

## FUNDAMENTAL EQUATIONS

## §1. The strain tensor

THE mechanics of solid bodies, regarded as continuous media, forms the content of the *theory of elasticity*.<sup>†</sup>

Under the action of applied forces, solid bodies exhibit deformation to some extent, i.e. they change in shape and volume. The deformation of a body is described mathematically in the following way. The position of any point in the body is defined by its radius vector  $\mathbf{r}$  (with components  $x_1 = x$ ,  $x_2 = y$ ,  $x_3 = z$ ) in some co-ordinate system. When the body is deformed, every point in it is in general displaced. Let us consider some particular point; let its radius vector before the deformation be  $\mathbf{r}$ , and after the deformation have a different value  $\mathbf{r}'$  (with components  $x'_i$ ). The displacement of this point due to the deformation is then given by the vector  $\mathbf{r}' - \mathbf{r}$ , which we shall denote by  $\mathbf{u}$ :

$$u_i = x'_i - x_i. \quad (1.1)$$

The vector  $\mathbf{u}$  is called the *displacement vector*. The co-ordinates  $x'_i$  of the displaced point are, of course, functions of the co-ordinates  $x_i$  of the point before displacement. The displacement vector  $u_i$  is therefore also a function of the co-ordinates  $x_i$ . If the vector  $\mathbf{u}$  is given as a function of  $x_i$ , the deformation of the body is entirely determined.

When a body is deformed, the distances between its points change. Let us consider two points very close together. If the radius vector joining them before the deformation is  $dx_i$ , the radius vector joining the same two points in the deformed body is  $dx'_i = dx_i + du_i$ . The distance between the points is  $dl = \sqrt{(dx_1^2 + dx_2^2 + dx_3^2)}$  before the deformation, and  $dl' = \sqrt{(dx'_1{}^2 + dx'_2{}^2 + dx'_3{}^2)}$  after it. Using the general summation rule,<sup>‡</sup> we can write  $dl^2 = dx_i^2$ ,  $dl'^2 = dx'_i{}^2 = (dx_i + du_i)^2$ . Substituting  $du_i = (\partial u_i / \partial x_k) dx_k$ , we can write

$$dl'^2 = dl^2 + 2 \frac{\partial u_i}{\partial x_k} dx_i dx_k + \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_l} dx_k dx_l.$$

Since the summation is taken over both suffixes  $i$  and  $k$  in the second term on the right, we can put  $(\partial u_i / \partial x_k) dx_i dx_k = (\partial u_k / \partial x_i) dx_i dx_k$ . In the third

<sup>†</sup> The basic equations of elasticity theory were established in the 1820's by CAUCHY and by POISSON.

<sup>‡</sup> In accordance with the usual rule, we omit the sign of summation over vector and tensor suffixes. Summation over the values 1, 2, 3 is understood with respect to all suffixes which appear twice in a given term.

term, we interchange the suffixes  $i$  and  $l$ . Then  $dl'^2$  takes the final form

$$dl'^2 = dl^2 + 2u_{ik} dx_i dx_k, \quad (1.2)$$

where the tensor  $u_{ik}$  is defined as

$$u_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i} \frac{\partial u_l}{\partial x_k} \right). \quad (1.3)$$

These expressions give the change in an element of length when the body is deformed.

The tensor  $u_{ik}$  is called the *strain tensor*. We see from its definition that it is symmetrical, i.e.

$$u_{ik} = u_{ki}. \quad (1.4)$$

This result has been obtained by writing the term  $2(\partial u_i / \partial x_k) dx_i dx_k$  in  $dl'^2$  in the explicitly symmetrical form

$$\left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) dx_i dx_k.$$

Like any symmetrical tensor,  $u_{ik}$  can be *diagonalised* at any given point. This means that, at any given point, we can choose co-ordinate axes (the *principal axes* of the tensor) in such a way that only the diagonal components  $u_{11}$ ,  $u_{22}$ ,  $u_{33}$  of the tensor  $u_{ik}$  are different from zero. These components, the *principal values* of the strain tensor, will be denoted by  $u^{(1)}$ ,  $u^{(2)}$ ,  $u^{(3)}$ . It should be remembered, of course, that, if the tensor  $u_{ik}$  is diagonalised at any point in the body, it will not in general be diagonal at any other point.

If the strain tensor is diagonalised at a given point, the element of length (1.2) near it becomes

$$\begin{aligned} dl'^2 &= (\delta_{ik} + 2u_{ik}) dx_i dx_k \\ &= (1 + 2u^{(1)}) dx_1^2 + (1 + 2u^{(2)}) dx_2^2 + (1 + 2u^{(3)}) dx_3^2. \end{aligned}$$

We see that the expression is the sum of three independent terms. This means that the strain in any volume element may be regarded as composed of independent strains in three mutually perpendicular directions, namely those of the principal axes of the strain tensor. Each of these strains is a simple extension (or compression) in the corresponding direction: the length  $dx_1$  along the first principal axis becomes  $dx'_1 = \sqrt{1 + 2u^{(1)}} dx_1$ , and similarly for the other two axes. The quantity  $\sqrt{1 + 2u^{(i)}} - 1$  is consequently equal to the relative extension  $(dx'_i - dx_i) / dx_i$  along the  $i$ th principal axis.

In almost all cases occurring in practice, the strains are small. This means that the change in any distance in the body is small compared with the distance itself. In other words, the relative extensions are small compared with unity. In what follows we shall suppose that all strains are small.

If a body is subjected to a small deformation, all the components of the strain tensor are small, since they give, as we have seen, the relative changes in lengths in the body. The displacement vector  $u_i$ , however, may sometimes be large, even for small strains. For example, let us consider a long thin rod. Even for a large deflection, in which the ends of the rod move

a considerable distance, the extensions and compressions in the rod itself will be small.

Except in such special cases,† the displacement vector for a small deformation is itself small. For it is evident that a three-dimensional body (i.e. one whose dimension in no direction is small) cannot be deformed in such a way that parts of it move a considerable distance without the occurrence of considerable extensions and compressions in the body.

Thin rods will be discussed in Chapter II. In other cases  $u_i$  is small for small deformations, and we can therefore neglect the last term in the general expression (1.3), as being of the second order of smallness. Thus, for small deformations, the strain tensor is given by

$$u_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right). \quad (1.5)$$

The relative extensions of the elements of length along the principal axes of the strain tensor (at a given point) are, to within higher-order quantities,  $\sqrt{(1+2u^{(i)})} - 1 \approx u^{(i)}$ , i.e. they are the principal values of the tensor  $u_{ik}$ .

Let us consider an infinitesimal volume element  $dV$ , and find its volume  $dV'$  after the deformation. To do so, we take the principal axes of the strain tensor, at the point considered, as the co-ordinate axes. Then the elements of length  $dx_1, dx_2, dx_3$  along these axes become, after the deformation,  $dx'_1 = (1+u^{(1)}) dx_1$ , etc. The volume  $dV$  is the product  $dx_1 dx_2 dx_3$ , while  $dV'$  is  $dx'_1 dx'_2 dx'_3$ . Thus  $dV' = dV(1+u^{(1)})(1+u^{(2)})(1+u^{(3)})$ . Neglecting higher-order terms, we therefore have  $dV' = dV(1+u^{(1)}+u^{(2)}+u^{(3)})$ . The sum  $u^{(1)}+u^{(2)}+u^{(3)}$  of the principal values of a tensor is well known to be invariant, and is equal to the sum of the diagonal components  $u_{ii} = u_{11} + u_{22} + u_{33}$  in any co-ordinate system. Thus

$$dV' = dV(1+u_{ii}). \quad (1.6)$$

We see that the sum of the diagonal components of the strain tensor is the relative volume change  $(dV' - dV)/dV$ .

It is often convenient to use the components of the strain tensor in spherical or cylindrical co-ordinates. We give here, for reference, the corresponding formulae, which express the components in terms of the derivatives of the components of the displacement vector in the same co-ordinates. In spherical co-ordinates  $r, \theta, \phi$ , we have

$$\left. \begin{aligned} u_{rr} &= \frac{\partial u_r}{\partial r}, & u_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, & u_{\phi\phi} &= \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{r} \cot \theta + \frac{u_r}{r}, \\ 2u_{\theta\phi} &= \frac{1}{r} \left( \frac{\partial u_\phi}{\partial \theta} - u_\phi \cot \theta \right) + \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi}, & 2u_{r\theta} &= \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta}, \\ 2u_{\phi r} &= \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}. \end{aligned} \right\} \quad (1.7)$$

† Which include, besides deformations of thin rods, those of thin plates to form cylindrical surfaces.

