Nuclear Masses and Mass Excess: Q values for Nuclear Reactions

Nuclear masses for a given atomic weight (A) and nuclear charge/atomic number (Z) can be approximated by the **semi-empirical mass formula**

$$\begin{aligned} M(A,Z) &= (A-Z)m_n + Z(m_p+m_e) \\ &\quad - a_1 A + a_2 A^{2/3} + a_3\frac{(A/2-Z)^2}{A} + a_4\frac{Z^2}{A^{1/3}} + a_5\frac{\delta}{A^{3/4}}\,. \end{aligned} \tag{1}$$

Note that implicit in eq. (1) is that the mass density of the nucleus is a constant ($\sim 2.6\times10^{14}$ g cm^{-3}) and that the radius of the spherical nucleus is $\sim$1.2 $\times$ 10^{-13} cm (= 1.2 fermi). The first term ($(A-Z)m_n + Z(m_p+m_e)$) is just the rest masses of the nucleons (and electrons), the second is the **bulk** nuclear term proportional to the number of nucleons in the nucleus, the third term is the bulk **"surface tension"** term proportional to the surface area of the nucleus, the fourth term is the **"symmetry energy"** term that is proportional to the neutron-proton asymmetry in the nucleus, the fifth term is the **Coulomb** term ($\propto Z^2e^2/a$), and the last term is the **pairing** term. This formula was originally inspired by the **"liquid drop"** model of the nucleus and is quite classical and empirical. The original theory of nuclear fission involved the relative values of the surface tension and Coulomb terms. It was determined that a nucleus would fission in twain if Z^2/A was greater than roughly $\sim$45. For values of this *fission parameter* lower than $\sim$37, the mean time for spontaneous fission is longer than the age of the Universe.

The difference between the **atomic mass** and the **nuclear mass** is the atomic binding energy of the electrons, plus the rest mass of the electron. Take care to determine which mass you use from tables in the literature. Generally, the binding energy of the electrons in the atom ($\sim$eVs to keVs) are too small to compete with the nuclear terms ($\sim$MeVs) and you needn't worry, but for nuclei with the largest atomic numbers the atomic correction can be $\sim$50-100 keV. In a strong or electromagnetic nuclear reaction, since total nuclear charge is conserved, the Zm_e term in eq. (1) drops out when determining Q values. For a weak interaction, since the nuclear charge changes, the extra m_e must be accounted for explicitly.

More specifically, the coefficients in eq. (1) have the following approximate values measured in MeV/c^2:

$$a_1 = 15.53,\ a_2 = 17.804,\ a_3 = 94.77,\ a_4 = 0.7103,\ a_5 = 33.6.$$

The nucleon and electron masses (in MeV) are: $m_n \approx 939.57$, $m_p \approx 938.27$, and $m_e \approx 0.511$ MeV/c^2. The term in a_1 represents an increase in the binding energy (*i.e.*, decrease in nuclear mass) due to nearest-neighbor interactions between nucleons: to lowest order, nuclei are rather like drops of liquid, in which the interactions are very short range, and are attractive at low pressure, but are strongly repulsive under compression. Consequently, the liquid prefers to maintain a roughly constant density and the volume of the nucleus $\propto A$. Just as in most liquids, there is a positive energy associated with the surface area of the drop because particles near the surface have fewer neighbors to bond with: this is represented by the positive term $a_2 A^{2/3}$. The next term (in a_3) favors comparable numbers of neutrons $(A-Z)$ and protons (Z). However, the term in a_4 is the electrostatic energy of the protons, which causes a shift toward neutron-rich nuclei (*i.e.*, $A - Z > Z$) with increasing atomic number. The last term reflects an attractive pairing between like nucleons: $\delta = -1$ if the number of protons and the number of neutrons are both even, $\delta = 0$ if A or Z but not both is odd, and $\delta = +1$ if both are odd. Nuclei with larger binding energies per nucleon

$$\frac{B}{A} \equiv -\frac{M(A,Z) - (A-Z)m_n - Z(m_p + m_e)}{A} \tag{2}$$

tend of course to be stabler. ^{56}Fe is the most bound common nucleus, ^{56}Ni is the most bound nucleus for symmetric $(A - Z = Z)$ nuclei, and ^{62}Ni is the most bound nucleus of all. Notice that all these "most bound nuclei" are at the **"iron peak."**

