WHITE DWARFS (DEGENERATE DWARFS)

White dwarfs are stars supported by pressure of degenerate electron gas, i.e. in their interiors thermal energy kT is much smaller then Fermi energy E_F. We shall derive the equations of structure of white dwarfs, sometimes called degenerate dwarfs, in the limiting case when their thermal pressure may be neglected, but the degenerate electron gas may be either non-relativistic, somewhat relativistic, or ultra-relativistic.

The mass - radius relation for white dwarfs may be estimated using the usual algebraic approximation to the differential equations of stellar structure and an analytical approximation to the equation of state for degenerate electron gas. The equations of stellar structure may be approximated with:

$$\frac{M}{R} \approx R^2 \rho, \qquad \text{i.e.} \qquad \rho \approx \frac{M}{R^3}, \tag{wd.1}$$

$$\frac{P}{R} \approx \frac{GM}{R^2}\rho \approx \frac{GM^2}{R^5}, \qquad \text{i.e.} \qquad P \approx \frac{GM^2}{R^4}. \tag{wd.2}$$

The equation of state may be approximated as

$$P \approx \left[\left(K_1 \rho^{5/3} \right)^{-2} + \left(K_2 \rho^{4/3} \right)^{-2} \right]^{-1/2}. \tag{wd.3}$$

Equations (wd.1) , (wd.2) , and (wd.3) may be combined to obtain

$$P^{-2} \approx \frac{R^8}{G^2 M^4} \approx \frac{R^{10}}{K_1^2 M^{10/3}} + \frac{R^8}{K_2^2 M^{8/3}}. \tag{wd.4}$$

This may be rearranged to have

$$R \approx \frac{K_1}{GM^{1/3}} \left[1 - \frac{G^2 M^{4/3}}{K_2^2} \right]^{1/2}. \tag{wd.5}$$

The last equation should have the correct asymptotic form, but there may be dimensionless coefficients of the order unity that our approximate analysis cannot provide. However, we may recover the coefficients noticing that in the two limiting cases, $\rho \ll 10^6 \text{ g cm}^{-3}$, and $\rho \gg 10^6 \text{ g cm}^{-3}$, the equation of state is very well approximated with a polytrope with index $n = 1.5$ and $n = 3$, respectively. In these two limiting cases we have exact mass - radius relations:

$$R = \frac{K_1}{0.4242 \ GM^{1/3}} \qquad \text{for} \ \ n = 1.5, \tag{wd.6}$$

$$M = \left(\frac{K_2}{0.3639 \ G} \right)^{1.5} = 1.142 \times 10^{34} \ \mu_e^{-2} \ \ (\text{g}) = 5.82 \ M_\odot \mu_e^{-2}, \qquad \text{for} \ \ n = 3. \tag{wd.7}$$

Note that the value of the polytropic constant N_3 is 0.3639 and

$$K_2 = \frac{1}{4} \left(3\pi^2 \right)^{\frac{1}{3}} \hbar c \left(N_A Y_e \right)^{4/3} .$$

Combining these equations, we may write

$$R \approx \frac{K_1}{0.4242 \ GM^{1/3}} \left[1 - \left(\frac{M}{M_{Ch}} \right)^{4/3} \right]^{1/2} = \tag{wd.8}$$

$$0.0126 \; R_\odot \left(\frac{2}{\mu_e}\right)^{5/3} \left(\frac{M}{M_\odot}\right)^{-1/3} \left[1 - \left(\frac{M}{M_{Ch}}\right)^{4/3}\right]^{1/2} ,$$

with the **Chandrasekhar mass**

$$M_{Ch} = 1.456 \; M_\odot \left(\frac{2}{\mu_e}\right)^2 = 1.456 \; M_\odot \left(2Y_e\right)^2 , \qquad \text{where } \mu_e = \frac{1}{Y_e}. \tag{wd.9}$$

Importantly, the Chandrasekhar mass is a function only of fundamental constants:

$$M_{Ch} \sim (\hbar c/G)^{\frac{3}{2}} \, Y_e^2/m_p^2.$$

The analytical formula (eq. wd.8) approximates the exact numerical mass - radius relation for white dwarfs with an error smaller than 15% for masses near Chandrasekhar limit, and much better accuracy at lower masses.

The following table gives a comparison between the numerical and analytical values of white dwarf radii for $\mu_e = 2$ ($Y_e = 0.5$). The first column gives the logarithm of central density, the second white dwarf mass in units of $M_\odot$, the third and fourth give numerical and analytical white dwarf radii, respectively, in units of $R_\odot$, and the fifth column gives the fractional error of the analytical radii.

$\log \rho_c$	$M/M_\odot$	$R/R_\odot$		error
		numerical	analytical	
4	.04811	.03448	.03446	.0008
5	.14600	.02339	.02335	.0015
6	.39366	.01566	.01558	.0048
7	.80146	.01013	.00997	.0158
8	1.16176	.00619	.00593	.0411
9	1.34619	.00353	.00325	.0803
10	1.41096	.00188	.00165	.1230

Variational Principle: The radius-mass relation for a white dwarf (or any star) can be derived from a variational principle. Writing the total energy as the sum of the internal and the gravitational energies, and writing each using $\rho \sim M/R^3$ as functions of the total mass and radius, one can derive $R(M)$ by taking the derivative of the energy with respect to R and setting the result to zero. In this way, one can derive the $R \sim M^{-1/3}$ relation in the NR limit and the fact that there is a singular mass (the Chandrasekhar mass) in the relativistic limit. This approach also provides insight into the stability of the fundamental radial mode. In the relativistic limit, the star is neutrally (un)stable. In the NR limit, the period of oscillation is the dynamical time ($\sim 1/\sqrt{G\rho}$).