In cylindrical co-ordinates  $r, \phi, z$ ,

$$\left. \begin{aligned} u_{rr} &= \frac{\partial u_r}{\partial r}, & u_{\phi\phi} &= \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r}, & u_{zz} &= \frac{\partial u_z}{\partial z}, \\ 2u_{\phi z} &= \frac{1}{r} \frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z}, & 2u_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \\ 2u_{r\phi} &= \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \phi}. \end{aligned} \right\} (1.8)$$

## §2. The stress tensor

In a body that is not deformed, the arrangement of the molecules corresponds to a state of thermal equilibrium. All parts of the body are in mechanical equilibrium. This means that, if some portion of the body is considered, the resultant of the forces on that portion is zero.

When a deformation occurs, the arrangement of the molecules is changed, and the body ceases to be in its original state of equilibrium. Forces therefore arise which tend to return the body to equilibrium. These internal forces which occur when a body is deformed are called *internal stresses*. If no deformation occurs, there are no internal stresses.

The internal stresses are due to molecular forces, i.e. the forces of interaction between the molecules. An important fact in the theory of elasticity is that the molecular forces have a very short range of action. Their effect extends only to the neighbourhood of the molecule exerting them, over a distance of the same order as that between the molecules, whereas in the theory of elasticity, which is a macroscopic theory, the only distances considered are those large compared with the distances between the molecules. The range of action of the molecular forces should therefore be taken as zero in the theory of elasticity. We can say that the forces which cause the internal stresses are, as regards the theory of elasticity, "near-action" forces, which act from any point only to neighbouring points. Hence it follows that the forces exerted on any part of the body by surrounding parts act only on the surface of that part.

The following reservation should be made here. The above assertion is not valid in cases where the deformation of the body results in macroscopic electric fields in it (pyroelectric and piezoelectric bodies). We shall not discuss such bodies in this book, however.

Let us consider the total force on some portion of the body. Firstly, this total force is equal to the sum of all the forces on all the volume elements in that portion of the body, i.e. it can be written as the volume integral  $\int \mathbf{F} dV$ , where  $\mathbf{F}$  is the force per unit volume and  $\mathbf{F}dV$  the force on the volume element  $dV$ . Secondly, the forces with which various parts of the portion considered act on one another cannot give anything but zero in the total resultant force, since they cancel by NEWTON'S third law. The required total force can therefore be regarded as the sum of the forces exerted on the given portion of the

body by the portions surrounding it. From above, however, these forces act on the surface of that portion, and so the resultant force can be represented as the sum of forces acting on all the surface elements, i.e. as an integral over the surface.

Thus, for any portion of the body, each of the three components  $\int F_i dV$  of the resultant of all the internal stresses can be transformed into an integral over the surface. As we know from vector analysis, the integral of a scalar over an arbitrary volume can be transformed into an integral over the surface if the scalar is the divergence of a vector. In the present case we have the integral of a vector, and not of a scalar. Hence the vector  $F_i$  must be the divergence of a tensor of rank two, i.e. be of the form

$$F_i = \partial\sigma_{ik}/\partial x_k. \quad (2.1)$$

Then the force on any volume can be written as an integral over the closed surface bounding that volume:†

$$\int F_i dV = \int \frac{\partial\sigma_{ik}}{\partial x_k} dV = \oint \sigma_{ik} df_k, \quad (2.2)$$

where  $df_i$  are the components of the surface element vector  $d\mathbf{f}$ , directed (as usual) along the outward normal.‡

The tensor  $\sigma_{ik}$  is called the *stress tensor*. As we see from (2.2),  $\sigma_{ik}df_k$  is the  $i$ th component of the force on the surface element  $d\mathbf{f}$ . By taking elements of area in the planes of  $xy$ ,  $yz$ ,  $zx$ , we find that the component  $\sigma_{ik}$  of the stress tensor is the  $i$ th component of the force on unit area perpendicular to the  $x_k$ -axis. For instance, the force on unit area perpendicular to the  $x$ -axis, normal to the area (i.e. along the  $x$ -axis) is  $\sigma_{xx}$ , and the tangential forces (along the  $y$  and  $z$  axes) are  $\sigma_{yx}$  and  $\sigma_{zx}$ .

The following remark should be made concerning the sign of the force  $\sigma_{ik}df_k$ . The surface integral in (2.2) is the force exerted on the volume enclosed by the surface by the surrounding parts of the body. The force which this volume exerts on the surface surrounding it is the same with the opposite sign. Hence, for example, the force exerted by the internal stresses on the surface of the body itself is  $-\oint \sigma_{ik}df_k$ , where the integral is taken over the surface of the body and  $d\mathbf{f}$  is along the outward normal.

Let us determine the moment of the forces on a portion of the body. The moment of the force  $\mathbf{F}$  can be written as an antisymmetrical tensor of rank two, whose components are  $F_i x_k - F_k x_i$ , where  $x_i$  are the co-ordinates of the

† The integral over a closed surface is transformed into one over the volume enclosed by the surface by replacing the surface element  $df_i$  by the operator  $dV\partial/\partial x_i$ .

‡ Strictly speaking, to determine the total force on a deformed portion of the body we should integrate, not over the old co-ordinates  $x_i$ , but over the co-ordinates  $x'_i$  of the points of the deformed body. The derivatives (2.1) should therefore be taken with respect to  $x'_i$ . However, in view of the smallness of the deformation, the derivatives with respect to  $x_i$  and  $x'_i$  differ only by higher-order quantities, and so the derivatives can be taken with respect to the co-ordinates  $x_i$ .

point where the force is applied.† Hence the moment of the forces on the volume element  $dV$  is  $(F_i x_k - F_k x_i) dV$ , and the moment of the forces on the whole volume is  $M_{ik} = \int (F_i x_k - F_k x_i) dV$ . Like the total force on any volume, this moment can be expressed as an integral over the surface bounding the volume. Substituting the expression (2.1) for  $F_i$ , we find

$$\begin{aligned} M_{ik} &= \int \left( \frac{\partial \sigma_{il}}{\partial x_l} x_k - \frac{\partial \sigma_{kl}}{\partial x_l} x_i \right) dV \\ &= \int \frac{\partial (\sigma_{il} x_k - \sigma_{kl} x_i)}{\partial x_l} dV - \int \left( \sigma_{il} \frac{\partial x_k}{\partial x_l} - \sigma_{kl} \frac{\partial x_i}{\partial x_l} \right) dV. \end{aligned}$$

In the second term we use the fact that the derivative of a co-ordinate with respect to itself is unity, and with respect to another co-ordinate is zero (since the three co-ordinates are independent variables). Thus  $\partial x_k / \partial x_l = \delta_{kl}$ , where  $\delta_{kl}$  is the unit tensor; the multiplication gives  $\sigma_{il} \delta_{kl} = \sigma_{ik}$ ,  $\sigma_{kl} \delta_{il} = \sigma_{ki}$ . In the first term, the integrand is the divergence of a tensor; the integral can be transformed into one over the surface. The result is  $M_{ik} = \oint (\sigma_{il} x_k - \sigma_{kl} x_i) df_l + \int (\sigma_{ki} - \sigma_{ik}) dV$ . If  $M_{ik}$  is to be an integral over the surface only, the second term must vanish identically, i.e. we must have

$$\sigma_{ik} = \sigma_{ki}. \quad (2.3)$$

Thus we reach the important result that the stress tensor is symmetrical. The moment of the forces on a portion of the body can then be written simply as

$$M_{ik} = \int (F_i x_k - F_k x_i) dV = \oint (\sigma_{il} x_k - \sigma_{kl} x_i) df_l. \quad (2.4)$$

It is easy to find the stress tensor for a body undergoing uniform compression from all sides (*hydrostatic* compression). In this case a pressure of the same magnitude acts on every unit area on the surface of the body, and its direction is along the inward normal. If this pressure is denoted by  $p$ , a force  $-p df_i$  acts on the surface element  $df_i$ . This force, in terms of the stress tensor, must be  $\sigma_{ik} df_k$ . Writing  $-p df_i = -p \delta_{ik} df_k$ , we see that the stress tensor in hydrostatic compression is

$$\sigma_{ik} = -p \delta_{ik}. \quad (2.5)$$

Its non-zero components are simply equal to the pressure.