Q Values

The Q value of a nuclear reaction is the difference between the sum of the masses of the initial reactants and the sum of the masses of the final products, in energy units (usually in MeV). This is also the corresponding difference of the binding energies of the nuclei (not per nucleon), since nucleon number is conserved in a reaction. The masses may be provided in a table of **mass excesses** ($\Delta M(A, Z)$), which is the value of $M(A, Z) - Am_u$ (usually in MeV), but relative to the corresponding number for the isotope ^{12}C. ($m_u \equiv m_{amu}$.) A useful table of mass excesses can be found in Clayton's book, p. 289 (his Table 4.1). Hence, the "mass excess" of ^{12}C is defined to be zero. For instance, for the reaction $^{12}\mathrm{C}(\alpha, \gamma)\,^{16}\mathrm{O}$, the Q value is

$$\begin{aligned} Q &= 931.478 \text{ MeV } \left(M(^{12}\mathrm{C}) + M(^{4}\mathrm{He}) - M(^{16}\mathrm{O})\right) \\ &= 7.1613 \text{ MeV}\,, \end{aligned} \tag{3}$$

where the mass excess of ^{16}O is negative (oxygen is more bound than carbon). For the triple-α process, one finds $Q/A = 0.606$ MeV/nucleon, for hydrogen burning it is 26.73 MeV, or ~6.7 MeV per nucleon. Note that 1 MeV/nucleon is equivalent to 0.965×10^{18} erg g^{-1}, a very useful conversion factor. Note also that for hydrogen burning the efficiency of conversion of mass into energy is **~6.7/931≈0.007**, less than but near 1%. This is the core fact of fusion as the source of energy for stars.

Magic Numbers and Abundances

There is a pronounced tendency for elements with odd Z to be less abundant than those with even Z, reflecting the influence of the last term in eq. (1). Although it is not accounted for in that formula, nuclei whose Z or $A - Z$ is one of the **magic numbers** $2, 8, 20, 28, 50, 82, 126$ are more strongly bound than their neighbors, and doubly magic nuclei such as ^{4}He, ^{16}O ($Z = 8$), ^{40}Ca ($Z = 20$), and ^{208}Pb ($Z = 82$) are particularly strongly bound, because they represent closed shells of nucleons orbiting within the mean-field nuclear potential. These "magic" isotopes are also more abundant than their neighbors. So the binding energy per nucleon clearly has some explanatory power. Note that the dominate stable isotopes tend to be the "α-nuclei" up to ^{40}Ca, after which they trend to more neutron-rich species. This is due to the increasing importance of the Coulomb term, which disfavors protons and requires that there be more neutrons as glue. The region in the (A, Z) plane of stability against nucleon (β) decay is called the **valley of beta stability**.

The reason that the heavier nuclei are less abundant is of course that the Coulomb barrier makes them difficult to form by fusion except at extreme temperatures. In the cores of massive stars at the end of their evolution, the temperature approaches $10^{10}\,\mathrm{K} \approx 1\,\mathrm{MeV}/k_{\mathrm{B}}$, so that elements up to the iron group (meaning $24 \leq Z \leq 28$, chromium through nickel) do form. Since these cores are strongly gravitationally bound, much of the heavier elements remain locked up in neutron stars or black holes, but some is returned to the interstellar medium in supernovae.

In general, the elements and isotopes of Nature are produced predominantly in specific environments, by specific processes. These include the **Big Bang**, **cosmic-ray spallation**, the **triple-α** process and $^{12}\mathrm{C}(\alpha, \gamma)\,^{16}\mathrm{O}$, the **CNO** cycle (e.g., ^{14}N), **supernovae**, the **s-process**, the **r-process**, the **p-process**, the **rp-process**, and the **ν-process**. Can you identifiy which isotopes come from which processes?

The Physics of Fusion in Stars and Elsewhere

Most stars derive their luminosity from the conversion of hydrogen to helium. The rest mass of one ^{4}He atom is about 0.71% less than the combined rest masses of four hydrogen atoms (note that the electrons are included in the atomic masses here). The difference, or about 26.7 MeV, is released as heat, except for ≈ 0.6 MeV worth of neutrinos (in the *pp* chain). There are two paths from $4\,^1H$ to ^4He: **the *pp* cycle**, which predominates in the Sun and cooler stars, and **the CNO cycle**, which predominates in stars with slightly higher central temperatures.

pp	**CNO**
$p + p \rightarrow {}^2\mathrm{H} + e^+ + \nu_e$	${}^{12}\mathrm{C} + p \rightarrow {}^{13}\mathrm{N} + \gamma$
${}^2\mathrm{H} + p \rightarrow {}^3\mathrm{He}$	${}^{13}\mathrm{N} \rightarrow {}^{13}\mathrm{C} + e^+ + \nu_e$
${}^3\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + 2p$	${}^{13}\mathrm{C} + p \rightarrow {}^{14}\mathrm{N} + \gamma$
	${}^{14}\mathrm{N} + p \rightarrow {}^{15}\mathrm{O} + \gamma$ (rate limiting)
	${}^{15}\mathrm{O} \rightarrow {}^{15}\mathrm{N} + e^+ + \nu_e$
	${}^{15}\mathrm{N} + p \rightarrow {}^{12}\mathrm{C} + {}^4\mathrm{He}$

Table 1: *The main channels of the pp and CNO cyles (Bahcall 1989).*

In Table 1, the isotopic designations refer to nuclei rather than whole atoms, so that ^{1}H would be equivalent to a proton, p. In some books, the helium nucleus is denoted by α instead of ^{4}He, and the deuterium nucleus by d instead of ^{2}H.

