PERSPECTIVE

Astronomical reach of fundamental physics

Adam S. Burrows[1] and Jeremiah P. Ostriker

Department of Astrophysical Sciences, Princeton University, Princeton, NJ 08544

Edited by Neta A. Bahcall, Princeton University, Princeton, NJ, and approved January 7, 2014 (received for review September 23, 2013)

Using basic physical arguments, we derive by dimensional and physical analysis the characteristic masses and sizes of important objects in the universe in terms of just a few fundamental constants. This exercise illustrates the unifying power of physics and the profound connections between the small and the large in the cosmos we inhabit. We focus on the minimum and maximum masses of normal stars, the corresponding quantities for neutron stars, the maximum mass of a rocky planet, the maximum mass of a white dwarf, and the mass of a typical galaxy. To zeroth order, we show that all these masses can be expressed in terms of either the Planck mass or the Chandrasekar mass, in combination with various dimensionless quantities. With these examples, we expose the deep interrelationships imposed by nature between disparate realms of the universe and the amazing consequences of the unifying character of physical law.

Fundamental Physical Constants from Which to Build the Universe

One of the profound insights of modern science in general, and of physics in particular, is that not only is everything connected, but that everything is connected quantitatively. Another is that there are physical constants of nature that by their units, constancy in space and time, and magnitude encapsulate nature's laws. The constant speed of light delimits and defines the fundamental character of kinematics and dynamics. Planck's constant tells us that there is something special about angular momentum and the products of length and momentum and of energy and time. What is more, in combination, the constants of nature set the scales (lengths, times, and masses) for all objects and phenomena in the universe, because scales are dimensioned entities and the only building blocks from which to construct them are the fundamental constants around which all nature rotates. Although in part merely dimensional analysis, such constructions encapsulate profound insights into diverse physical phenomena.

Hence, the masses of nuclei, atoms, stars, and galaxies are set by a restricted collection of basic constants that embody the finite number of core natural laws. In this paper, we demonstrate this reduction to fundamentals by deriving the characteristic masses of important astronomical objects in terms of just a few fundamental constants. In doing so, we are less interested in precision than illumination and focus on the orders of magnitude. Most of our arguments are not original (see refs. 1–5 and references therein), although some individual arguments are. This exercise will be valuable to the extent that it provides a unified discussion of the physical scales found in the astronomical world. Those interested in the fundamental connections between the small and the large in this universe we jointly inhabit are invited to contemplate the examples we assembled here.

Reducing, even in approximate fashion, the properties of the objects of the universe to their fundamental dependencies requires first and foremost a choice of irreducible fundamental constants. Various combinations of those constants of nature can also be useful, so the word "irreducible" is used here with great liberty. We could choose among the following:

$$\hbar, c, e, G, m_p, m_\pi, m_e, \quad [1]$$

where $\hbar$ is the reduced Planck's constant $(h/2\pi)$, c is the speed of light, e is the elementary electron charge, G is Newton's gravitational constant, m_p is the proton mass, m_π is the pion mass, and m_e is the electron mass. It is in principle possible that these masses can be reduced to one fiducial mass, with mass ratios derived from some overarching theory, but this is currently beyond the state of the art in particle physics. However, dimensionless combinations of fundamental constants emerge naturally in the variety of contexts in which they are germane. Examples are $\alpha = e^2/\hbar c$, the fine structure constant $\left(\sim \frac{1}{137}\right)$; $\alpha_g = Gm_p^2/\hbar c(\sim 10^{-38})$, the corresponding gravitational coupling constant; and $m_{pl} = (\hbar c/G)^{1/2}$, the Planck mass $(\sim 2\times 10^{-5}\ \mathrm{g})$.

As noted, if our fundamental theory was complete, we would be able to express all physical quantities in terms of only three dimensioned quantities, one each for mass, length, and time. Many would associate this fundamental theory with the Planck scale, so everything could be written in terms of m_{pl}, the Planck length $(R_{pl} = (G\hbar/c^3)^{1/2} \sim 10^{-33}\ \mathrm{cm})$, and the Planck time $(G\hbar/c^5)^{1/2}(\sim 10^{-43}\ s)$. The remaining quantities of relevance would be dimensionless ratios derivable in this fundamental theory. For instance, these ratios could be

$$\begin{aligned} \eta_p &= m_{pl}/m_p(\sim 10^{19}) \\ \eta_e &= m_{pl}/m_e(\sim 10^{22}) \\ \eta_\pi &= m_{pl}/m_\pi(\sim 10^{20}), \end{aligned} \quad [2]$$

and α, and, in principle, these ratios could be derived in the hypothetical fundamental theory. We show in this paper that we can write astronomical masses in terms of m_{pl} or m_p, with mass ratios and dimensionless combinations of fundamental constants setting the corresponding relative scale factors. We thereby reduce all masses to combinations of only five quantities: m_p, m_e, m_π, α, and α_g or m_{pl}, η_p, η_e, η_π, and α. For radii, $\hbar$ and c are explicitly needed. Note that $\alpha_g = 1/\eta_p^2$ and that $\alpha/\alpha_g \sim 10^{36}$. The latter is a rather large number, a fact with significant consequences.

We assume in this simplified treatment that the proton and neutron masses are the same and equal to the atomic mass unit, itself the reciprocal of Avogadro's constant (N_A). The pion mass can be the mass of any of the three pions, sets the length scale for the nucleus, and helps set the energy scale of nuclear binding energies (6). However, for specificity and for sanity's sake, we will assume that the number of spatial dimensions is three. Then, with only five constants we

Author contributions: A.S.B. and J.P.O. performed research and wrote the paper.

The authors declare no conflict of interest.

This article is a PNAS Direct Submission.

[1]To whom correspondence should be addressed. E-mail: burrows@astro.princeton.edu.

can proceed to explain, in broad outline, objects in the universe. In this essay, we emphasize stars (and planets) and galaxies, but in principle, life and the universe are amenable to similar analyses (3, 7, 8).

Before we proceed, we make an aside on nuclear scales and why we have introduced the pion mass and η_π. Nucleons (protons and neutrons) are comprised of three quarks, and pions are comprised of quark-antiquark pairs. Quantum chromodynamics (QCD) (9) is the fundamental theory of the quark and gluon interactions and determines the properties, such as masses, of composite particles. The proton mass (~938 MeV/c^2) is approximately equal to $\sim 3\times\Lambda_{QCD}$, where $\Lambda_{QCD}(\sim 300$ MeV) is the QCD energy scale (10), and 3 is its number of constituent quarks (11). The pion, however, is a pseudo-Goldstone boson, and the square of its mass is proportional to $\Lambda_{QCD}(m_u+m_d)$, where m_u and m_d are the up and down bare quark masses, respectively. m_u+m_d is very approximately equal to 10 MeV (quite small). In principle, all the hadron masses and physical dimensions can be determined from the bare quark masses and the QCD energy scale (12), because the associated running coupling constant (which sets, for example, $g^2_{\pi NN}/4\pi$) perforce approaches unity (large values) on the very spatial scales that set particle size.