In the general case of an arbitrary deformation, the non-diagonal components of the stress tensor are also non-zero. This means that not only a normal force but also tangential (shearing) stresses act on each surface element. These latter stresses tend to move the surface elements relative to each other.

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† The moment of the force  $\mathbf{F}$  is defined as the vector product  $\mathbf{F} \times \mathbf{r}$ , and we know from vector analysis that the components of a vector product form an antisymmetrical tensor of rank two as written here.

In equilibrium the internal stresses in every volume element must balance, i.e. we must have  $F_i = 0$ . Thus the equations of equilibrium for a deformed body are

$$\partial\sigma_{ik}/\partial x_k = 0. \quad (2.6)$$

If the body is in a gravitational field, the sum  $\mathbf{F} + \rho\mathbf{g}$  of the internal stresses and the force of gravity ( $\rho\mathbf{g}$  per unit volume) must vanish;  $\rho$  is the density† and  $\mathbf{g}$  the gravitational acceleration vector, directed vertically downwards. In this case the equations of equilibrium are

$$\partial\sigma_{ik}/\partial x_k + \rho g_i = 0. \quad (2.7)$$

The external forces applied to the surface of the body (which are the usual cause of deformation) appear in the boundary conditions on the equations of equilibrium. Let  $\mathbf{P}$  be the external force on unit area of the surface of the body, so that a force  $\mathbf{P} df$  acts on a surface element  $df$ . In equilibrium, this must be balanced by the force  $-\sigma_{ik} df_k$  of the internal stresses acting on that element. Thus we must have  $P_i df - \sigma_{ik} df_k = 0$ . Writing  $df_k = n_k df$ , where  $\mathbf{n}$  is a unit vector along the outward normal to the surface, we find

$$\sigma_{ik}n_k = P_i. \quad (2.8)$$

This is the condition which must be satisfied at every point on the surface of a body in equilibrium.

We shall derive also a formula giving the mean value of the stress tensor in a deformed body. To do so, we multiply equation (2.6) by  $x_k$  and integrate over the whole volume:

$$\int \frac{\partial\sigma_{il}}{\partial x_l} x_k dV = \int \frac{\partial(\sigma_{il}x_k)}{\partial x_l} dV - \int \sigma_{il} \frac{\partial x_k}{\partial x_l} dV = 0.$$

The first integral on the right is transformed into a surface integral; in the second integral we put  $\partial x_k/\partial x_l = \delta_{kl}$ . The result is  $\oint \sigma_{il}x_k df_l - \int \sigma_{ik} dV = 0$ . Substituting (2.8) in the first integral, we find  $\oint P_i x_k df = \int \sigma_{ik} dV = V\bar{\sigma}_{ik}$ , where  $V$  is the volume of the body and  $\bar{\sigma}_{ik}$  the mean value of the stress tensor. Since  $\sigma_{ik} = \sigma_{ki}$ , this formula can be written in the symmetrical form

$$\bar{\sigma}_{ik} = (1/2V) \oint (P_i x_k + P_k x_i) df. \quad (2.9)$$

Thus the mean value of the stress tensor can be found immediately from the external forces acting on the body, without solving the equations of equilibrium.

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† Strictly speaking, the density of a body changes when it is deformed. An allowance for this change, however, involves higher-order quantities in the case of small deformations, and is therefore unimportant.

### §3. The thermodynamics of deformation

Let us consider some deformed body, and suppose that the deformation is changed in such a way that the displacement vector  $u_i$  changes by a small amount  $\delta u_i$ ; and let us determine the work done by the internal stresses in this change. Multiplying the force  $F_i = \partial\sigma_{ik}/\partial x_k$  by the displacement  $\delta u_i$  and integrating over the volume of the body, we have  $\int \delta R \, dV = \int (\partial\sigma_{ik}/\partial x_k) \delta u_i \, dV$ , where  $\delta R$  denotes the work done by the internal stresses per unit volume. We integrate by parts, obtaining

$$\int \delta R \, dV = \oint \sigma_{ik} \delta u_i \, df_k - \int \sigma_{ik} \frac{\partial \delta u_i}{\partial x_k} \, dV.$$

By considering an infinite medium which is not deformed at infinity, we make the surface of integration in the first integral tend to infinity; then  $\sigma_{ik} = 0$  on the surface, and the integral is zero. The second integral can, by virtue of the symmetry of the tensor  $\sigma_{ik}$ , be written

$$\begin{aligned} \int \delta R \, dV &= -\frac{1}{2} \int \sigma_{ik} \left( \frac{\partial \delta u_i}{\partial x_k} + \frac{\partial \delta u_k}{\partial x_i} \right) \, dV \\ &= -\frac{1}{2} \int \sigma_{ik} \delta \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \, dV \\ &= -\int \sigma_{ik} \delta u_{ik} \, dV. \end{aligned}$$

Thus we find

$$\delta R = -\sigma_{ik} \delta u_{ik}. \quad (3.1)$$

This formula gives the work  $\delta R$  in terms of the change in the strain tensor.

If the deformation of the body is fairly small, it returns to its original undeformed state when the external forces causing the deformation cease to act. Such deformations are said to be *elastic*. For large deformations, the removal of the external forces does not result in the total disappearance of the deformation; a *residual deformation* remains, so that the state of the body is not that which existed before the forces were applied. Such deformations are said to be *plastic*. In what follows we shall consider only elastic deformations.

We shall also suppose that the process of deformation occurs so slowly that thermodynamic equilibrium is established in the body at every instant, in accordance with the external conditions. This assumption is almost always justified in practice. The process will then be thermodynamically reversible.

In what follows we shall take all such thermodynamic quantities as the entropy  $S$ , the internal energy  $\mathcal{E}$ , etc., relative to unit volume of the body,†

† The following remark should be made here. Strictly speaking, the unit volumes before and after the deformation should be distinguished, since they in general contain different amounts of matter. We shall always relate the thermodynamic quantities to unit volume of the undeformed body, i.e. to the amount of matter therein, which may occupy a different volume after the deformation. Accordingly, the total energy of the body, for example, is obtained by integrating  $\mathcal{E}$  over the volume of the undeformed body.



and not relative to unit mass as in fluid mechanics, and denote them by the corresponding capital letters.

An infinitesimal change  $d\mathcal{E}$  in the internal energy is equal to the difference between the heat acquired by the unit volume considered and the work  $dR$  done by the internal stresses. The amount of heat is, for a reversible process,  $TdS$ , where  $T$  is the temperature. Thus  $d\mathcal{E} = TdS - dR$ ; with  $dR$  given by (3.1), we obtain

$$d\mathcal{E} = TdS + \sigma_{ik} du_{ik}. \quad (3.2)$$

This is the fundamental thermodynamic relation for deformed bodies.

In hydrostatic compression, the stress tensor is  $\sigma_{ik} = -p\delta_{ik}$  (2.5). Then  $\sigma_{ik} du_{ik} = -p\delta_{ik} du_{ik} = -p du_{ii}$ . We have seen, however (cf. (1.6)), that the sum  $u_{ii}$  is the relative volume change due to the deformation. If we consider unit volume, therefore,  $u_{ii}$  is simply the change in that volume, and  $du_{ii}$  is the volume element  $dV$ . The thermodynamic relation then takes its usual form

$$d\mathcal{E} = TdS - pdV.$$

Introducing the free energy of the body,  $F = \mathcal{E} - TS$ , we find the form

$$dF = -SdT + \sigma_{ik} du_{ik} \quad (3.3)$$

of the relation (3.2). Finally, the thermodynamic potential  $\Phi$  is defined as

$$\Phi = \mathcal{E} - TS - \sigma_{ik}u_{ik} = F - \sigma_{ik}u_{ik}. \quad (3.4)$$

This is a generalisation of the usual expression  $\Phi = \mathcal{E} - TS + pV$ .† Substituting (3.4) in (3.3), we find

$$d\Phi = -SdT - u_{ik} d\sigma_{ik}. \quad (3.5)$$

The independent variables in (3.2) and (3.3) are respectively  $S$ ,  $u_{ik}$  and  $T$ ,  $u_{ik}$ . The components of the stress tensor can be obtained by differentiating  $\mathcal{E}$  or  $F$  with respect to the components of the strain tensor, for constant entropy  $S$  or temperature  $T$  respectively:

$$\sigma_{ik} = (\partial\mathcal{E}/\partial u_{ik})_S = (\partial F/\partial u_{ik})_T. \quad (3.6)$$

Similarly, by differentiating  $\Phi$  with respect to the components  $\sigma_{ik}$ , we can obtain the components  $u_{ik}$ :

$$u_{ik} = -(\partial\Phi/\partial\sigma_{ik})_T. \quad (3.7)$$

† For hydrostatic compression, the expression (3.4) becomes  $\Phi = F + pu_{ii} = F + p(V - V_0)$ , where  $V - V_0$  is the volume change resulting from the deformation. Hence we see that the definition of  $\Phi$  used here differs by a term  $-pV_0$  from the usual definition  $\Phi = F + pV$ .

