Pre-Algebra
Class 10 Geometry I

1. Points, Lines, and Angles

The word *geometry* means: the description and measure of space. Let's start with a **point**. This is a dot with zero **size** (or **dimension**) but with a definite **location**. For example, when you were drawing graphs in Class 9, you could locate a point by its (x,y) coordinates. A **line** connects two **points** in space and has one dimension, **length**. A **plane** has two dimensions, **length** and **width**. We're going to start with **plane geometry**, i.e. geometry on a flat plane, for example the page of your notebook. When you were plotting graphs, you plotted points on the **(x,y) plane**.

(a) Lines and Points

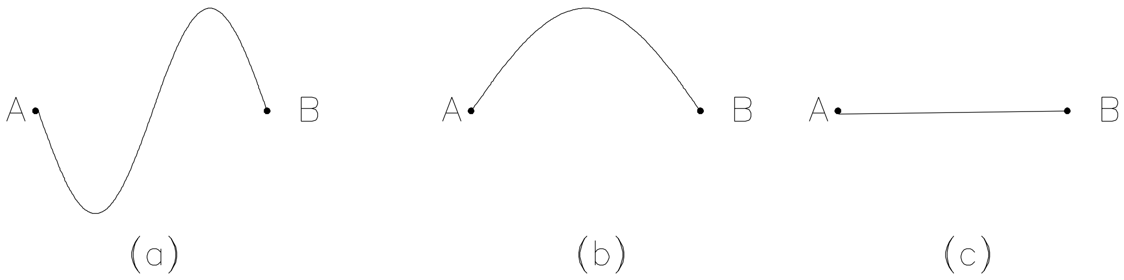

Figure 1. Lines joining the points A and B

Consider two points on a plane, A and B, and join them by a **line**. Some examples are shown in Figure 1. The line, or line segment, is called AB. The line might be the path you take walking from A to B. Note that there is an infinite number of lines which can join A to B. There's one special one. Suppose we stand at A and hold a string tied to B and pull it taut. This is the shortest distance between A and B and is called a **straight line** (see Figure 1c). **A straight line is the shortest distance between two points**. Note that the linear relationships you were graphing in Class 9, things like:

$$y = ax + b$$

are straight lines.

Now suppose we have two straight lines, AB and CD, which cross, or intersect, at E. Figure 2 shows the **intersection** of two straight lines. **Straight lines intersect only once**.

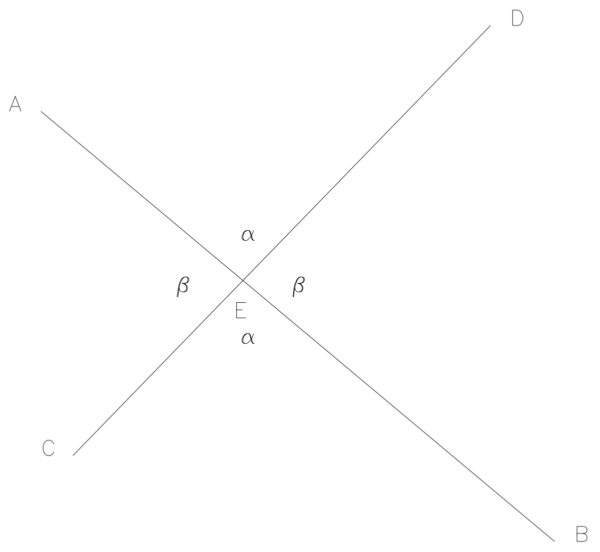

Figure 2. Intersecting straight lines. The greek letters α and β denote the measures of the angles

(b) Angles

Now suppose that you walk from C towards D, i.e. along the straight line CD. At E you stop, turn, and walk towards A. The change of direction is called the **angle** between straight lines CE and EA. This angle is called $C\hat{E}A$. The little angle sign over the E means that the point of the angle is at point E. Angles are sometimes denoted by $\angle CEA$, that is with the angle sign in front. It's my opinion that this is less clear, and I'm not going to use it in these notes.

Angles are **measured** in **degrees**, denoted like this: $37° = 37$ degrees. The number of degrees in one complete turn (i.e. you end up facing the same way you started) is 360°. Note from Figure 2 that there are four angles at the intersection of a straight line. 360° divided by 4 is 90°, so that any two lines which cross each other at 90° have all four angles equal to each other , and the lines are **perpendicular** to each other. The **size**, or **measure**, of each angle is 90°. Angles of 90° are called **right** or **perpendicular** angles, and you denote them by the little square as shown in Figure 3. When you were drawing graphs, you plotted points by plotting positions relative to the (x,y) axes, which are **perpendicular** to each other.