For our purposes, we assume there are two fundamental masses in the theory. These could be Λ_{QCD} and m_u+m_d, but they could also be m_p and m_π. We opt for the latter. Importantly, pion exchange mediates the strong force between nucleons, and the nucleon-nucleon interaction determines nuclear binding energies and nuclear energy yields. Moreover, since the pion interaction is a derivative interaction (11), the corresponding nuclear energy scale is not m_π (~135 MeV/c^2), but is proportional to $(g^2_{\pi NN}/4\pi)m_\pi(\eta_p/\eta_\pi)^2$. The upshot is that the binding energies of nuclei scale as $m_\pi(\eta_p/\eta_\pi)^2$ and the size of the nucleon is $\sim\hbar/m_\pi c$, about one fermi for the measured value of m_π. Hence, the two fundamental masses, m_π and m_p, determine the density of the nucleus and nuclear binding energies, both useful quantities. This fact is the reason we include m_π (or η_π) in our list of fundamental constants with which we describe the Cosmos—reasons for including m_p (or η_p) are more self-evident. An ultimate theory would provide the ratios of all the couplings (QCD, electro-weak, gravitational) and mass ratios, as well as all other dimensionless numbers of nature. Because we are currently absent of such a theory, we proceed using the quantities described.

Maximum Mass of a White Dwarf: The Chandrasekhar Mass

Stars are objects in hydrostatic equilibrium for which inward forces of gravity are balanced by outward forces due to pressure gradients. The equation of hydrostatic equilibrium is

$$\frac{dP}{dr} = -\rho\frac{GM(r)}{r^2}, \qquad [3]$$

where P is the pressure, r is the radius, ρ is the mass density, and $M(r)$ is the interior mass (the volume integral of ρ). Dimensional analysis thus yields for the average pressure or the central pressure (P_c) an approximate relation, $P_c\sim\frac{GM^2}{R^4}$, where M is the total stellar mass and R is the stellar radius, and we used the crude relation $\rho\sim\frac{M}{R^3}$.

For a given star, its equation of state, connnecting pressure with temperature, density, and composition, is also known independently. Setting the central pressure derived using hydrostatic equilibrium equal to the central pressure from thermodynamics or statistical physics can yield a useful relation between M and R (for a given composition). For example, if the pressure is a power-law function of density alone (i.e., a polytrope), such that $P=\kappa\rho^\gamma$, then using $\rho\sim\frac{M}{R^3}$ and setting $\kappa\rho^\gamma$ equal to $\frac{GM^2}{R^4}$ gives us

$$M\propto\left(\frac{\kappa}{G}\right)^{\frac{1}{2-\gamma}}R^{(4-3\gamma)/(2-\gamma)}. \qquad [4]$$

One notices immediately that M and R are decoupled for $\gamma=4/3$. In fact, one can show from energy arguments that if γ is the adiabatic gamma and not merely of structural significance, then at $\gamma=4/3$ the star is neutrally stable—changing its radius at a given mass requires no work or energy. In other words, the star is unstable to collapse for $\gamma<4/3$ and stable to perturbation and pulsation at $\gamma>4/3$. For $\gamma=4/3$, there is only one mass, and it is $\left(\frac{\kappa}{G}\right)^{3/2}$.

The significance of this is that white dwarf stars are supported by electron degeneracy pressure, fermionic zero-point motion, which is independent of temperature. Such degenerate objects are created at the terminal stages of the majority of stars and are what remains after such a star's thermonuclear life. The pressure, which like all pressures resembles an energy density, is approximately given by the average energy per electron ($\langle\varepsilon\rangle$) times the number density of electrons (n_e). If the electrons are nonrelativistic, $\langle\varepsilon\rangle\sim P_F^2/2m_e$, where P_F is the fermi momentum, and simple quantum mechanical phase-space arguments for fermions give $P_F=(3\pi^2\hbar^3 n_e)^{1/3}$. Because $n_e=N_A\rho Y_e$, where Y_e is the number of electrons per baryon (~0.5), it is easy to show that the pressure is proportional to $\rho^{5/3}$, that the associated $\gamma=5/3$, and that the star is, therefore, a polytrope. By the arguments above, such stars are stable.

However, as the mass increases, the central density increases, thereby increasing the fermi momentum. Above $cP_F\sim m_e c^2$, the electrons become relativistic. The formula for the fermi momentum is unchanged, but the fermi energy is now linear (not quadratic) in P_F. This fact means that $\langle\varepsilon\rangle\times n_e$ is proportional to $\rho^{4/3}$ and that the white dwarf becomes unstable (formally neutrally stable, but slightly unstable if general relativistic effects are included). Hence, the onset of relativistic electron motion throughout a large fraction of the star, the importance of quantum mechanical degeneracy, and the inexorable effects of gravity conspire to yield a maximum mass for a white dwarf (13, 14). This mass, at the confluence of relativity (c), quantum mechanics ($\hbar$), and gravitation (G), is the Chandrasekhar mass, named after one of its discoverers. Its value can be derived quite simply using $\left(\frac{\kappa}{G}\right)^{3/2}$ (Eq. 4) and $\kappa=\frac{3\pi^2}{4}\frac{\hbar c}{m_p^{4/3}}Y_e^{4/3}$, where we have set $N_A=1/m_p$. The result is

$$M_{Ch}\sim\left(\frac{\hbar c}{G}\right)^{3/2}\frac{Y_e^2}{m_p^2}. \qquad [5]$$

Note that the electron mass does not occur and that the proper prefactor is ~3.1. For $Y_e=0.5$, $3.1Y_e^2$ is of order unity and $M_{Ch}=1.46$ M$_\odot$. We recovered our first significant result. There is a maximum mass for a white dwarf, and its value depends only on fundamental constants $\hbar$, c, G, and m_p, most of which are of the microscopic world. We can rewrite this result in numerous ways, some of which are

$$M_{Ch}\sim m_p\left(\frac{\hbar c}{Gm_p^2}\right)^{3/2}=\frac{m_p}{\alpha_g^{3/2}}$$
$$M_{Ch}\sim m_p\eta_p^3 \qquad [6]$$
$$M_{Ch}\sim m_{pl}\eta_p^2.$$

Hence, the Chandrasekhar mass can be considered to be a function of the Planck mass and the ratio, η_p, or the proton mass and the small dimensionless gravitational coupling constant, $\alpha_g\left(=\frac{Gm_p^2}{\hbar c}\right)$. Such relationships are impressive in their compactness and in their implications for the power of science to explain nature using fundamental arguments.

As an aside, we can use the arguments above to derive a relation for the characteristic radius of a white dwarf. To do this, we return to Eq. **4**, but insert $\gamma = 5/3$, the nonrelativistic value relevant for most of the family. The degeneracy pressure is then given by $P = \kappa\rho^{5/3}$, where

$$\kappa = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e m_p^{5/3}}Y_e^{5/3}, \qquad [7]$$

which is an expression easily shown to follow from the general fact that pressure is average particle energy times number density, but for the nonrelativistic relation between momentum and energy. We will find this relation useful elsewhere, so we highlight it. Eq. **4** then yields $R_{wd} \sim \frac{\kappa}{GM^{1/3}}$. We now use some sleight-of-hand. Noting that a white dwarf's mass changes little as the electron's fermi energy transitions from nonrelativistic to relativistic and the mass transitions from $\sim 0.5\ M_\odot$ to M_{ch}, and using Eq. **7**, we obtain

$$\begin{aligned} R_{wd} &\sim \frac{\hbar}{m_e c}\eta_p \\ &\sim R_{pl}\eta_e\eta_p \\ &\sim 10^4 \text{km} \end{aligned} \qquad [8]$$

where we have set $M \sim M_{Ch}$. This relation powerfully connects the radius of a white dwarf to the product of the electron Compton wavelength and the ratio of the Planck mass to the proton mass. It also provides the white dwarf radius in terms of the Planck radius. Putting numbers in yields something of order a few thousand kilometers, the actual measured value. We will later compare this radius with that obtained for neutron stars and derive an important result.