#### §4. Hooke's law

In order to be able to apply the general formulae of thermodynamics to any particular case, we must know the free energy  $F$  of the body as a function of the strain tensor. This expression is easily obtained by using the fact that the deformation is small and expanding the free energy in powers of  $u_{ik}$ . We shall at present consider only isotropic bodies. The corresponding results for crystals will be obtained in §10.

In considering a deformed body at some temperature (constant throughout the body), we shall take the undeformed state to be the state of the body in the absence of external forces and at the same temperature; this last condition is necessary on account of the thermal expansion (see §6). Then, for  $u_{ik} = 0$ , the internal stresses are zero also, i.e.  $\sigma_{ik} = 0$ . Since  $\sigma_{ik} = \partial F / \partial u_{ik}$ , it follows that there is no linear term in the expansion of  $F$  in powers of  $u_{ik}$ .

Next, since the free energy is a scalar, each term in the expansion of  $F$  must be a scalar also. Two independent scalars of the second degree can be formed from the components of the symmetrical tensor  $u_{ik}$ : they can be taken as the squared sum of the diagonal components ( $u_{ii}^2$ ) and the sum of the squares of all the components ( $u_{ik}^2$ ). Expanding  $F$  in powers of  $u_{ik}$ , we therefore have as far as terms of the second order

$$F = F_0 + \frac{1}{2}\lambda u_{ii}^2 + \mu u_{ik}^2. \quad (4.1)$$

This is the general expression for the free energy of a deformed isotropic body. The quantities  $\lambda$  and  $\mu$  are called *Lamé coefficients*.

We have seen in §1 that the change in volume in the deformation is given by the sum  $u_{ii}$ . If this sum is zero, then the volume of the body is unchanged by the deformation, only its shape being altered. Such a deformation is called a *pure shear*.

The opposite case is that of a deformation which causes a change in the volume of the body but no change in its shape. Each volume element of the body retains its shape also. We have seen in §1 that the tensor of such a deformation is  $u_{ik} = \text{constant} \times \delta_{ik}$ . Such a deformation is called a *hydrostatic compression*.

Any deformation can be represented as the sum of a pure shear and a hydrostatic compression. To do so, we need only use the identity

$$u_{ik} = (u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}) + \frac{1}{3}\delta_{ik}u_{ll}. \quad (4.2)$$

The first term on the right is evidently a pure shear, since the sum of its diagonal terms is zero ( $\delta_{ii} = 3$ ). The second term is a hydrostatic compression.

As a general expression for the free energy of a deformed isotropic body, it is convenient to replace (4.1) by another formula, using this decomposition of an arbitrary deformation into a pure shear and a hydrostatic compression. We take as the two independent scalars of the second degree the sums of the

squared components of the two terms in (4.2). Then  $F$  becomes†

$$F = \mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll})^2 + \frac{1}{2}Ku_{ll}^2. \quad (4.3)$$

The quantities  $K$  and  $\mu$  are called respectively the *bulk modulus* or *modulus of hydrostatic compression* (or simply the *modulus of compression*) and the *shear modulus* or *modulus of rigidity*.  $K$  is related to the Lamé coefficients by

$$K = \lambda + \frac{2}{3}\mu. \quad (4.4)$$

In a state of thermodynamic equilibrium, the free energy is a minimum. If no external forces act on the body, then  $F$  as a function of  $u_{ik}$  must have a minimum for  $u_{ik} = 0$ . This means that the quadratic form (4.3) must be positive. If the tensor  $u_{ik}$  is such that  $u_{ll} = 0$ , only the first term remains in (4.3); if, on the other hand, the tensor is of the form  $u_{ik} = \text{constant} \times \delta_{ik}$ , then only the second term remains. Hence it follows that a necessary (and evidently sufficient) condition for the form (4.3) to be positive is that each of the coefficients  $K$  and  $\mu$  is positive. Thus we conclude that the moduli of compression and rigidity are always positive:

$$K > 0, \quad \mu > 0. \quad (4.5)$$

We now use the general thermodynamic relation (3.6) to determine the stress tensor. To calculate the derivatives  $\partial F/\partial u_{ik}$ , we write the total differential  $dF$  (for constant temperature):

$$dF = Ku_{ll} du_{ll} + 2\mu(u_{ik} - \frac{1}{3}u_{ll}\delta_{ik}) d(u_{ik} - \frac{1}{3}u_{ll}\delta_{ik}).$$

In the second term, multiplication of the first parenthesis by  $\delta_{ik}$  gives zero, leaving  $dF = Ku_{ll} du_{ll} + 2\mu(u_{ik} - \frac{1}{3}u_{ll}\delta_{ik}) du_{ik}$ , or writing  $du_{ll} = \delta_{ik} du_{ik}$ ,

$$dF = [Ku_{ll}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}u_{ll}\delta_{ik})] du_{ik}.$$

Hence the stress tensor is

$$\sigma_{ik} = Ku_{ll}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}). \quad (4.6)$$

This expression determines the stress tensor in terms of the strain tensor for an isotropic body. It shows, in particular, that, if the deformation is a pure shear or a pure hydrostatic compression, the relation between  $\sigma_{ik}$  and  $u_{ik}$  is determined only by the modulus of rigidity or of hydrostatic compression respectively.

It is not difficult to obtain the converse formula which expresses  $u_{ik}$  in terms of  $\sigma_{ik}$ . To do so, we find the sum  $\sigma_{ii}$  of the diagonal terms. Since this sum is zero for the second term of (4.6), we have  $\sigma_{ii} = 3Ku_{ii}$ , or

$$u_{ii} = \sigma_{ii}/3K. \quad (4.7)$$

† The constant term  $F_0$  is the free energy of the undeformed body, and is of no further interest. We shall therefore omit it, for brevity, taking  $F$  to be only the free energy of the deformation (the elastic free energy, as it is called).

Substituting this expression in (4.6) and so determining  $u_{ik}$ , we find

$$u_{ik} = \delta_{ik}\sigma_{ll}/9K + (\sigma_{ik} - \frac{1}{3}\delta_{ik}\sigma_{ll})/2\mu, \quad (4.8)$$

which gives the strain tensor in terms of the stress tensor.

Equation (4.7) shows that the relative change in volume ( $u_{ii}$ ) in any deformation of an isotropic body depends only on the sum  $\sigma_{ii}$  of the diagonal components of the stress tensor, and the relation between  $u_{ii}$  and  $\sigma_{ii}$  is determined only by the modulus of hydrostatic compression. In hydrostatic compression of a body, the stress tensor is  $\sigma_{ik} = -p\delta_{ik}$ . Hence we have in this case, from (4.7),

$$u_{ii} = -p/K. \quad (4.9)$$

Since the deformations are small,  $u_{ii}$  and  $p$  are small quantities, and we can write the ratio  $u_{ii}/p$  of the relative volume change to the pressure in the differential form  $(1/V)(\partial V/\partial p)_T$ . Thus

$$\frac{1}{K} = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T.$$

The quantity  $1/K$  is called the *coefficient of hydrostatic compression* (or simply the *coefficient of compression*).

