$ and get

$$\begin{aligned} x &= \$17.20 \times \frac{5}{4} \\ &= \$17\frac{1}{5} \times \frac{5}{4} \\ &= \$\frac{86}{5} \times \frac{5}{4} \\ &= \$\frac{43}{2} = \$21\frac{1}{2} = \$21.50 \end{aligned}$$

So the list price of this algebra book is \$ 21.50.

**(d)** Jane started a trip with a full tank of gas. During the trip she used $\frac{3}{4}$ of the tank. She then filled the tank with $12\frac{1}{2}$ gallons of gas. How many gallons does her tank hold?

Let $x$ be the volume of the tank in gallons. During the trip Jane used $\frac{3}{4}x$ gallons and replaced it with $12\frac{1}{2}$. Therefore, the equation is

$$\frac{3}{4}x = 12\frac{1}{2}$$

Dividing both sides of this equation by $\frac{3}{4}$ we get

$$x = 12\frac{1}{2} \div \frac{3}{4} = \frac{25}{2} \times \frac{4}{3} = \frac{50}{3} = 16\frac{2}{3}$$

Jane's tank holds $16\frac{2}{3}$ gallons of gas.