Maximum Mass of a Neutron Star

Neutron stars are supported against gravity by pressure due to the strong repulsive nuclear force between nucleons. Atomic nuclei are fragmented into their component subatomic particles and experience a phase transition to nucleons near and above the density of the nucleus. As we have argued, the interparticle spacing in the nucleus is set by the range of the strong force, which is the reduced pion Compton wavelength $\lambda_\pi = \hbar/m_\pi c$. Therefore, because the repulsive nuclear force is so strong, the average density of a neutron star will be set (very approximately) by the density of the nucleus $\left(\rho_{nuc} \sim \frac{m_p}{4\pi\lambda_\pi^3/3}\right)$. Although nucleons are fermions, the maximum mass of a neutron star is not determined by the special relativistic physics that sets the Chandrasekhar mass (15). Rather, the maximum mass of a neutron star is set by a general relativistic (GR) instability. GR gravity is stronger than Newtonian gravity, and at high enough pressures, pressure itself becomes a source, along with mass, to amplify gravity even more. The result is that increasing pressure eventually becomes self-defeating when trying to support the more massive neutron stars and the object collapses. Whereas the collapse of a Chandrasekhar mass white dwarf leads to a neutron star, the collapse of a critical neutron star leads to a black hole.

The critical mass of a neutron star is reached as its radius shrinks to near its Schwarzschild radius, $\frac{2GM}{c^2}$, say to within a factor of two (but call this factor β_n). GR physics introduces no new fundamental constants and involves only G and c. Therefore, the condition for GR instability is $R = \frac{2GM\beta_n}{c^2}$. We can derive the corresponding mass by using the relation $\rho \sim \frac{3M}{4\pi R^3}$. The result is

$$\begin{aligned} M_{NS} &\sim \left(\frac{\hbar c}{G}\right)^{3/2}\frac{1}{m_p^2}\left(\frac{\eta_\pi}{2\beta_n\eta_p}\right)^{3/2} \\ &\sim M_{Ch}\left(\frac{\eta_\pi}{2\beta_n\eta_p}\right)^{3/2} \\ &\propto \frac{m_p}{\alpha_g^{3/2}}\left(\frac{\eta_\pi}{2\beta_n\eta_p}\right)^{3/2} \\ &\propto m_{pl}\eta_p^2\left(\frac{\eta_\pi}{2\beta_n\eta_p}\right)^{3/2}. \end{aligned} \qquad [9]$$

Dropping β_n $(\sim O(1))$, M_{NS} is also even more simply written as $M_{Ch}\left(\frac{\eta_\pi}{\eta_p}\right)^{3/2}$. The Planck mass and the Chandrasekhar mass appear again, but this time accompanied by the proton-pion mass ratio and unaccompanied by $3.1Y_e^2$. For $\beta_n \sim 2$, given the values of m_p and m_π with which we are familiar, M_{NS} is actually close to M_{Ch}. Details of the still-unknown nuclear equation of state, GR, and the associated density profiles yield values for M_{NS} between ~ 1.5 and $\sim 2.5\ M_\odot$, where eventually we would need to distinguish between the gravitational mass and the baryonic mass. We note here that the fact that the two common types of degenerate stars—end products of stellar evolution—have nearly the same mass derives in part from the rough similarity of the pion and proton masses.

Of course, the pion mass, m_π, enters through its role in the nuclear density, as does $\hbar$ and m_p. c enters through GR and through its role in setting the range of the Yukawa potential. G enters due the perennial combat of matter with gravity in stars. All but m_π are actors in the Chandrasekhar saga, so we should not be surprised to see M_{Ch} once again, albeit with a modifying factor. Note that, because neutron stars are formed when a white dwarf achieves the Chandrasekhar mass and critical neutron stars collapse to black holes, the existence of stable neutron stars requires that $M_{NS} > M_{Ch}$, with many, many details. When those details are accounted for we find that M_{NS} (baryonic) is about twice M_{Ch}, the ratio being comfortably greater than, but at the same time uncomfortably close to, unity. Suffice it to say, stable neutron stars do exist in our universe.

The radius of a neutron star is set by some multiple of the Schwarzschild radius. The baryonic mass of a neutron star is set by its formation and accretion history, but it is bounded by the minimum possible Chandrasekhar mass (recall this is set by Y_e). Therefore, if we set a neutron star's mass equal to its maximum, we will not be far off. The upshot is the formula

$$R_{ns} \sim 2\frac{2G}{c^2}\left(\frac{\hbar c}{G}\right)^{3/2}\frac{1}{m_p^2} \sim \frac{\hbar}{m_p c}\eta_p \sim R_{pl}\eta_p^2.$$

This equation is very much like Eq. **8** for the radius of a white dwarf, with m_p substituted for m_e. Therefore, the ratio of the radius of a white dwarf to the radius of a neutron star is simply the ratio of the proton mass to the electron mass (η_e/η_p), within factors of order unity. In our universe, η_e/η_p is ~1,836, but we can call this "three orders of magnitude." Ten-kilometer neutron stars quite naturally imply $\sim 5\times10^3$ to $\sim 10^4$-km white dwarfs. By now, we should not be surprised that this ratio arises in elegant fashion from fundamental quantities using basic physical arguments.

Minimum Mass of a Neutron Star

The minimum possible mass (M_{ns}) for a stable neutron star may not be easily realized in nature. Current theory puts it at $\sim 0.1\ M_\odot$ (16), and there is no realistic mechanism by which a white dwarf progenitor near such a mass, inside a star in its terminal stages or in a tight binary, can be induced to collapse to neutron star densities. The Chandrasekhar instability is the natural agency, but M_{Ch} is much larger than $\sim 0.1\ M_\odot$. However, such an object might form via gravitational instability in the accretion disks of rapidly rotating neutron stars of canonical mass. Mass loss after the collapse of a Chandrasekhar core is possible by Roche-lobe overflow in the tight binary, but the associated tidal potential is quite different from the spherical potential for which M_{ns} is derived.

Nevertheless, a first-principles estimate of M_{ns} is of some interest. What is the physics that determines it? A white dwarf is supported by the electron degeneracy pressure of free electrons, and its baryons are sequestered in nuclei. A neutron star is supported by the repulsive strong force between degenerate free nucleons, and most of its nuclei are dissociated. Both are gravitationally bound. One can ask the question: for a given mass, which of the two, neutron star or white dwarf, is the lower energy state? Note that to transition to a neutron star, the nuclei of a white dwarf must be dissociated into nucleons, and the binding energy of the nucleus must be paid. Note also that a neutron star, with its much smaller radius, is the more gravitationally bound. Therefore, we see that when the specific (per nucleon) gravitational binding energy is equal to the specific nuclear binding energy of the individual nuclei, we are at M_{ns}. Above this mass, the conversion of an extended white dwarf into a compact neutron star and the concomitant release of gravitational binding energy can then pay the necessary nuclear breakup penalty. However, of course, a substantial potential barrier must be overcome.