We see from (4.8) that the strain tensor  $u_{ik}$  is a linear function of the stress tensor  $\sigma_{ik}$ . That is, the deformation is proportional to the applied forces. This law, valid for small deformations, is called *Hooke's law*.†

We may give also a useful form of the expression for the free energy of a deformed body, which is obtained immediately from the fact that  $F$  is quadratic in the strain tensor. According to EULER'S theorem,  $u_{ik}\partial F/\partial u_{ik} = 2F$ , whence, since  $\partial F/\partial u_{ik} = \sigma_{ik}$ , we have

$$F = \frac{1}{2}\sigma_{ik}u_{ik}. \quad (4.10)$$

If we substitute in this formula the  $u_{ik}$  as linear combinations of the components  $\sigma_{ik}$ , the elastic energy will be represented as a quadratic function of the  $\sigma_{ik}$ . Again applying EULER'S theorem, we obtain  $\sigma_{ik}\partial F/\partial \sigma_{ik} = 2F$ , and a comparison with (4.10) shows that

$$u_{ik} = \partial F/\partial \sigma_{ik}. \quad (4.11)$$

It should be emphasised, however, that, whereas the formula  $\sigma_{ik} = \partial F/\partial u_{ik}$  is a general relation of thermodynamics, the inverse formula (4.11) is applicable only if HOOKE'S law is valid.

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† HOOKE'S law is actually applicable to almost all elastic deformation. The reason is that deformations usually cease to be elastic when they are still so small that HOOKE'S law is a good approximation. Substances such as rubber form an exception.

### §5. Homogeneous deformations

Let us consider some simple cases of what are called *homogeneous deformations*, i.e. those in which the strain tensor is constant throughout the volume of the body. For example, the hydrostatic compression already considered is a homogeneous deformation.

We first consider a *simple extension* (or compression) of a rod. Let the rod be along the  $x$ -axis, and let forces be applied to its ends which stretch it in both directions. These forces act uniformly over the end surfaces of the rod; let the force on unit area be  $p$ .

Since the deformation is homogeneous, i.e.  $u_{ik}$  is constant through the body, the stress tensor  $\sigma_{ik}$  is also constant, and so it can be determined at once from the boundary conditions (2.8). There is no external force on the sides of the rod, and therefore  $\sigma_{ik}n_k = 0$ . Since the unit vector  $\mathbf{n}$  on the side of the rod is perpendicular to the  $x$ -axis, i.e.  $n_x = 0$ , it follows that all the components  $\sigma_{ik}$  except  $\sigma_{zz}$  are zero. On the end surface we have  $\sigma_{zi}n_i = p$ , or  $\sigma_{zz} = p$ .

From the general expression (4.8) which relates the components of the strain and stress tensors, we see that all the components  $u_{ik}$  with  $i \neq k$  are zero. For the remaining components we find

$$u_{xx} = u_{yy} = -\frac{1}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right)p, \quad u_{zz} = \frac{1}{3}\left(\frac{1}{3K} + \frac{1}{\mu}\right)p. \quad (5.1)$$

The component  $u_{zz}$  gives the relative lengthening of the rod. The coefficient of  $p$  is called the *coefficient of extension*, and its reciprocal is the *modulus of extension* or *Young's modulus*,  $E$ :

$$u_{zz} = p/E, \quad (5.2)$$

where

$$E = 9K\mu/(3K + \mu). \quad (5.3)$$

The components  $u_{xx}$  and  $u_{yy}$  give the relative compression of the rod in the transverse direction. The ratio of the transverse compression to the longitudinal extension is called *Poisson's ratio*,  $\sigma$ :†

$$u_{xx} = -\sigma u_{zz}, \quad (5.4)$$

where

$$\sigma = \frac{1}{2}(3K - 2\mu)/(3K + \mu). \quad (5.5)$$

† The use of  $\sigma$  to denote POISSON'S ratio and  $\sigma_{ik}$  to denote the components of the stress tensor cannot lead to ambiguity, since the latter always have suffixes.

Since  $K$  and  $\mu$  are always positive, Poisson's ratio can vary between  $-1$  (for  $K = 0$ ) and  $\frac{1}{2}$  (for  $\mu = 0$ ). Thus †

$$-1 \leq \sigma \leq \frac{1}{2}. \quad (5.6)$$

Finally, the relative increase in the volume of the rod is

$$u_{ii} = p/3K. \quad (5.7)$$

The free energy of a stretched rod can be obtained immediately from formula (4.10). Since only the component  $\sigma_{zz}$  is not zero, we have  $F = \frac{1}{2}\sigma_{zz}u_{zz}$ , whence

$$F = p^2/2E. \quad (5.8)$$

In what follows we shall, as is customary, use  $E$  and  $\sigma$  instead of  $K$  and  $\mu$ . Inverting formulae (5.3) and (5.5), we have‡

$$\mu = E/2(1+\sigma), \quad K = E/3(1-2\sigma). \quad (5.9)$$

We shall write out here the general formulae of §4, with the coefficients expressed in terms of  $E$  and  $\sigma$ . The free energy is

$$F = \frac{E}{2(1+\sigma)} \left( u_{ik}^2 + \frac{\sigma}{1-2\sigma} u_{ii}^2 \right). \quad (5.10)$$

The stress tensor is given in terms of the strain tensor by

$$\sigma_{ik} = \frac{E}{1+\sigma} \left( u_{ik} + \frac{\sigma}{1-2\sigma} u_{ii} \delta_{ik} \right). \quad (5.11)$$

Conversely,

$$u_{ik} = [(1+\sigma)\sigma_{ik} - \sigma\sigma_{ii}\delta_{ik}]/E. \quad (5.12)$$

Since formulae (5.11) and (5.12) are in frequent use, we shall give them also in component form:

$$\left. \begin{aligned} \sigma_{xx} &= \frac{E}{(1+\sigma)(1-2\sigma)} [(1-\sigma)u_{xx} + \sigma(u_{yy} + u_{zz})], \\ \sigma_{yy} &= \frac{E}{(1+\sigma)(1-2\sigma)} [(1-\sigma)u_{yy} + \sigma(u_{xx} + u_{zz})], \\ \sigma_{zz} &= \frac{E}{(1+\sigma)(1-2\sigma)} [(1-\sigma)u_{zz} + \sigma(u_{xx} + u_{yy})], \\ \sigma_{xy} &= \frac{E}{1+\sigma} u_{xy}, \quad \sigma_{xz} = \frac{E}{1+\sigma} u_{xz}, \quad \sigma_{yz} = \frac{E}{1+\sigma} u_{yz}, \end{aligned} \right\} \quad (5.13)$$

† In practice, POISSON'S ratio varies only between 0 and  $\frac{1}{2}$ . There are no substances known for which  $\sigma < 0$ , i.e. which would expand transversely when stretched longitudinally. It may be mentioned that the inequality  $\sigma > 0$  corresponds to  $\lambda > 0$ , where  $\lambda$  is the Lamé coefficient appearing in (4.1); in other words, both terms in (4.1), as well as in (4.3), are always positive in practice, although this is not thermodynamically necessary. Values of  $\sigma$  close to  $\frac{1}{2}$  (e.g. for rubber) correspond to a modulus of rigidity which is small compared with the modulus of compression.

‡ The second Lamé coefficient is  $\lambda = E\sigma/(1-2\sigma)(1+\sigma)$ .

and conversely

$$\begin{aligned}
 u_{xx} &= \frac{1}{E}[\sigma_{xx} - \sigma(\sigma_{yy} + \sigma_{zz})], \\
 u_{yy} &= \frac{1}{E}[\sigma_{yy} - \sigma(\sigma_{xx} + \sigma_{zz})], \\
 u_{zz} &= \frac{1}{E}[\sigma_{zz} - \sigma(\sigma_{xx} + \sigma_{yy})], \\
 u_{xy} &= \frac{1 + \sigma}{E}\sigma_{xy}, \quad u_{xz} = \frac{1 + \sigma}{E}\sigma_{xz}, \quad u_{yz} = \frac{1 + \sigma}{E}\sigma_{yz}.
 \end{aligned}
 \tag{5.14}$$

Let us now consider the compression of a rod whose sides are fixed in such a way that they cannot move. The external forces which cause the compression of the rod are applied to its ends and act along its length, which we again take to be along the  $z$ -axis. Such a deformation is called a *unilateral compression*. Since the rod is deformed only in the  $z$ -direction, only the component  $u_{zz}$  of  $u_{ik}$  is not zero. Then we have from (5.11)

$$\sigma_{xx} = \sigma_{yy} = \frac{E}{(1 + \sigma)(1 - 2\sigma)}u_{zz}, \quad \sigma_{zz} = \frac{E(1 - \sigma)}{(1 + \sigma)(1 - 2\sigma)}u_{zz}.$$

Again denoting the compressing force by  $p$  ( $\sigma_{zz} = p$ , which is negative for a compression), we have

$$u_{zz} = p(1 + \sigma)(1 - 2\sigma)/E(1 - \sigma). \tag{5.15}$$

The coefficient of  $p$  is called the *coefficient of unilateral compression*. For the transverse stresses we have

$$\sigma_{xx} = \sigma_{yy} = p\sigma/(1 - \sigma). \tag{5.16}$$

Finally, the free energy of the rod is

$$F = p^2(1 + \sigma)(1 - 2\sigma)/2E(1 - \sigma). \tag{5.17}$$

## §6. Deformations with change of temperature

Let us now consider deformations which are accompanied by a change in the temperature of the body; this can occur either as a result of the deformation process itself, or from external causes.