Note from Figure 3 that the angle between two opposite directions on a straight line, for example between EB and EA, is $\boxed{90° + 90° = 180°}$. At any point on a straight line, e.g. point F in Figure 3, **the angle** (denoted by the little arc around point F), is 180°. This is another definition of a straight line - the angle about any point on the line is 180°.

Angles of measure $< 90°$ are called **acute angles**.

Angles of measure $= 90°$ are called **right angles**.

Angles of measure $> 90°$ are called **obtuse angles**.

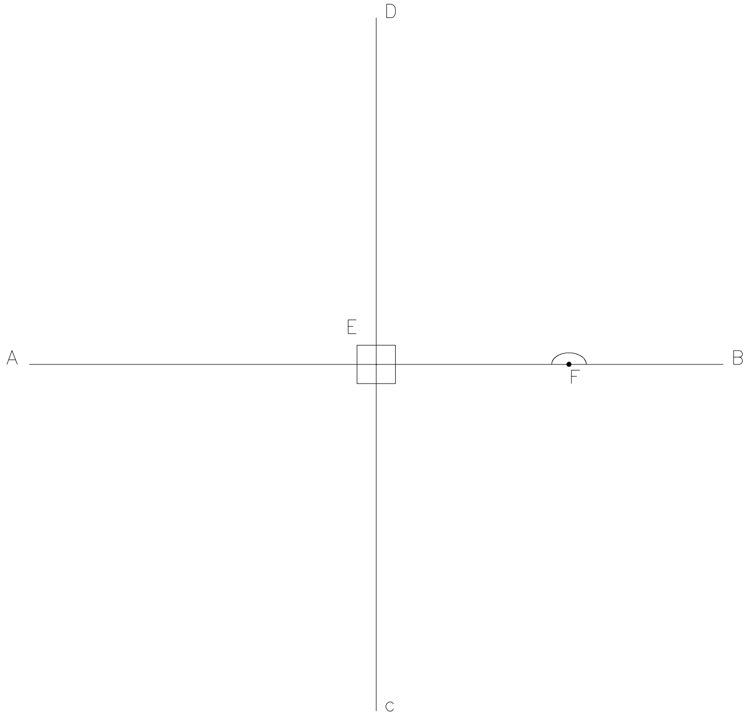

Figure 3. Perpendicular lines

Let's look at an example:

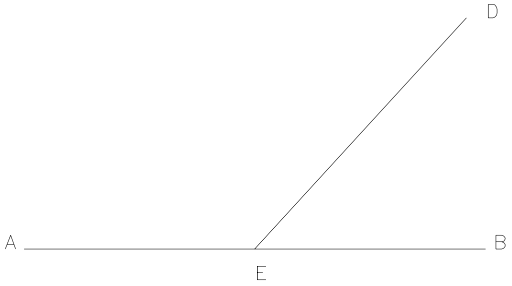

Figure 4

Question 1: What kind of angle is AÊD? DÊB?
Question 2: What is the measure of angle DÊB if
(a) AÊD = 90° ?
(b) AÊD = 60° ?
(c) AÊD = 180° ?
(d) AÊD = 120° ?

Question 3: You walk down the path towards class, and realize you've forgotten your book and have to go back to get it. Through what angle do you turn to start walking back?

Theorem 1: Opposite angles where two intersecting lines cross have the same measure.
Not all intersecting lines cross at 90° - see Figure 2 for an example. Now note that:
$A\hat{E}D + D\hat{E}B = 180°$, because AB is a straight line
$D\hat{E}B + B\hat{E}C = 180°$, because CD is a straight line
$\therefore A\hat{E}D = B\hat{E}C$
and $A\hat{E}C = D\hat{E}B$
Hint: in proving geometry theorems, like the one we just proved, it's helpful to draw the diagram (Figure 2 in this case) and mark equal angles and equal line lengths. Angular measures are often denoted by greek letters (α, β, γ, etc.) to avoid confusion with the regular letters which label the points. The two pairs of equal angles are denoted α and β in Figure 2.
Question 4: Look at Figure 2. If $A\hat{E}D = 120°$, what is the measure of: $D\hat{E}B$? $C\hat{E}B$?