In equation form, this can be stated as $\frac{GM^2}{R} \sim \Delta Mc^2$, where ΔMc^2 is the binding energy of the nucleus. This equation can be rewritten as $\frac{GM}{Rc^2} \sim \frac{\Delta M}{M} = f$, where f is the binding energy per mass, divided by c^2. From the fact that the nucleon is bound in a nucleus with an energy $\sim m_\pi c^2 \left(\frac{\eta_p}{\eta_\pi}\right)^2$, we obtain $f \sim \left(\frac{\eta_p}{\eta_\pi}\right)^3$ (an appropriate extra factor might be ∼3). In our universe, f is ∼0.01–0.02. Now, notice that we can rewrite this condition as $\frac{2GM}{Rc^2} \sim 2f$, which is the same as the condition for the maximum neutron star mass, with $\frac{1}{2f}$ substituted for β_n in Eq. **9**. Using exactly the same manipulations, we then derive

$$\begin{aligned} M_{ns} &\sim M_{Ch} f^{3/2} \left(\frac{\eta_\pi}{\eta_p}\right)^{3/2} \\ &\sim M_{Ch} \left(\frac{\eta_p}{\eta_\pi}\right)^3 \\ &\sim m_{pl} \eta_p^2 \left(\frac{\eta_p}{\eta_\pi}\right)^3. \end{aligned} \quad [10]$$

The Planck mass and M_{Ch} emerge again (quite naturally), but this time accompanied by the small factor $\left(\frac{\eta_p}{\eta_\pi}\right)^3$ $\left(\left(\frac{m_\pi}{m_p}\right)^3\right)$. Note that $M_{ns}/M_{NS} \sim (2\beta_n f)^{3/2} \propto \left(\frac{m_\pi}{m_p}\right)^{4.5}$, which fortunately is a number less than one. However, to derive the precise value for M_{ns} requires retaining dropped factors and much more precision, but the basic dependence on fundamental constants is clear and revealing.

Maximum Mass of a Rocky Planet

A rocky planet such as the Earth, Venus, Mars, or Mercury is comprised of silicates and/or iron and is in hydrostatic equilibrium. If its composition was uniform, its density would be constant at the solid's laboratory value. This value is set by the Coulomb interaction and quantum mechanics (which establishes a characteristic radius, the Bohr radius). As its mass increases, the interior pressures increase and the planet's matter is compressed, at first barely and slowly (because of the strength of materials), but later (at higher masses) substantially. Eventually, for the highest masses, many of the electrons in the constituent atoms would be released into the conduction band, the material would be metalized (17), and the supporting pressure would be due to degenerate electrons. Objects supported by degenerate electrons are white dwarfs. Therefore, at low masses and pressures, the (constant) density of materials and the Coulomb interaction imply $R \propto M^{1/3}$, whereas at high masses, the object acts like a white dwarf supported by electron-degeneracy pressure and $R \propto M^{-1/3}$. This behavior indicates that there is a mass at which the radius is a maximum for a given composition (18). It also implies that there is a mass range over which the radius of the object is independent of mass and $\frac{dR}{dM} \sim 0$. For hydrogen-dominated objects, this state, for which Coulomb and degeneracy effects balance, is where Jupiter and Saturn reside, but one can contemplate the same phenomenon for rocky and iron planets. It is interesting to ask the following question: what is the maximum mass (M_{rock}) of a rocky planet, above which its radius decreases with increasing mass in white-dwarf-like fashion, and below which it behaves more like a member of a constant-density class of objects? This definition provides a reasonable upper bound to the mass of a rocky or solid planet. With the anticipated discovery of many exoplanet "Super-Earths" in coming years, this issue is of more than passing interest.

There are two related methods to derive M_{rock}. The first is to set the expression for nonrelativistic degeneracy pressure ($P_0 = \kappa\rho_0^{5/3}$, where κ is given by Eq. **7**) equal to the corresponding expression for the central pressure of a constant density object in hydrostatic equilibrium. The result is

$$P = \frac{1}{2}\left(\frac{4\pi}{3}\right)^{1/3} GM^{2/3}\rho_0^{4/3} \quad [11]$$

The other method is to use the radius-mass relations for both white dwarfs and constant-density planets and to set the radius obtained using one relation equal to that using the other. Stated in equation form, this is $R_{wd} \sim \frac{\kappa}{GM^{1/3}} \sim R_{rock} \sim \left(\frac{3M}{4\pi\rho_0}\right)^{1/3}$. For both methods, one needs the density, ρ_0, and using either method one gets almost exactly the same result. ρ_0 is set by the Bohr radius, $a_B = \frac{\hbar^2}{m_e e^2}$, and one then obtains

$$\rho_0 \sim \frac{3m_p\mu}{4\pi a_B^3}, \quad [12]$$

where μ is the mean molecular weight of the constituent silicate or iron (divided by m_p). A bit of manipulation yields

$$\begin{aligned} M_{rock} &\sim m_p \left(\frac{e^2}{Gm_p^2}\right)^{3/2} \frac{6}{\pi} Y_e^{5/2}\mu^{1/3} \\ &\sim m_p \left(\frac{\alpha}{\alpha_g}\right)^{3/2} \\ &\sim M_{Ch}\alpha^{3/2} \\ &\sim m_{pl}\eta_p^2\alpha^{3/2}. \end{aligned} \quad [13]$$

Note that M_{rock} is actually independent of $\hbar$, c, and m_e.

This equation expresses another extraordinary result: M_{rock} can be written to scale with the Planck mass and η_p or the Chandrasekhar mass, but multiplied by a power of the fine-structure constant. The latter comes from the role of the Coulomb interaction in setting the size scale of atoms and also indicates that M_{rock} is much smaller than M_{Ch}, as one might expect. As Eq. **13** also shows, M_{rock} scales with m_p, amplified by the ratio of the electromagnetic to the gravitational fine structure constants to the 3/2 power. Plugging numbers into either of these formulas, one finds that M_{rock} is $\sim 2 \times 10^{30}$ g $\sim 300\ M_{Earth} \sim 1\ M_{Jup}$. Detailed calculations only marginally improve on this estimate. Note that reinstating the $Y_e^{5/2}\mu^{1/3}$ dependence yields roughly the same value for both hydrogen and iron planets.

Maximum Mass of a Star

The fractional contribution of radiation pressure [$P_{rad} = (1/3)a_{rad}T^4$, where $a_{rad} = \frac{\pi^2 k_B^4}{15\hbar^3 c^3}$

and k_B is Boltzmann's constant] to the total supporting pressure in stars increases with mass. At sufficiently large mass, the gas becomes radiation dominated, and P_{rad} exceeds the ideal gas contribution $P_{IG} = \frac{\rho k_B T}{\mu m_p}$. The entropy per baryon of the star also rises with mass. If we set $\beta P \equiv P_{IG}$, where $P = P_{IG} + P_{rad}$, thereby defining β, we derive (19)

$$P = \left(\frac{1-\beta}{\beta^4}\right)^{1/3} \left(\frac{3k_B}{\mu m_p a_{rad}}\right)^{1/3} \frac{k_B}{\mu m_p} \rho^{4/3}. \quad [14]$$

The upshot is that the effective polytropic gamma (defined assuming $P \sim \kappa\rho^\gamma$) decreases from ∼5/3 toward ∼4/3 as the stellar mass increases. As we argued in the section on the Chandrasekhar mass, the onset of relativity makes a star susceptible to gravitational instability. Photons are relativistic particles. A radiation pressure–dominated stellar envelope can easily be ejected if perturbed and at the very least is prone to pulsation if coaxed. Such coaxing and/or perturbation could come from nuclear burning or radiation-driven winds. With radiation domination, and given the high opacity of envelopes sporting heavy elements with high Z, radiation pressure–driven winds can blow matter away and in this manner limit the mass that can accumulate. This physics sets the maximum mass (M_S) of a stable star on the hydrogen-burning main sequence.