We shall regard as the undeformed state the state of the body in the absence of external forces at some given temperature  $T_0$ . If the body is at a temperature  $T$  different from  $T_0$ , then, even if there are no external forces, it will in general be deformed, on account of thermal expansion. In the expansion of the free energy  $F(T)$ , there will therefore be terms linear, as well as quadratic, in the strain tensor. From the components of the tensor  $u_{ik}$ , of rank two,

we can form only one linear scalar quantity, the sum  $u_{ii}$  of its diagonal components. We shall also assume that the temperature change  $T - T_0$  which accompanies the deformation is small. We can then suppose that the coefficient of  $u_{ii}$  in the expansion of  $F$  (which must vanish for  $T = T_0$ ) is simply proportional to the difference  $T - T_0$ . Thus we find the free energy to be (instead of (4.3))

$$F(T) = F_0(T) - K\alpha(T - T_0)u_{ii} + \mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{ii})^2 + \frac{1}{2}Ku_{ii}^2, \quad (6.1)$$

where the coefficient of  $T - T_0$  has been written as  $-K\alpha$ . The quantities  $\mu$ ,  $K$  and  $\alpha$  can here be supposed constant; an allowance for their temperature dependence would lead to terms of higher order.

Differentiating  $F$  with respect to  $u_{ik}$ , we obtain the stress tensor:

$$\sigma_{ik} = -K\alpha(T - T_0)\delta_{ik} + Ku_{ii}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{ii}). \quad (6.2)$$

The first term gives the additional stresses caused by the change in temperature. In free thermal expansion of the body (external forces being absent), there can be no internal stresses. Equating  $\sigma_{ik}$  to zero, we find that  $u_{ik}$  is of the form constant  $\times \delta_{ik}$ , and

$$u_{ii} = \alpha(T - T_0). \quad (6.3)$$

But  $u_{ii}$  is the relative change in volume caused by the deformation. Thus  $\alpha$  is just the *thermal expansion coefficient* of the body.

Among the various (thermodynamic) types of deformation, isothermal and adiabatic deformations are of importance. In isothermal deformations, the temperature of the body does not change. Accordingly, we must put  $T = T_0$  in (6.1), returning to the usual formulae; the coefficients  $K$  and  $\mu$  may therefore be called *isothermal moduli*.

A deformation is adiabatic if there is no exchange of heat between the various parts of the body (or, of course, between the body and the surrounding medium). The entropy  $S$  remains constant. It is the derivative  $-\partial F/\partial T$  of the free energy with respect to temperature. Differentiating the expression (6.1), we have as far as terms of the first order in  $u_{ik}$

$$S(T) = S_0(T) + K\alpha u_{ii}. \quad (6.4)$$

Putting  $S$  constant, we can determine the change of temperature  $T - T_0$  due to the deformation, which is therefore proportional to  $u_{ii}$ . Substituting this expression for  $T - T_0$  in (6.2), we obtain for  $\sigma_{ik}$  an expression of the usual kind,

$$\sigma_{ik} = K_{ad}u_{ii}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}\delta_{ik}u_{ii}), \quad (6.5)$$

with the same modulus of rigidity  $\mu$  but a different modulus of compression  $K_{ad}$ . The relation between the adiabatic modulus  $K_{ad}$  and the ordinary isothermal modulus  $K$  can also be found directly from the thermodynamic formula

$$\left(\frac{\partial V}{\partial p}\right)_S = \left(\frac{\partial V}{\partial p}\right)_T + \frac{T(\partial V/\partial T)_p^2}{C_p},$$



where  $C_p$  is the specific heat per unit volume at constant pressure. If  $V$  is taken to be the volume occupied by matter which before the deformation occupied unit volume, the derivatives  $\partial V/\partial T$  and  $\partial V/\partial p$  give the relative volume changes in heating and compression respectively. That is,

$$(\partial V/\partial T)_p = \alpha, \quad (\partial V/\partial p)_S = -1/K_{\text{ad}}, \quad (\partial V/\partial p)_T = -1/K.$$

Thus we find the relation between the adiabatic and isothermal moduli to be

$$1/K_{\text{ad}} = 1/K - T\alpha^2/C_p, \quad \mu_{\text{ad}} = \mu. \quad (6.6)$$

For the adiabatic YOUNG'S modulus and POISSON'S ratio we easily obtain

$$E_{\text{ad}} = \frac{E}{1 - ET\alpha^2/9C_p}, \quad \sigma_{\text{ad}} = \frac{\sigma + ET\alpha^2/9C_p}{1 - ET\alpha^2/9C_p}. \quad (6.7)$$

In practice,  $ET\alpha^2/C_p$  is usually small, and it is therefore sufficiently accurate to put

$$E_{\text{ad}} = E + E^2T\alpha^2/9C_p, \quad \sigma_{\text{ad}} = \sigma + (1 + \sigma)ET\alpha^2/9C_p. \quad (6.8)$$

In isothermal deformation, the stress tensor is given in terms of the derivatives of the free energy:

$$\sigma_{ik} = (\partial F/\partial u_{ik})_T.$$

For constant entropy, on the other hand, we have (see (3.6))

$$\sigma_{ik} = (\partial \mathcal{E}/\partial u_{ik})_S,$$

where  $\mathcal{E}$  is the internal energy. Accordingly, the expression analogous to (4.3) determines, for adiabatic deformations, not the free energy but the internal energy per unit volume:

$$\mathcal{E} = \frac{1}{2}K_{\text{ad}}u_{ll}^2 + \mu(u_{ik} - \frac{1}{3}u_{ll}\delta_{ik})^2. \quad (6.9)$$

## §7. The equations of equilibrium for isotropic bodies

Let us now derive the equations of equilibrium for isotropic solid bodies. To do so, we substitute in the general equations (2.7)

$$\partial\sigma_{ik}/\partial x_k + \rho g_i = 0$$

the expression (5.11) for the stress tensor. We have

$$\frac{\partial\sigma_{ik}}{\partial x_k} = \frac{E\sigma}{(1 + \sigma)(1 - 2\sigma)} \frac{\partial u_{ll}}{\partial x_i} + \frac{E}{1 + \sigma} \frac{\partial u_{ik}}{\partial x_k}.$$

Substituting

$$u_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right),$$

we obtain the equations of equilibrium in the form

$$\frac{E}{2(1+\sigma)} \frac{\partial^2 u_i}{\partial x_k^2} + \frac{E}{2(1+\sigma)(1-2\sigma)} \frac{\partial^2 u_i}{\partial x_i \partial x_l} + \rho g_i = 0. \quad (7.1)$$

These equations can be conveniently rewritten in vector notation. The quantities  $\partial^2 u_i / \partial x_k^2$  are components of the vector  $\Delta \mathbf{u}$ , and  $\partial u_i / \partial x_i \equiv \text{div } \mathbf{u}$ . Thus the equations of equilibrium become

$$\Delta \mathbf{u} + \frac{1}{1-2\sigma} \mathbf{grad} \text{ div } \mathbf{u} = -\rho \mathbf{g} \frac{2(1+\sigma)}{E}. \quad (7.2)$$

It is sometimes useful to transform this equation by using the vector identity  $\mathbf{grad} \text{ div } \mathbf{u} = \Delta \mathbf{u} + \mathbf{curl} \text{ curl } \mathbf{u}$ . Then (7.2) becomes

$$\begin{aligned} \mathbf{grad} \text{ div } \mathbf{u} - \frac{1-2\sigma}{2(1-\sigma)} \mathbf{curl} \text{ curl } \mathbf{u} \\ = -\rho \mathbf{g} \frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}. \end{aligned} \quad (7.3)$$

We have written the equations of equilibrium for a uniform gravitational field, since this is the body force most usually encountered in the theory of elasticity. If there are other body forces, the vector  $\rho \mathbf{g}$  on the right-hand side of the equation must be replaced accordingly.