(c) Parallel Lines

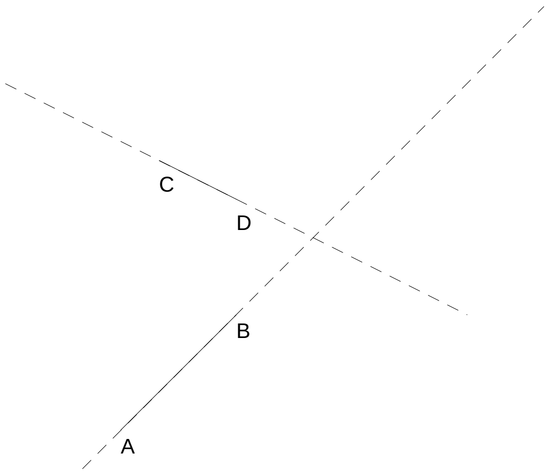

Figure 5

If the plane is infinite in size, you can extend or **produce** a line in both directions, as shown by the dotted lines in Figure 5. Eventually, as shown, the lines will intersect. There's one special case which is an exception: lines which, no matter how far you produce them in either direction, will never cross. These are lines going in the same direction, and are called **parallel lines** - see Figure 6. The definition is: parallel lines do not intersect, no matter how far you produce them in either direction.
In Figure 6, AB and CD are parallel lines. They are denoted by the little arrow sign > on the lines. Draw line EF which intersects them both, at X and Y. Since the lines are parallel, $E\hat{X}B = E\hat{Y}D$, as shown.

Question 5: if angle $E\hat{Y}B = 60°$, what is the measure of:
$E\hat{X}B$?
$A\hat{X}E$?
$A\hat{X}Y$?
$C\hat{Y}F$?
$F\hat{Y}D$?

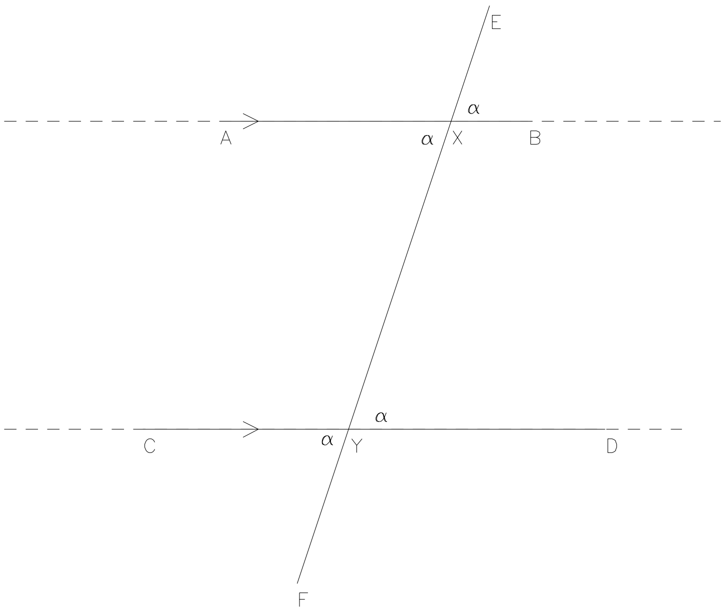

Figure 6. Parallel Lines

2. Triangles, Squares, Rectangles, Polygons

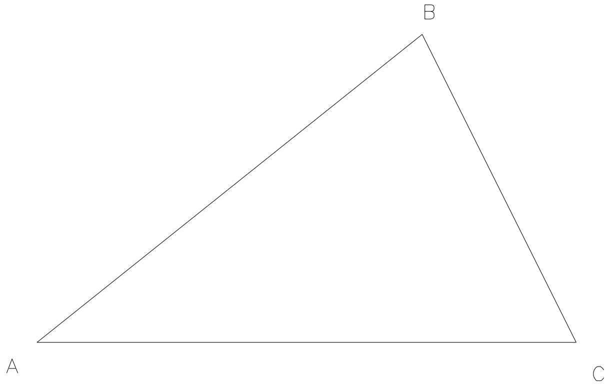

Figure 7. Triangle ABC

(a) Triangles

Triangles are figures on a plane where three points on a plane are joined by three straight lines. For example, Figure 7 shows triangle ABC. A triangle has three **points**, or **vertices**; three **sides**, and three **angles**. (The word *triangle* of course means *three angles*; the prefix “tri” means “three” in Latin, and you often hear it used for where we live: – the “tri-state” area, meaning New Jersey, New York and Connecticut. Many of the words for three are descended from *tri*: *three* itself of course; *trois* in French; *tres* in Spanish; *três* in Portuguese; *tre* in Italian; and so on.) Now we’re going to figure out the sum of the angles of a triangle. Let’s draw a line through *vertex* B, parallel to the base AC, as shown in Figure 8.