Using Eq. **4**, with $\gamma = 4/3$, Eq. **14**, and the same arguments by which we derived the Chandrasekhar mass, we find the only mass for a given κ

$$\begin{aligned} M_S &\sim \left(\frac{\kappa_{rad}}{G}\right)^{3/2} \\ &\sim \left(\frac{1-\beta}{\beta^4}\right)^{1/2} \left(\frac{45}{\pi^2}\right)^{1/2} \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2 \mu^2} \\ &\sim \left(\frac{1-\beta}{\beta^4}\right)^{1/2} \left(\frac{45}{\pi^2}\right)^{1/2} \frac{M_{Ch}}{\mu^2} \\ &\sim \left(\frac{1-\beta}{\beta^4}\right)^{1/2} \left(\frac{45}{\pi^2}\right)^{1/2} m_{pl} (\eta_p/\mu)^2, \end{aligned} \quad [15]$$

where M_{Ch} is our Y_e-free Chandrasekhar mass. Because k_B has no meaning independent of temperature scale, it does not appear in Eq. **15**. That M_{Ch} occurs in M_S should not be surprising, because in both cases the onset of relativity in the hydrostatic context, for electrons in one case and via photons in the other, is the salient aspect of the respective limits (20). In determining both M_{Ch} and M_S, relativity (c), quantum mechanics ($\hbar$), and gravity (G) play central roles.

The ratio of M_S to $M_{Ch}(Y_e)$ depends only on μ, Y_e, β, and some dimensionless constants. Therefore, it is a universal number. For $\beta = 1/2$, $Y_e = 0.5$, and $\mu \sim 1$, we find $M_S/M_{Ch} \sim 20$, but for smaller βs (a greater degree of radiation domination), the ratio is larger $\left(\propto \frac{1}{\beta^2}\right)$. The relevant β and actual ratio depend on the details of formation and wind mass loss (and, hence, heavy element abundance), and are quite uncertain. Nevertheless, indications are that the latter ratio could range from ∼50 to ∼150 (21).

As an aside, we note that one can derive another (larger) maximum mass associated with general relativity by using the fact that the fundamental mode for spherical stellar pulsation occurs at a critical thermodynamic gamma (γ_1).† Including the destabilizing effect of general relativity, this critical γ_1 equals $4/3 + K\frac{GM}{Rc^2}$, where $K \sim 1$. A mixture of radiation with ideal gas for which radiation is dominant has a γ_1 of $\sim 4/3 + \beta/6$ (6). One could argue that, if the temperature is high enough to produce electron-positron pairs, then γ_1 would plummet and the star would be unstable to collapse. Hydrostatic equilibrium suggests that the stellar temperature times Boltzmann's constant when pairs start to become important would then be some fraction of $\sim m_e c^2$ and would equal $\frac{GMm_p}{R}$. This correspondence gives us an expression for $\frac{GM}{Rc^2}$ ($\sim \eta_p/\eta_e$) and, therefore, that $\beta/6 \sim m_e/m_p = \eta_p/\eta_e$. Using Eq. **15**, we find a maximum stellar mass due to general-relativistic instability and pair production near $\sim 10^6$ $M_\odot$, close to the expected value (22). Because other instabilities intervene before this mass is reached, this limit is not relevant for the main sequence limit.

However, the facts that (*i*) M_S is much greater than M_{Ch} and (*ii*) their ratio is a large constant are interesting. Therefore, we can take some comfort in noting that because M_S is much larger than M_{Ch}, white dwarfs, neutron stars, and stellar-mass black holes, all stellar "residues" that would be birthed in stars are allowed to exist.

Minimum Mass of a Star

Stars are assembled from interstellar medium gas by gravitational collapse. As they contract, they radiate thermal energy and the compact protostar that first emerges is in quasi-hydrostatic equilibrium. Before achieving the hydrogen-burning main sequence, the protostar becomes opaque and radiates from a newly formed photosphere at a secularly evolving luminosity. The progressive loss of energy from its surface occasions further gradual quasi-hydrostatic contraction. In parallel, the central temperature increases via what is referred to as the "negative specific heat" effect in stars. Energy loss leads to temperature increase. From energy conservation, the change in gravitational energy (loosely $-\frac{GM^2}{2R}$) due to the shrinkage is equal to the sum of the photon losses and the increase in thermal energy, in rough equipartition. Using either hydrostatic equilibrium or the Virial theorem and the ideal gas law connecting pressure with temperature $\left(P = \frac{\rho k_B T}{\mu m_p}\right)$, we obtain

$$k_B T_{int} \sim \frac{GMm_p\mu}{3R} \sim G\mu m_p M^{2/3}\rho^{1/3}, \quad [16]$$

where μ is the mean molecular weight (of order unity), and $R \sim \left(\frac{3M}{4\pi\rho}\right)^{1/3}$ has been used. We note that Eq. **16** demonstrates that the temperature increases as the star shrinks.

However, the negative specific heat is an indirect consequence of the ideal gas law. As the protostar contracts and heats, its density rises. In doing so, the core entropy decreases (despite the temperature increase) and the core becomes progressively more electron degenerate. When it becomes degenerate, because degeneracy pressure is asymptotically independent of temperature, further energy loss does not lead to a temperature increase but a decrease. Hence, there is a peak in the core temperature that manifests itself at the onset of core degeneracy. The standard argument states that if at this peak temperature (T_{max}) the integrated core thermonuclear power is smaller than the surface luminosity (also a power), then the star will not achieve the hydrogen main sequence. It will become a brown dwarf, which will cool inexorably into obscurity, but over Gigayear timescales. The mass at which T_{max} is just sufficent for core power to balance surface losses is the minimum stellar mass (M_s). Below this mass is the realm of the brown dwarf. Above it reside canonical, stably burning stars (23).

We can use Eq. **16** to derive the critical mass in terms of T_{max}. A gas becomes degenerate when quantum statistics emerges to be important. For an electron, this is when the deBroglie wavelength of the electron ($\lambda = \frac{h}{m_e v}$, where v is the average particle speed) approaches the interparticle spacing $\left(\frac{\mu m_p}{\rho}\right)^{1/3}$. It is also when the simple expression for the ideal gas pressure $\left(= \frac{\rho k_B T}{\mu m_p}\right)$ equals the simple expression for the degeneracy pressure ($P = \kappa\rho^{5/3}$, where κ is given by Eq. **7**). We use

†γ_1 is the logarithmic derivative of the pressure with respect to the mass density at constant entropy.

the latter condition and derive an expression for T_{max}

$$k_B T_{max} \sim \frac{\mu m_p}{\kappa} G^2 M^{4/3} \sim \frac{G^2 \mu m_p^{8/3} m_e}{\hbar^2 Y_e^{5/3}} M^{4/3}. \quad [17]$$

The textbooks state that M_s is then derived by setting T_{max} to some ignition temperature (frequently set to 10^6 K), and then solving for M in Eq. **17**. In this way, using measured constants and retaining a few dropped coefficients, one derives $M_s \sim 0.1\ M_\odot$ and this number is rather accurate.

However, this procedure leaves unexplained the origin of $T_{max} \sim 10^6$ K[‡] and here we depart from the traditional explanation to introduce our own. There are two things to note. First, M_s is not determined solely by thermonuclear considerations—photon opacities, temperatures, densities, and elemental abundances (metallicity) at the stellar surface set the emergent luminosity that is to be balanced by core thermonuclear power. Second, the ignition temperature is not some fundamental quantity but is in part determined by the specific nuclear physics of the relevant thermonuclear process, in this case the low-temperature exothermic proton-proton reactions to deuterium, ^{3}He, and ^{4}He. Hence, we would need the details of the interaction physics of the proton-proton chain, integrated over the Maxwell-Boltzmann distribution of the relative proton-proton energies and over the temperature and density profiles of the stellar core. However, a simpler approach is possible (20). We can do this because the thermonuclear interaction rate is greatest at large particle kinetic energies, which are Boltzmann suppressed ($\propto e^{-E/k_B T}$), and because Coulomb repulsion between the protons requires that they barrier penetrate (quantum tunnel) to within range of the nuclear force. This fact introduces the Gamow factor (24)