A very important case is that where the deformation of the body is caused, not by body forces, but by forces applied to its surface. The equation of equilibrium then becomes

$$(1-2\sigma)\Delta \mathbf{u} + \mathbf{grad} \text{ div } \mathbf{u} = 0 \quad (7.4)$$

or

$$2(1-\sigma) \mathbf{grad} \text{ div } \mathbf{u} - (1-2\sigma) \mathbf{curl} \text{ curl } \mathbf{u} = 0. \quad (7.5)$$

The external forces appear in the solution only through the boundary conditions.

Taking the divergence of equation (7.4) and using the identity

$$\text{div } \mathbf{grad} \equiv \Delta,$$

we find

$$\Delta \text{ div } \mathbf{u} = 0, \quad (7.6)$$

i.e.  $\text{div } \mathbf{u}$  (which determines the volume change due to the deformation) is a harmonic function. Taking the Laplacian of equation (7.4), we then obtain

$$\Delta \Delta \mathbf{u} = 0, \quad (7.7)$$

i.e. in equilibrium the displacement vector satisfies the *biharmonic equation*. These results remain valid in a uniform gravitational field (since the right-hand side of equation (7.2) gives zero on differentiation), but not in the general case of external forces which vary through the body.

The fact that the displacement vector satisfies the biharmonic equation does not, of course, mean that the general integral of the equations of equilibrium (in the absence of body forces) is an arbitrary biharmonic vector; it must be remembered that the function  $\mathbf{u}(x, y, z)$  also satisfies the lower-order differential equation (7.4). It is possible, however, to express the general integral of the equations of equilibrium in terms of the derivatives of an arbitrary biharmonic vector (see Problem 10).

If the body is non-uniformly heated, an additional term appears in the equation of equilibrium. The stress tensor must include the term

$$-K\alpha(T - T_0)\delta_{ik}$$

(see (6.2)), and  $\partial\sigma_{ik}/\partial x_k$  accordingly contains a term

$$-K\alpha\partial T/\partial x_i = -[E\alpha/3(1-2\sigma)]\partial T/\partial x_i.$$

The equation of equilibrium thus takes the form

$$\frac{3(1-\sigma)}{1+\sigma} \mathbf{grad} \operatorname{div} \mathbf{u} - \frac{3(1-2\sigma)}{2(1+\sigma)} \mathbf{curl} \operatorname{curl} \mathbf{u} = \alpha \mathbf{grad} T. \quad (7.8)$$

Let us consider the particular case of a *plane deformation*, in which one component of the displacement vector ( $u_z$ ) is zero throughout the body, while the components  $u_x, u_y$  depend only on  $x$  and  $y$ . The components  $u_{zz}, u_{xz}, u_{yz}$  of the strain tensor then vanish identically, and therefore so do the components  $\sigma_{xz}, \sigma_{yz}$  of the stress tensor (but not the longitudinal stress  $\sigma_{zz}$ , the existence of which is implied by the constancy of the length of the body in the  $z$ -direction). †

Since all quantities are independent of the co-ordinate  $z$ , the equations of equilibrium (in the absence of external body forces)  $\partial\sigma_{ik}/\partial x_k = 0$  reduce in this case to two equations:

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0, \quad \frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0. \quad (7.9)$$

The most general functions  $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$  satisfying these equations are of the form

$$\sigma_{xx} = \partial^2\chi/\partial y^2, \quad \sigma_{xy} = -\partial^2\chi/\partial x\partial y, \quad \sigma_{yy} = \partial^2\chi/\partial x^2, \quad (7.10)$$

where  $\chi$  is an arbitrary function of  $x$  and  $y$ . It is easy to obtain an equation which must be satisfied by this function. Such an equation must exist, since the three quantities  $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$  can be expressed in terms of the two quantities  $u_x, u_y$ , and are therefore not independent. Using formulae (5.13), we find, for a plane deformation,

$$\sigma_{xx} + \sigma_{yy} = E(u_{xx} + u_{yy})/(1+\sigma)(1-2\sigma).$$

† The use of the theory of functions of a complex variable provides very powerful methods of solving plane problems in the theory of elasticity. See N. I. MUSKHELISHVILI, *Some Basic Problems of the Mathematical Theory of Elasticity*, 2nd English ed., P. Noordhoff, Groningen 1963.

But

$$\sigma_{xx} + \sigma_{yy} = \Delta\chi, \quad u_{xx} + u_{yy} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \equiv \operatorname{div} \mathbf{u},$$

and, since by (7.6)  $\operatorname{div} \mathbf{u}$  is harmonic, we conclude that the function  $\chi$  satisfies the equation

$$\Delta \Delta \chi = 0, \quad (7.11)$$

i.e. it is biharmonic. This function is called the *stress function*. When the plane problem has been solved and the function  $\chi$  is known, the longitudinal stress  $\sigma_{zz}$  is determined at once from the formula

$$\sigma_{zz} = \sigma E(u_{xx} + u_{yy}) / (1 + \sigma)(1 - 2\sigma) = \sigma(\sigma_{xx} + \sigma_{yy}),$$

or

$$\sigma_{zz} = \sigma \Delta \chi. \quad (7.12)$$

#### PROBLEMS

**PROBLEM 1.** Determine the deformation of a long rod (of length  $l$ ) standing vertically in a gravitational field.

**SOLUTION.** We take the  $z$ -axis along the axis of the rod, and the  $xy$ -plane in the plane of its lower end. The equations of equilibrium are  $\partial \sigma_{xi} / \partial x_i = \partial \sigma_{yi} / \partial x_i = 0$ ,  $\partial \sigma_{zi} / \partial x_i = \rho g$ . On the sides of the rod all the components  $\sigma_{ik}$  except  $\sigma_{zz}$  must vanish, and on the upper end ( $z = l$ )  $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$ . The solution of the equations of equilibrium satisfying these conditions is  $\sigma_{zz} = -\rho g(l - z)$ , with all other  $\sigma_{ik}$  zero. From  $\sigma_{ik}$  we find  $u_{ik}$  to be  $u_{zx} = u_{yy} = \sigma \rho g(l - z) / E$ ,  $u_{zz} = -\rho g(l - z) / E$ ,  $u_{xy} = u_{xz} = u_{yz} = 0$ , and hence by integration we have the components of the displacement vector,  $u_x = \sigma \rho g(l - z)x / E$ ,  $u_y = \sigma \rho g(l - z)y / E$ ,  $u_z = -(\rho g / 2E)\{l^2 - (l - z)^2 - \sigma(x^2 + y^2)\}$ . The expression for  $u_z$  satisfies the boundary condition  $u_z = 0$  only at one point on the lower end of the rod. Hence the solution obtained is not valid near the lower end.

**PROBLEM 2.** Determine the deformation of a hollow sphere (of external and internal radii  $R_2$  and  $R_1$ ) with a pressure  $p_1$  inside and  $p_2$  outside.