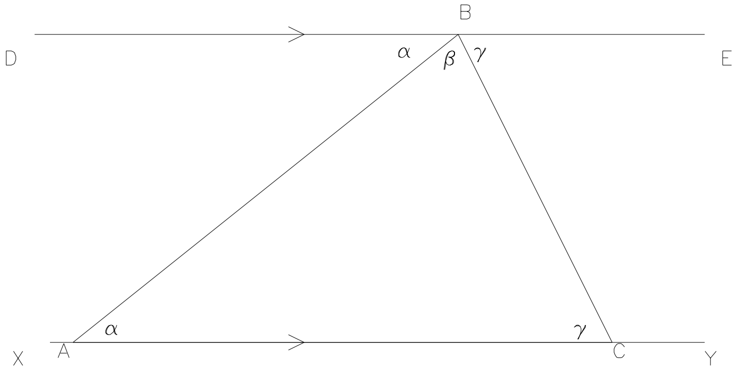

Figure 8. Proof that the angles of a triangle sum to 180°.

In Figure 8, DE and XY are parallel lines. We make the line XY by producing the base of the triangle (the side of the triangle at the bottom of the figure) in either direction. We draw parallel line through vertex B. Now, since DE and XY are parallel lines:
$\mathrm{C\hat{A}B} = \mathrm{D\hat{B}A}$ (1)
$\mathrm{A\hat{C}B} = \mathrm{C\hat{B}E}$ (2)

But because DBE is a straight line:
$\mathrm{D\hat{B}A} + \mathrm{A\hat{B}C} + \mathrm{C\hat{B}E} = 180^\circ$ (3)
Substitute (1) and (2) in (3) to get:
$\mathrm{C\hat{A}B} + \mathrm{A\hat{B}C} + \mathrm{B\hat{C}A} = 180^\circ$
Theorem 2: **The sum of the angles of a triangle is 180°.**
Or see the notations α, β, and γ for the angle measures in Figure 8.

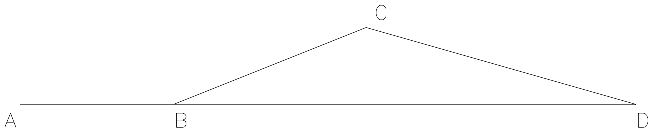

Figure 9. Proof that the exterior angle of a triangle equals the sum of the two opposite interior angles

Question 6: see Figure 9:
Prove that $\mathrm{A\hat{B}C} = \mathrm{B\hat{C}D} + \mathrm{C\hat{D}B}$.

Triangles have various shapes, see Figure 10:

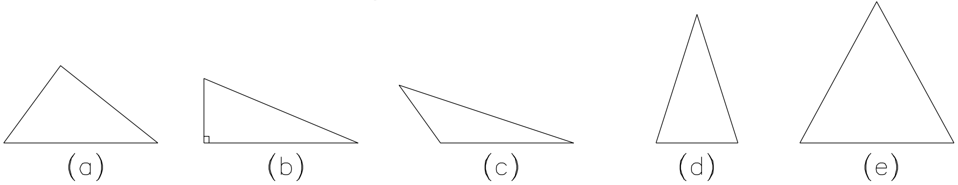

Figure 10. Triangles

(a) Acute: all angles < 90°
(b) Right: one angle = 90°
(c) Scalene: one angle > 90°
(d) Isosceles: two sides equal, two angles equal
(3) Equilateral: all sides equal, all angles equal

Question 7: what is the measure of each angle in an equilateral triangle?

(b) Polygons

Triangles have three vertices joined by three straight lines. **Polygons** are many-sided figures (*poly* is a greek word meaning many) with n vertices, n sides and n angles, where n is a number greater than 2.
n = 3: triangle
n = 4: quadrilateral
n = 5: pentagon
n = 6: hexagon
n = 7: septagon, etc.

Regular polygons have all sides and all angles equal. A regular triangle is an equilateral triangle. A regular quadrilateral is a square. There are, of course, irregular polygons. We'll come back to squares in a minute, but first, let's prove an interesting theorem.
Theorem 3: For an n-sided polygon, the sum of the measures of the interior ang;es is (n - 2) ×180°.
Proof: for example, see Figure 11. Take vertex A, and draw straight lines joining it to all the other vertices. Joining to B and C just goes along two of the existing sides. Joining to each other vertex makes a triangle for each line joined. Thus you can slice the polygon into (n - 2) triangles. For example, the 8-sided figure (octagon) in Figure 11 can be divided into 6 triangles (count them). So the sum of the angles of the polygon is (n - 2) times the sum of the angles in each triangle. Since the sum of the angles in each triangle is 180°, the sum of the angles in an n-sided polygon is (n - 2) ×180°.