$$e^{-\frac{2e^2}{\hbar v}} = e^{-\frac{2\alpha c}{v}} = e^{-\frac{b}{\sqrt{E}}}, \quad [18]$$

where $E = 1/2 \mu_p v^2$, μ_p here is the reduced proton mass $(m_p/2)$, v is the relative speed, and this equation defines b as $\alpha c \sqrt{\mu_p/2}$. The reaction rate contains the product of the Boltzmann and Gamow exponentials and the necessary integral over the thermal Maxwell-Boltzmann distribution yields another exponential of a function of T. That function is determined by the method of steepest descent, whereby the product of the Boltzmann and Gamow factors is approximated by the exponential of the extremal value of the argument: $-\frac{E}{k_B T} - \frac{b}{\sqrt{E}}$. That Gamow peak energy (E_{gam}) is $(bk_B T/2)^{2/3}$. The result is a term

$$e^{-\frac{E}{k_B T}} e^{-\frac{b}{\sqrt{E}}} \rightarrow e^{-\frac{3E_{gam}}{k_B T}}, \quad [19]$$

in the thermonuclear rate. Although the surface luminosity depends on metallicity and details of the opacity, that dependence is not very strong (23). Furthermore, due to the Boltzmann and Gamow exponentials (Eq. **19**), any characteristic temperature we might derive (which might be set equal to T_{max}) depends only logarithmically on the problematic rate prefactor. This argument allows us to focus on the argument of the exponential in Eq. **19**. Therefore, we set $\frac{3E_{gam}}{k_B T}$ in Eq. **19** equal to a dimensionless number f_g of order unity (but larger) and presume that it, whatever its value, is a universal dimensionless number, to within logarithmic terms. The resulting equation for the critical T is

$$k_B T_{max} = \frac{27}{16 f_g^3} \alpha^2 m_p c^2 \quad [20]$$

In Eq. **20**, both α and m_p come from the $\frac{2e^2}{\hbar v}$ term associated with Coulomb repulsion in Gamow tunneling physics. Setting T_{max} in Eq. **20** equal to T_{max} in Eq. **17**, we obtain

$$\begin{aligned} M_s &\sim m_{pl} \left(\frac{\eta_p}{f_g^{9/8}} \right)^2 \left(\frac{\eta_e}{\eta_p} \alpha^2 \right)^{3/4} \left(\frac{27\pi^2}{8} \right)^{3/4} \frac{Y_e^{5/4}}{\mu^{3/4}} \\ &\sim \frac{M_{Ch}}{f_g^{9/4}} \left(\frac{\eta_e}{\eta_p} \alpha^2 \right)^{3/4} \left(\frac{27\pi^2}{8} \right)^{3/4} \frac{Y_e^{5/4}}{\mu^{3/4}} \\ &\sim m_p \left(\frac{e^2}{G m_p^2} \right)^{3/2} \left(\frac{\eta_e}{\eta_p} \right)^{3/4} \left(\frac{27\pi^2}{8 f_g^3} \right)^{3/4} \frac{Y_e^{5/4}}{\mu^{3/4}} \\ &\propto m_p \left(\frac{\alpha}{\alpha_g} \right)^{3/2} \left(\frac{\eta_e}{\eta_p} \right)^{3/4} \\ &\propto m_{pl} \eta_p^2 \alpha^{3/2} \left(\frac{\eta_e}{\eta_p} \right)^{3/4}. \end{aligned} \quad [21]$$

If we now set $f_g \sim 5$ and use reasonable values for Y_e and μ, we finally obtain a value of $\sim 0.1\ M_\odot$ for M_s.

Importantly, we tethered the minimum main sequence mass to the fundamental constants G, e, m_e, and m_p, but in fact c and $\hbar$ have cancelled! M_s depends only on m_p and the ratio α/α_g in a duel between electromagnetism and gravity. The last expression for M_s in Eq. **21** reveals something else. It is the same as the expression in Eq. **13** for the mass of the largest rocky planet $\left(m_p \left(\frac{\alpha}{\alpha_g} \right)^{3/2} \right)$, but with an additional factor of $(\eta_e/\eta_p)^{3/4}$. This factor is ∼300 and is, as one would expect, much larger than one. As a result, we find that, given our simplifying assumptions and specific values for Y_e and μ, M_s/M_{rock} depends only on $(\eta_e/\eta_p)^{3/4}$. For measured values of m_p and m_e, M_s/M_{rock} is then ∼100; the mass of the lowest mass star exceeds that of the most massive rocky planet. If we now scale M_s to the detailed theoretical value of M_{rock}, we obtain a value for M_s that is within a factor of 2–3 of the correct value. This quantitative correspondence is a bit better than might have been expected but is heartening and illuminating nevertheless.

Characteristic Mass of a Galaxy

For stars and planets, we were not concerned with whether they could be formed in the universe (nature seems to have been quite fecund, in any case), but with the masses (say, maximum and minimum) that circumscribed and constrained their existence. However, because of the character of galaxies (as accumulations of stars, dark matter, and gas), the mass we derive for them here is that for the typical galaxy in the context of galaxy formation, and we find this approach to be the most productive, informative, and useful.

First, we ask the following. Is it obvious on simple physical grounds that a galaxy mass will be much greater than a stellar mass, i.e. that a galaxy will contain many stars? The distribution of the stellar masses of galaxies $(M_{star,gal})$ is empirically known to satisfy the so-called Schechter form with probabilities distributed as

$$dP \propto \left(M_{star,gal}/M^* \right)^{-\alpha_p} e^{-M_{star,gal}/M^*} \times d\left(M_{star,gal}/M^* \right), \quad [22]$$

with α_p typically having the value of ∼1.2 and M^* a constant with units of mass. Given this form, with a weak power law at low masses and an exponential cutoff at high masses, there is a characteristic mass for galaxies, $\langle M_{star,gal} \rangle$, the value being somewhat greater than M^*, or roughly $10^{11}\ M_\odot$; galaxies much more massive than this are exponentially rare. At the other extreme, there is very little mass bound up in galaxies having masses much less than $10^7\ M_\odot$. The observed range of masses seems to be set by fundamental physics in that it does not appear to be very dependent

‡A better number for T_{max} is $\sim 3.5 \times 10^6$ K.

on the epoch of galaxy formation or the environment in which the galaxies are formed (e.g., in clusters, groups, or the field).

The theory of galaxy formation is by now fairly well developed, with *ab initio* hydrodynamic computations based on the standard cosmological model providing reasonably good fits to the formation epochs, masses, sizes and spatial distributions of galaxies, although this theory still provides rather poor representations of their detailed interior structures. For a recent review of some of the outstanding problems, see Ostriker and Naab (25). In the standard ΛCDM cosmological model, the mean density of matter at high redshifts is slightly less than the critical density for bound objects to form. However, there is a spectrum of perturbations such that those that are several σ more dense than average are gravitationally bound and will collapse, with the dark matter and the baryons forming self-gravitating lumps of radius R_{halo}, determined by the requirement that the density of the self-gravitating system formed in the collapse is several hundred times the mean density of the universe at the time of the collapse (26). The temperature of the gas (absent cooling) will be the virial temperature ($C^2 \sim GM_{tot}/R_{halo}$, where C is the speed of sound). If the gas can cool via radiative processes given its temperature and density, it will further collapse to the center of the dark matter halo within which it had been embedded and will form a galaxy, some fraction of the gaseous mass being formed into stars and a comparable fraction ejected by feedback processes subsequent to star formation.