**SOLUTION.** We use spherical co-ordinates, with the origin at the centre of the sphere. The displacement vector  $\mathbf{u}$  is everywhere radial, and is a function of  $r$  alone. Hence  $\operatorname{curl} \mathbf{u} = 0$ , and equation (7.5) becomes  $\operatorname{grad} \operatorname{div} \mathbf{u} = 0$ . Hence

$$\operatorname{div} \mathbf{u} = \frac{1}{r^2} \frac{d(r^2 u)}{dr} = \text{constant} \equiv 3a,$$

or  $u = ar + b/r^2$ . The components of the strain tensor are (see formulae (1.7))  $u_{rr} = a - 2b/r^3$ ,  $u_{\theta\theta} = u_{\phi\phi} = a + b/r^3$ . The radial stress is

$$\sigma_{rr} = \frac{E}{(1 + \sigma)(1 - 2\sigma)} \{(1 - \sigma)u_{rr} + 2\sigma u_{\theta\theta}\} = \frac{E}{1 - 2\sigma} a - \frac{2E}{1 + \sigma} \frac{b}{r^3}.$$

The constants  $a$  and  $b$  are determined from the boundary conditions:  $\sigma_{rr} = -p_1$  at  $r = R_1$ , and  $\sigma_{rr} = -p_2$  at  $r = R_2$ . Hence we find

$$a = \frac{p_1 R_1^3 - p_2 R_2^3}{R_2^3 - R_1^3} \cdot \frac{1 - 2\sigma}{E}, \quad b = \frac{R_1^3 R_2^3 (p_1 - p_2)}{R_2^3 - R_1^3} \cdot \frac{1 + \sigma}{2E}.$$

For example, the stress distribution in a spherical shell with a pressure  $p_1 = p$  inside and  $p_2 = 0$  outside is given by

$$\sigma_{rr} = \frac{pR_1^3}{R_2^3 - R_1^3} \left( 1 - \frac{R_2^3}{r^3} \right), \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{pR_1^3}{R_2^3 - R_1^3} \left( 1 + \frac{R_2^3}{r^3} \right).$$

For a thin spherical shell of thickness  $h = R_2 - R_1 \ll R$  we have approximately

$$u = pR^2(1 - \sigma)/2Eh, \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{1}{2}pR/h, \quad \bar{\sigma}_{rr} = \frac{1}{2}p,$$

where  $\bar{\sigma}_{rr}$  is the mean value of the radial stress over the thickness of the shell.

The stress distribution in an infinite elastic medium with a spherical cavity (of radius  $R$ ) subjected to hydrostatic compression is obtained by putting  $R_1 = R$ ,  $R_2 = \infty$ ,  $p_1 = 0$ ,  $p_2 = p$ :

$$\sigma_{rr} = -p \left( 1 - \frac{R^3}{r^3} \right), \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p \left( 1 + \frac{R^3}{2r^3} \right).$$

At the surface of the cavity the tangential stresses  $\sigma_{\theta\theta} = \sigma_{\phi\phi} = -3p/2$ , i.e. they exceed the pressure at infinity.

**PROBLEM 3.** Determine the deformation of a solid sphere (of radius  $R$ ) in its own gravitational field.

**SOLUTION.** The force of gravity on unit mass in a spherical body is  $-gr/R$ . Substituting this expression in place of  $\mathbf{g}$  in equation (7.3), we obtain the following equation for the radial displacement:

$$\frac{E(1 - \sigma)}{(1 + \sigma)(1 - 2\sigma)} \frac{d}{dr} \left( \frac{1}{r^2} \frac{d(r^2 u)}{dr} \right) = \rho g \frac{r}{R}.$$

The solution finite for  $r = 0$  which satisfies the condition  $\sigma_{rr} = 0$  for  $r = R$  is

$$u = - \frac{g\rho R(1 - 2\sigma)(1 + \sigma)}{10E(1 - \sigma)} r \left( \frac{3 - \sigma}{1 + \sigma} - \frac{r^2}{R^2} \right).$$

It should be noticed that the substance is compressed ( $u_{rr} < 0$ ) inside a spherical surface of radius  $R\sqrt{\{(3 - \sigma)/3(1 + \sigma)\}}$  and stretched outside it ( $u_{rr} > 0$ ). The pressure at the centre of the sphere is  $(3 - \sigma)g\rho R/10(1 - \sigma)$ .

**PROBLEM 4.** Determine the deformation of a cylindrical pipe (of external and internal radii  $R_2$  and  $R_1$ ), with a pressure  $p$  inside and no pressure outside.†

**SOLUTION.** We use cylindrical co-ordinates, with the  $x$ -axis along the axis of the pipe. When the pressure is uniform along the pipe, the deformation is a purely radial displacement  $u_r = u(r)$ . Similarly to Problem 2, we have

$$\operatorname{div} \mathbf{u} = \frac{1}{r} \frac{d(ru)}{dr} = \text{constant} \equiv 2a.$$

Hence  $u = ar + b/r$ . The non-zero components of the strain tensor are (see formulae (1.8))  $u_{rr} = du/dr = a - b/r^2$ ,  $u_{\phi\phi} = u/r = a + b/r^2$ . From the conditions  $\sigma_{rr} = 0$  at  $r = R_2$ , and  $\sigma_{rr} = -p$  at  $r = R_1$ , we find

$$a = \frac{pR_1^2}{R_2^2 - R_1^2} \cdot \frac{(1 + \sigma)(1 - 2\sigma)}{E}, \quad b = \frac{pR_1^2 R_2^2}{R_2^2 - R_1^2} \cdot \frac{1 + \sigma}{E}.$$

† In Problems 4, 5 and 7 it is assumed that the length of the cylinder is maintained constant, so that there is no longitudinal deformation.

The stress distribution is given by the formulae

$$\sigma_{rr} = \frac{\rho R_1^2}{R_2^2 - R_1^2} \left( 1 - \frac{R_2^2}{r^2} \right), \quad \sigma_{\phi\phi} = \frac{\rho R_1^2}{R_2^2 - R_1^2} \left( 1 + \frac{R_2^2}{r^2} \right),$$

$$\sigma_{zz} = 2\rho\sigma R_1^2 / (R_2^2 - R_1^2).$$

**PROBLEM 5.** Determine the deformation of a cylinder rotating uniformly about its axis.

**SOLUTION.** Replacing the gravitational force in (7.3) by the centrifugal force  $\rho\Omega^2 r$  (where  $\Omega$  is the angular velocity), we have in cylindrical co-ordinates the following equation for the displacement  $u_r = u(r)$ :

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} \frac{d}{dr} \left( \frac{1}{r} \frac{d(ru)}{dr} \right) = -\rho\Omega^2 r.$$

The solution which is finite for  $r = 0$  and satisfies the condition  $\sigma_{rr} = 0$  for  $r = R$  is

$$u = \frac{\rho\Omega^2(1+\sigma)(1-2\sigma)}{8E(1-\sigma)} r [(3-2\sigma)R^2 - r^2].$$

**PROBLEM 6.** Determine the deformation of a non-uniformly heated sphere with a spherically symmetrical temperature distribution.

**SOLUTION.** In spherical co-ordinates, equation (7.8) for a purely radial deformation is

$$\frac{d}{dr} \left( \frac{1}{r^2} \frac{d(r^2 u)}{dr} \right) = \alpha \frac{1+\sigma}{3(1-\sigma)} \frac{dT}{dr}.$$

The solution which is finite for  $r = 0$  and satisfies the condition  $\sigma_{rr} = 0$  for  $r = R$  is

$$u = \alpha \frac{1+\sigma}{3(1-\sigma)} \left\{ \frac{1}{r^2} \int_0^r T(r) r^2 dr + \frac{2(1-2\sigma)}{1+\sigma} \frac{r}{R^3} \int_0^R T(r) r^2 dr \right\}.$$

The temperature  $T(r)$  is measured from the value for which the sphere, if uniformly heated, is regarded as undeformed. In the above formula the temperature in question is taken as that of the outer surface of the sphere, so that  $T(R) = 0$ .

**PROBLEM 7.** The same as Problem 6, but for a non-uniformly heated cylinder with an axially symmetrical temperature distribution.

**SOLUTION.** We similarly have in cylindrical co-ordinates

$$u = \alpha \frac{1+\sigma}{3(1-\sigma)} \left\{ \frac{1}{r} \int_0^r T(r) r dr + (1-2\sigma) \frac{r}{R^2} \int_0^R T(r) r dr \right\}.$$

**PROBLEM 8.** Determine the deformation of an infinite elastic medium with a given temperature distribution  $T(x, y, z)$  which is such that the temperature tends to a constant value  $T_0$  at infinity, there being no deformation there.

**SOLUTION.** Equation (7.8) has an obvious solution for which  $\text{curl } \mathbf{u} = 0$  and

$$\text{div } \mathbf{u} = \alpha(1+\sigma)[T(x, y, z) - T_0]/3(1-\sigma).$$