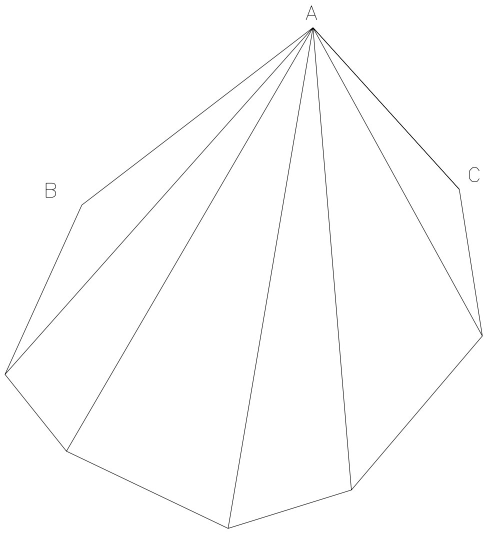

Figure 11. Polygon

One last piece of terminology. Figures of the same **size** and **shape** are called **congruent** - you can move them and rotate them as necessary and they will lie exactly on top of each other. Figures of the same shape but different sized are called *similar*.

(c) Squares, Rectangles and Area

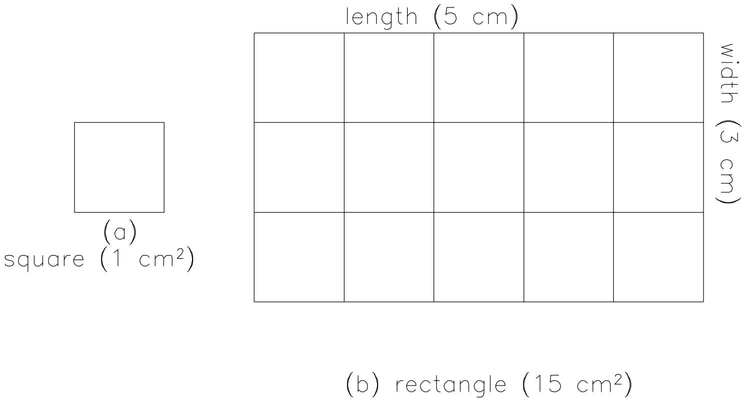

Figure 12. Square and Rectangle

Points have zero dimension, lines have one (length) and areas have two (length and width). Let's consider first **squares** and **rectangles**. The sum of the angles of any quadrilateral (4-sided) figure is (4 - 2) ×180° = 360°, as we showed above. **Squares** and **rectangles** are special quadrilaterals in which all four angles are 90° (360°/4 = 90°). Squares also have all four of their sides equal. Rectangles (literally "right-angled") have opposite sides equal (and parallel). A square and a rectangle are shown in Figure 12. The square (12a) has an area of 1 sq. cm. It takes 15 such squares to cover the rectangle, as shown, whose area is thus 15 sq cm. The length of the rectangle is 5 cm and the width is 3 cm. $5 \times 3 = 15$. So:

$\boxed{\text{Area} = \text{length} \times \text{width}}$

If the length is l and the width w, the area (A) is

$\boxed{A = l \times w}$

Now: what about the area of a triangle? Look at Figure 13:

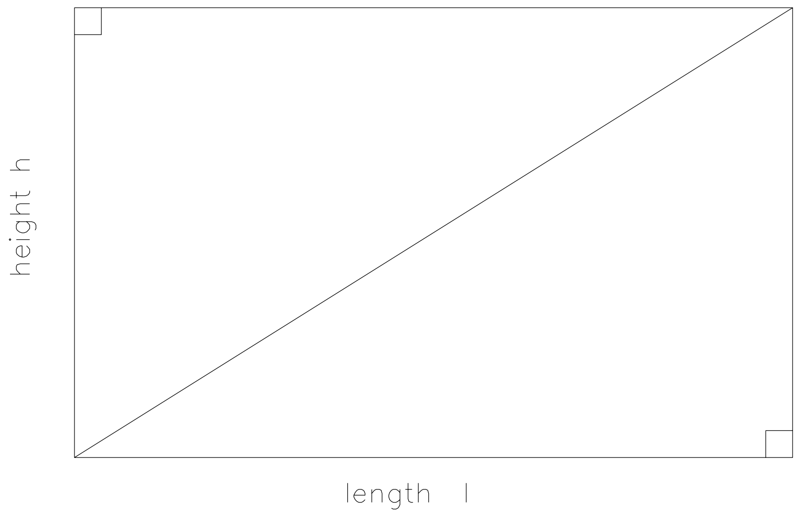

Figure 13. Area of a right-angled triangle

Figure 13 shows that the area of each right-angled triangle is half the area of the rectangle, or