The lower bound for normal galaxy masses is not well understood but is thought to be regulated by mechanical energy input processes such as stellar winds and supernovae, all primarily driven by the most massive ($\geq$20 $M_\odot$) stars comprising approximately one-sixth of the total stellar mass. One can easily show that roughly 10^{51} ergs in mechanical energy input per star is able to drive winds from star-forming galaxies with velocities of ~300 km/s, and Steidel et al. (27) observed such winds to be common. Because velocities of this magnitude are comparable to the gravitational escape velocities for systems less massive than our Milky Way, galaxy formation becomes increasingly inefficient for low-mass systems and essentially ceases when the sound speed of gas photoheated by the ultraviolet radiation from massive stars (10 → 20 km/s) approaches the escape velocity of low-mass dark matter halos. The result is a lower bound for normal galaxies and, in fact, the escape velocity from galaxies near the observed lower bound is of the order 30 km/s (26).

The physical argument for the upper bound and the typical mass is somewhat more complex. A collapsing proto-galactic clump has an evolving density (ρ) and temperature (T). Its collapse timescale (t_f) is set by the free-fall time due to gravitation and is proportional to $1/\sqrt{G\rho}$. As the clump collapses and the temperature and density rise, the gas radiates photons. Its associated characteristic cooling time (t_c) is the ratio of the internal energy density to the cooling rate. Because the cooling rate per gram (Eq. **23**) scales as density to a higher power than the free-fall rate, only the most overdense perturbations can cool on a time comparable to the free-fall time. Those for which $t_c \gg t_f$ will never form stars on either the free-fall time or the only somewhat (factor of 20) longer Hubble time. Moreover, because the highest-density regions are exponentially rare, perturbations having $t_c \ll t_f$ are uncommon. Thus, the condition $t_f \sim t_c$ sets a natural and preferred scale for galaxy formation. How does this scale compare with the Jeans mass (M_{Jeans}), the mass for which gravitational and thermal energies are in balance and which is proportional to $T^{3/2}/\rho^{1/2}$? In principle, the $t_f \sim t_c$ condition will imply a relationship between T and ρ that might yield $M_{star,gal}$s and M_{Jeans} that are functions of ρ or T and, hence, may not be universal. A wide range of values for M_{gal} would vitiate the concept of a preferred mass scale.

However, here nature comes to the rescue. The cooling rate (energy per volume per time) of an ideal gas of hydrogen can be approximated (following refs. 7, 8, and 28) by the formula

$$\Lambda_C \sim \left(A_{bf} + A_{ff} T\right) \frac{\rho^2}{T^{1/2}}, \qquad [23]$$

where A_{bf} is the bound-free (recombination) rate coefficient, and A_{ff} is the corresponding free-free (bremsstrahlung) rate coefficient. For the exploratory purposes of this study, these coefficients are

$$A_{ff} \sim \frac{2^{9/2}\pi^{1/2}}{3^{3/2}} \frac{e^4 \alpha k_B^{1/2}}{m_e^{3/2} m_p^2 c^2}$$
$$A_{bf} \sim \alpha^2 \frac{m_e c^2}{k_B} A_{ff}. \qquad [24]$$

In Eqs. **23** and **24**, m_e, e, and α appear due to the importance of electromagnetic radiation processes. As Eq. **23** suggests, free-free cooling exceeds bound-free cooling at high temperatures. Eq. **24** indicates that the cross-over temperature, below which recombination cooling predominates, is $\sim \alpha^2 \frac{m_e c^2}{k_B}$, which, using measured numbers, is $\sim 3 \times 10^5$ K. The temperatures of relevance during the incipient stages of galaxy formation are not much larger than this, so we can neglect the A_{ff} term in Eq. **23** and find that $\Lambda_C \propto \frac{\rho^2}{T^{1/2}}$. For an ideal gas, the internal energy density is $\frac{3/2 \rho k_B T}{\mu m_p}$. Therefore, $t_c \sim \frac{\rho k_B T}{m_p \Lambda_C} \propto T^{3/2}/\rho$. Because $t_f \propto 1/\rho^{1/2}$, t_c/t_f is proportional to $T^{3/2}/\rho^{1/2}$; this is proportional to M_{Jeans}, the Jeans mass! Therefore, we find that the $t_c/t_f \sim 1$ condition filters out a specific mass. What is its value? From $t_c/t_f \sim 1$ and the proper expression for M_{Jeans} $(=M_{gal})$, we derive

$$M_{gal} \sim m_p \frac{\alpha^5}{\alpha_g^2} \left(\frac{\eta_e}{\eta_p}\right)^{1/2}$$
$$\sim M_{Ch} \eta_p \alpha^5 \left(\frac{\eta_e}{\eta_p}\right)^{1/2} \qquad [25]$$
$$\sim m_{pl} \eta_p^3 \alpha^5 \left(\frac{\eta_e}{\eta_p}\right)^{1/2}.$$

Note that, with Eq. **25**, we obtained M_{gal} not only in terms of m_{pl}, η_p, η_e, and α, but should we wish to so express it, in terms of the familiar constants G, ħ, and c (as well as η_e, η_p, and α). Note also that η_p is a very large number and more than compensates for the smallness of α^5. The appearance of m_e, c, and α is a natural consequence of cooling's dependence on electromagnetic processes. As an indication of the importance of quantum mechanics in determining galaxy characteristics, ħ does not cancel.

For measured values of the fundamental constants and retaining the prefactor dropped in Eq. **25** $\left(\frac{2^{5/2}\pi^{7/2}}{27}\right)$, but retained in Silk (8), we find $M_{gal} \sim 10^{11}$ $M_\odot$, reassuringly close to the characteristic mass in stars (M^*) of the average L^* galaxy in our universe. Moreover, one can derive a Jeans length and, hence, a length scale for this average galaxy. It is

$$R_{gal} \sim \frac{\hbar}{m_e c} \alpha^3 \eta_p^2 \left(\frac{\eta_e}{\eta_p}\right)^{1/2} \sim R_{pl} \alpha^3 \left(\frac{\eta_e}{\eta_p}\right)^{3/2},$$

an expression that scales with the Compton wavelength of the electron. Plugging in measured numbers gives $R_{gal} \sim 50$ kiloparsecs, a number well within reason in our universe (and, in fact, for our own Milky Way).

Although it is reassuring that the critical mass that appears from our dimensional analysis corresponds well to the upper mass range of normal galaxies, we are left with two questions. First, there exist galaxies that are up to ∼10 times greater in mass than M^*; how do these form? Second, what happens to dark matter lumps that are much more massive than this critical mass and are dense enough to collapse? The answer to the first question is becoming observationally clear. All of these supergiant galaxies are brightest cluster galaxies (BCGs). We now know that they form at early times, reach a mass comparable to M^*, and cease star formation but keep on growing in mass (by roughly a factor of 2–3) and size (by roughly a factor of 4–8). The process by which this happens is "galactic cannibalism" (29), by which gravitationally induced dynamical friction causes the inspiral of satellite galaxies to merge with the central galaxy. Thus, the excessive mass of BCGs is caused by a distinct process of mass growth. With regard to the second question, the answer again lies in observations of groups and clusters. If these giant, self-gravitating units (dark matter halos) have total masses far above $(\Omega_{matter}/\Omega_{baryon})\ M^*$, i.e. greater than $10^{12}\ M_\odot$, then they host not one giant mass galaxy but rather a distribution of galaxy masses, the distribution of which is given by Eq. **22**.[§] Thus, we find it natural that most of the mass in the universe is in stellar systems containing roughly 10^{11} stars, each with mass between the limits M_s and M_S, which both scale with the Chandrasekhar mass.