$\boxed{A = \frac{1}{2}h \times l}$

Does this hold for all triangles? Sure: see Figure 14:

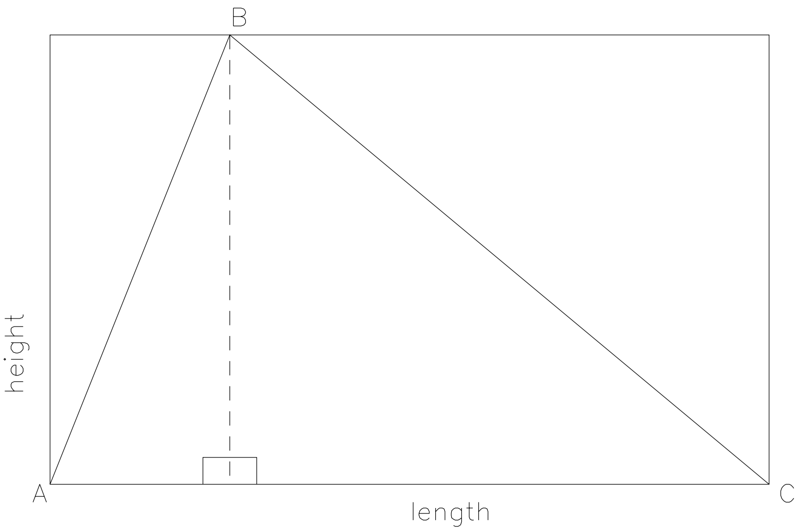

Figure 14. Area of a Triangle

Inspection of Figure 14 shows that

$\boxed{\text{Area of a triangle} = \text{half base} \times \text{height}}$

$\boxed{A = \frac{1}{2}l \times h}$

You'll prove this in the worksheet.

One last very important theorem: *Theorem 4: Pythagoras' Theorem*: If you have a right-angled triangle (see Figure 15), *the square of the hypotenuse is equal to the sum of the squares on the other two sides*. In Figure 15, c is the hypotenuse, the side opposite the right angle (and the longest side, since the right angle is the biggest angle). In mathematical symbols, Pythagoras' theorem states:

$$c^2 = a^2 + b^2$$

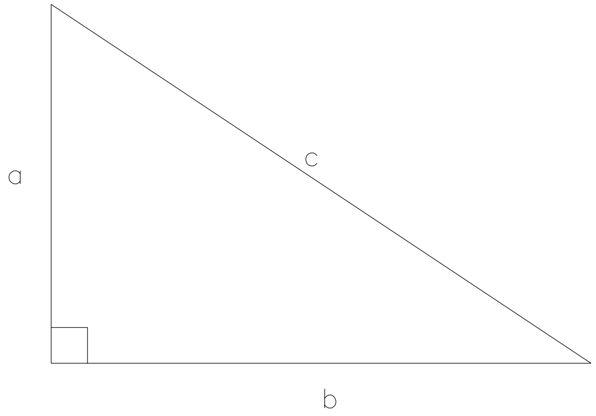

Figure 15. A right-angled triangle

This theorem is fundamental to calculating distances. There are many proofs, here's my favorite:

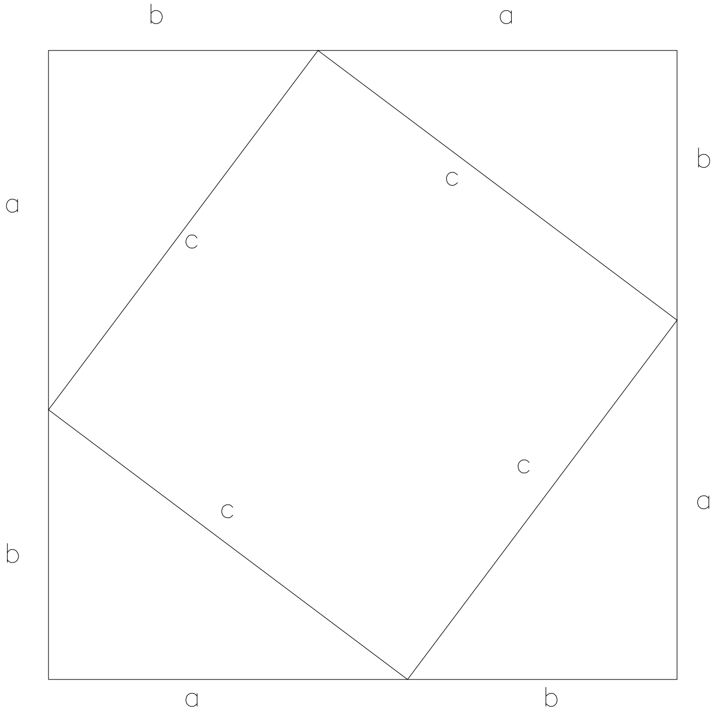

Figure 16. Proof of Pythagoras' Theorem

Draw a small square inside a big square, as shown in Figure 16. The length of each side of the big square is $a + b$, and the length of each side of the small square is c. There are six figures in Figure 16: the small square, the big square, and the four right-angled triangles (note, the squares are *similar* figures while the triangles are *congruent* figures).

Inspection of Figure 16 shows that the area of the big square equals the area of the small square plus the area of the four triangles. The area of the big square is $(a+b)^2$, the area of the small square is c^2, and the area of each triangle is 1/2 ab. So:
$(a+b)^2 = c^2 + 4 \times \frac{1}{2}ab$

$a^2 + 2ab + b^2 = c^2 + 2ab$

$\boxed{a^2 + b^2 = c^2}$
since you can subtract $2ab$ from both sides.

3. Circles

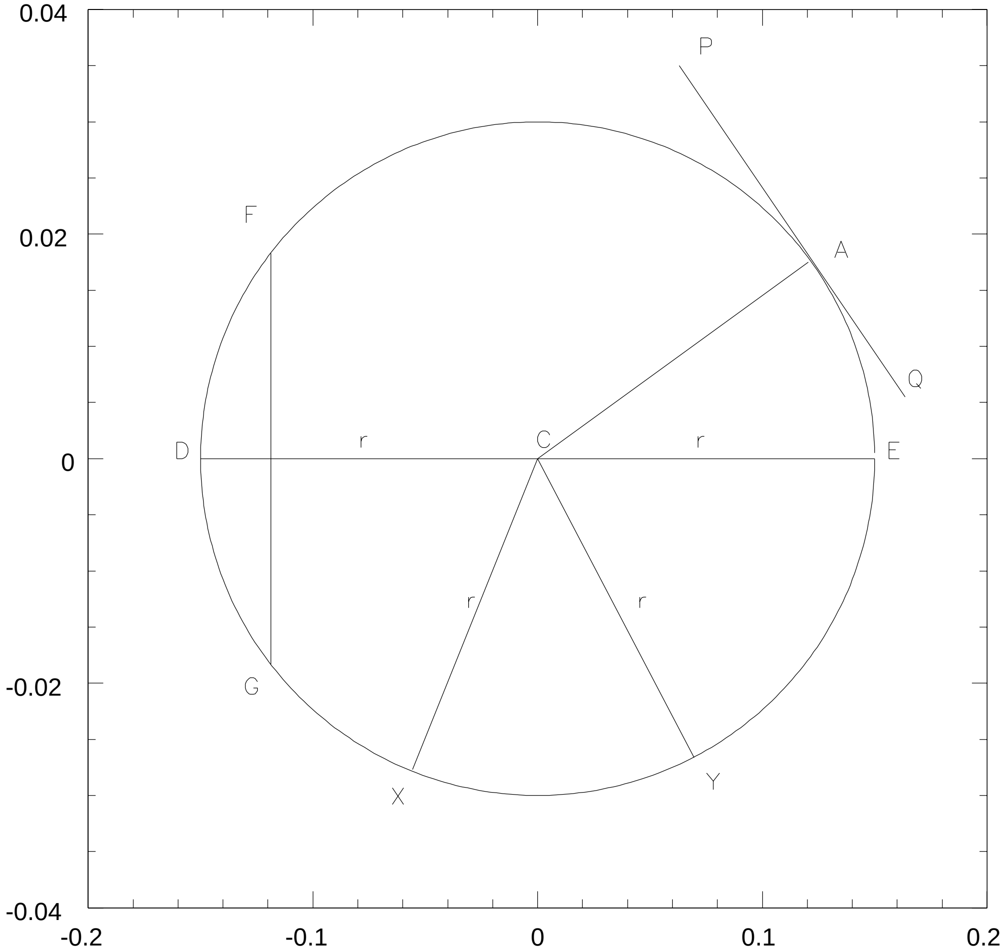

Figure 17. A circle

A *circle* is a closed figure in which all points are the same distance from the center. This distance is called the *radius* of the circle. In Figure 17, C is the center of the circle and all straight lines between C and the edge of the circle (for example CD, CE, CY, CA) all have length r, the *radius* of the circle. You can draw a circle by pinning one end of a string to your paper, tying the other end round a pencil, pulling the string taut and moving once round the circle (through an angle of 360°). Here are some definitions:

The length once round the circle (e.g. start at A and walk round to get back to A) is called the *circumference.*

If you draw a straight line from the center to the circumference, this distance is the *radius*, for example CA in Figure 17.