Conclusion

We now recapitulate in succinct form most of the masses discussed in this paper. First, we express our results in the most fundamental units

$$\begin{aligned} M_{rock} &\sim m_{pl}\eta_p^2\alpha^{3/2} \\ M_s &\sim m_{pl}\eta_p^2\alpha^{3/2}\left(\frac{\eta_e}{\eta_p}\right)^{3/4} \\ M_S &\sim 50\ m_{pl}\eta_p^2 \\ M_{Ch} &\sim m_{pl}\eta_p^2 \\ M_{NS} &\sim m_{pl}\eta_p^2\left(\frac{\eta_\pi}{2\beta_n\eta_p}\right)^{3/2} \\ M_{ns} &\sim m_{pl}\eta_p^2\left(\frac{\eta_p}{\eta_\pi}\right)^{3} \\ M_{gal} &\sim m_{pl}\eta_p^3\alpha^5\left(\frac{\eta_e}{\eta_p}\right)^{1/2}. \end{aligned} \qquad [26]$$

Eq. **26** reduces the maximum mass of a rocky planet (M_{rock}), the minimum mass of a star (M_s), the maximum mass of a star (M_S), the maximum mass of a white dwarf (M_{Ch}), the maximum mass of a neutron star (M_{max}), the minimum mass of a neutron star (M_{ns}), and the characteristic mass of a galaxy (M_{gal}) to only five simple quantities and makes clear a natural mass hierarchy related only to m_p, particle mass ratios, and the αs. We expressed the basics of important astronomical objects with only five constants, modulo some dimensionless numbers of order unity. Eq. **26** summarizes the interrelationships imposed by physics between disparate realms of the cosmos.

Alternatively, it is instructive to put these same relations into a somewhat more familiar form. We drop all dimensionless constants of order unity and summarize the relations derived for the masses of astronomical bodies in units of the Chandrasekhar mass (which is close to a solar mass), the three particle masses (m_p, m_π, and m_e), and the Planck mass, m_{pl}, with $M_{Ch} \sim m_p(m_{pl}/m_p)^3 \sim 1M_\odot$.

We found that neutron stars can exist within the range

$$(m_\pi/m_p)^3 < M/M_{Ch} < (m_p/m_\pi)^{3/2}.$$

Normal stars can exist within the range

$$\left((m_p/m_e)\alpha^2\right)^{3/4} < (M/M_{Ch}) < \sim 50,$$

and rocky planets can exist with masses

$$M/M_{Ch} < \alpha^{3/2},$$

which is comfortably smaller than the minimum mass of stars. Finally, normal galaxies have a characteristic mass

$$M/M_{Ch} \sim \alpha^5(m_{pl}/m_p)(m_p/m_e)^{1/2},$$

which is larger than the characteristic mass of both normal low mass stars and even the most massive stars by a very large factor.

Everything astronomical is indeed connected, and that the essence of an object can be reduced to a few central quantities is one of the amazing consequences of the unifying character of physical law. Indeed, in this exercise, we focused on stars, planets, and galaxies, avoided complexity, eschewed any hint that emergent phenomena might be of fundamental import, and ignored topics such as life and the ubiquitous complexity that clutters most experience. Rather, our goal here was to understand and articulate the simple connections inherent in the universal operation of a small number of physical principles and fundamental constants and to identify the ties between seemingly unrelated, but key, astronomical entities. We hope we conveyed to the reader that not only are these connections knowable and quantifiable but that they are both simple and profound.

ACKNOWLEDGMENTS. We thank Jeremy Goodman and Neta Bahcall for insightful comments on an earlier draft of this manuscript and Paul Langacker for stimulating conversations.

[§] Ω is the ratio of the density of a mass-energy component of the universe to the critical total density. Here, $\Omega_{matter}/\Omega_{baryon}$ is the ratio of the total matter density (dark matter plus regular matter) to the density of regular matter (baryons).

1. Weisskopf VF (1975) Of atoms, mountains, and stars: A study in qualitative physics. *Science* 187(4177):605–612.
2. Carr BJ, Rees MJ (1979) The anthropic principle and the structure of the physical world. *Nature* 278:605–612.
3. Rees M (1999) *Just Six Numbers: The Deep Forces That Shape the Universe* (Weidenfeld & Nicolson, London).
4. Press WH (1980) Man's size in terms of fundamental constants. *Am J Phys* 48(8):597–598.
5. Padmanahan T (2000) *Theoretical Astrophysics: Volume 1, Astrophysical Processes* (Cambridge Univ Press, Cambridge, UK).
6. Clayton DD (1968) *Principles of Stellar Evolution and Nucleosynthesis* (Univ of Chicago Press, Chicago).
7. Rees MJ, Ostriker JP (1977) Cooling, dynamics and fragmentation of massive gas clouds: Clues to the masses and radii of galaxies and clusters. *Mon Not R Astron Soc* 179:541–559.
8. Silk J (1977) On the fragmentation of cosmic gas clouds. I. The formation of galaxies and the first generation of stars. *Astrophys J* 211:638–648.
9. Greiner W, Schäfer A (1994) *Quantum Chromodynamics* (Springer, New York).
10. Wilczek F (1982) Quantum chromodynamics: The modern theory of strong interactions. *Ann Rev Nucl Part Sci* 32:177–209.
11. Weinberg S (1996) *The Quantum Theory of Fields, Vol. 2: Modern Applications* (Cambridge Univ Press, Cambridge, UK).
12. Dürr S, et al. (2008) Ab initio determination of light hadron masses. *Science* 322(5905):1224–1227.
13. Chandrasekhar S (1939) *Introduction to the Study of Stellar Structure* (Dover Publishing Co., New York).
14. Stoner EC (1930) The equilibrium of white dwarfs. *Philos Mag* 9:944–963.
15. Lattimer JM, Prakash M (2007) Neutron star observations: Prognosis for equation of state constraints. *Phys Rep* 442(1–6):109–165.
16. Haensel P, Zdunik JL, Douchin F (2002) Equation of state of dense matter and the minimum mass of cold neutron stars. *Astron Astrophys* 385:301–307.
17. Wigner E, Huntington HB (1935) On the possibility of a metallic modification of hydrogen. *J Chem Phys* 3:764–770.
18. Zapolsky HS, Salpeter EE (1969) The mass-radius relation for cold spheres of low mass. *Astrophys J* 158:809–813.
19. Eddington AS (1926) *The Internal Constitution of the Stars* (Cambridge Univ Press, Cambridge, UK).
20. Adams FC (2008) Stars in other universes: Stellar structure with different fundamental constants. *J Cosmol Astropart Phys* 8:10–38.
21. Humphreys RM, Davidson K (1984) The most luminous stars. *Science* 223(4633):243–249.
22. Hoyle F, Fowler WA (1964) Neutrino processes and pair formation in massive stars and supernovae. *Astrophys J* 9(Suppl):201–319.
23. Burrows A, Liebert J (1993) The science of brown dwarfs. *Rev Mod Phys* 65(2):301–336.
24. Gamow G (1928) On quantum theory of atomic nuclei. *Z Phys* 51:204–212.
25. Ostriker JP, Naab T (2012) Theoretical challenges in understanding galaxy evolution. *Phys Today* 65(8):43–49.
26. White SDM, Rees MJ (1978) Core condensation in heavy halos: A two-stage theory for galaxy formation and clustering. *Mon Not R Astron Soc* 183:341–358.
27. Steidel CC, et al. (2004) A survey of star-forming galaxies in the $1.4 \le Z \le 2.5$ redshift desert: Overview. *Astrophys J* 604(2):534–550.
28. Spitzer L (1948) The temperature of interstellar matter. *Astrophys J* 107:6.
29. Hausman MA, Ostriker JP (1978) Galactic cannibalism. III. The morphological evolution of galaxies and clusters. *Astrophys J* 224:320–336.