If you draw a line perpendicular to CA at A (i.e. line PQ in Figure 17) this is the *tangent* to the circle at point A.

A piece of the circle enclosed by part of the circumference and two lines to the center, e.g. CXY in Figure 17, is called a *sector.*

A straight line which runs from one side of the circle to the other and passes throught the center is called the *diameter* (line DCE in Figure 17).
A straight line which passes from one side of the circle to the other but does not go through the center is called a *chord* (e.g. FG in Figure 17). Obviously:

length of all chords < length of diameter

All circles have the same shape, i.e. are *similar.* They differ from one another only in the length of the radius and the location of the center.
The ratio of the circumference of a circle to its diameter is not a whole number: it's an irrational number called π. Numerically

$\pi = 3.14159265358979323846264338327950288841971$- - - -
We can approximate π as 3.0, 3.1, 3.14 ir 22/7. So the circumference of a circle is $\boxed{\pi\mathrm{D}, \text{ or } 2\pi\mathrm{r}}$.
Its area is $\boxed{\pi\mathrm{r}^2}$.
Formulae for a circle

$\boxed{\mathrm{D} = 2\mathrm{r}}$
$\boxed{\mathrm{C} = 2\pi\mathrm{r} = \pi\mathrm{D}}$
$\boxed{\mathrm{A} = \pi\mathrm{r}^2}$

Question: if Circle 1 has twice the diameter of Circle 2, what is the ratio of their areas? Of their circumferences?

4. Volume

We have seen that a point has zero dimensions, a line has one, and a plane has two. A *solid* has three: for example a rectangular solid, line a book, has dimensions *length*, *width*, and *height* (or thickness). The *volume* of this solid, i.e. the amount of space it takes up, is length x width x height. Volume is measured in cubic feet, cubic meters, cubic centimeters etc and is just the number of blocks of unit dimensions needed to fill the volume. A *cube* is a rectangular solid with all dimensions equal. Its volume is l^3, where l is the length of one side.

$\boxed{\text{Volume of a rectangular solid} = lwh \text{ (length x width x height)}}$

$\boxed{\text{Volume of a cube} = l^3 \text{ (read as l cubed)}}$

Note that in general, volume = area x heigth. The area of the top and bottom faces of a rectangular solid is A = l x w, so:

$\boxed{\text{Volume} = A \text{ x } h = Ah = l \text{ x } w \text{ x } h = lwh}$

Thus the volume of a *cylinder* (a can, for example), which has a circular base of radius r and has height h, is:

$\boxed{V = Ah = \pi r^2 h}$

The volume oh s *sphere* of radius r is

$\boxed{V = \frac{4}{3}\pi r^3}$

Note that any plane slice through a sphere is a circle. A circle whose center is also the center of the sphere is called a *great circle* - for example, the equator round the Earth, or lines of longitude. A circle whose center is not at the center of the sphere is called a *small circle.* Its radius is smaller than that of the sphere, and the circles get smaller the farther are their centers from the center of the sphere. Lines of latitude are small circles, except for the Equator, which is a great circle.
Here are a few interesting things about spheres:

(1) if you walk along the surface of a sphere, your path is curved in 3D
(2) The shortest distance between two points on the surface of a sphere is part of a great circle if you travel along the surface of a sphere. For example, airline flights between New York and Tokyo, Japan, fly almost over the North Pole.
(3) You have limited visibility of the surface of a sphere of you live on the surface: since light travels in a straight line, you can't see the whole surface from where you stand, even with the most powerful telescope. You can see only to the horizon. Even if you're very far away from a sphere, you can see only half of its surface.

5. Surface Area of a Solid

Just as a two-dimensional figure is enclosed by its circumference or *perimeter*, so a three-dimenional figure is enclosed by its *surface area.* For example, the surface area of one face of a cube is l^2. Since there are six faces of a cube, its surface area is $6l^2$.
Question: I have a rectangular solid of length l, width w, height h. What is its surface area?
The total surface area of a sphere is

$\boxed{A = 4\pi r^2}$