About 0.4% of *pp* reactions in the Sun start with $2p + e^- \rightarrow {}^2\mathrm{H} + \nu_e$ ("PEP" reaction) instead of the first reaction shown in the Table. About 15% involve

$$\begin{aligned} {}^3\mathrm{He} + {}^4\mathrm{He} &\rightarrow {}^7\mathrm{Be} + \gamma \\ {}^7\mathrm{Be} + e^- &\rightarrow {}^7\mathrm{Li} + \nu_e \\ {}^7\mathrm{Li} + p &\rightarrow {}^4\mathrm{He} + {}^4\mathrm{He} \end{aligned} \tag{4}$$

instead of the third reaction shown. Even more rarely (0.02%), the second and third reactions of (4) are replaced by

$$\begin{aligned} {}^7\mathrm{Be} + p &\rightarrow {}^8\mathrm{B} + \gamma \\ {}^8\mathrm{B} &\rightarrow {}^8\mathrm{Be}^* + e^+ + \nu_e \\ {}^8\mathrm{Be}^* &\rightarrow {}^4\mathrm{He} + {}^4\mathrm{He} \end{aligned} \tag{5}$$

in which ^{8}Be* is a metastable state. This last side chain is energetically negligible but experimentally important because it produces an exceptionally energetic neutrino (up to $\sim$14 MeV) which, though much rarer, is easier to detect than the paltry ≤ 0.420 MeV neutrino resulting from the first reaction in the Table.

The carbon, nitrogen, and oxygen in the CNO reactions serve as catalysts: no net production of these elements occurs. The second column of Table 1, for example, replaces the original ^{12}C nucleus. There is a side chain that goes through ^{16}O, but this also involves no net production of elements other than helium. Thus, even at high central temperatures, the CNO cycle could not have occurred in metal-free primordial high-mass stars.

On a per-proton basis, the *pp* and CNO cycles in stars proceed extremely slowly. Fusion has reduced the central hydrogen abundance of the Sun by about a factor of two in the 4.6 Gyr since its formation; thus the fusion rate per proton is $\approx 5\times10^{-18}\,\mathrm{s}^{-1}$. Let us compare this to a characteristic proton-proton collision rate, $n_p\sigma v_{\rm th}$, where $n \approx 6 \times 10^{25}\,\mathrm{cm}^{-3}$ is the central number density of protons and $v_{\rm th} = (3k_{\rm B}T_c/m_p)^{1/2} \approx 600\,\mathrm{km\,s}^{-1}$ is their thermal velocity. The choice of the collision cross section, σ, depends upon what one considers a collision. As will be seen later, a natural scale for cross sections is $\pi^2_{\rm dB}$ where $_{\rm dB} \equiv \hbar/mv$ is the reduced de Broglie wavelength. If $v = v_{\rm th}$ then $_{\rm dB} \approx 10^{-11}$ cm, and the collision rate $n_p\pi^2_{\rm dB}v \approx 10^{12}\,\mathrm{s}^{-1}$. Comparing this with the fusion rate estimated above, one sees that the probability of fusion per collision is $\sim 2 \times 10^{-31}$.

The rest of this lecture is devoted to explaining why the latter probability is so small. Actually, there are two principal reasons: the electrostatic repulsion between nuclei, and the weakness of the weak interactions. As a byproduct, we will see why the CNO cycle is so much more sensitive to temperature than the *pp* cycle.

Barrier Penetration: Non-Resonant Reactions

The strong force binds nucleons (protons and neutrons) in nuclei but has a limited range, of order one fermi: $1\,\mathrm{fm} \equiv 10^{-13}\,\mathrm{cm} = 10^{-15}\,\mathrm{m}$, so a fermi is also a femtometer. At separation r, the electrostatic energy between nuclei of charges $Z_1 e$ and $Z_2 e$ is $\approx 1.44 Z_1 Z_2\,\mathrm{MeV\,fm}/r$, whereas thermal energies are $\sim k_\mathrm{B}T \sim (T/10^7\,\mathrm{K})\,\mathrm{keV}$. Since the Boltzmann distribution falls off exponentially at $E \gg k_\mathrm{B}T$, and $T \approx 1.5\times 10^7\,\mathrm{K}$ at the center of the Sun, the probability that two colliding protons could approach within 1 fm would be $\sim e^{-670} \sim 10^{-290}$, if classical physics applied. Quantum-mechanical tunneling allows the protons to go "under" the Coulomb barrier with a probability that is much larger than this, though still exponentially suppressed.

WKB estimate of the penetration factor

If $U(r)$ is the Coulomb potential and $\ell = 0$, the particle is classically forbidden to be in this region if $r < R_E \equiv Z_1 Z_2 e^2/E$. We would like to calculate the radial probability current deep within the forbidden region where $r \sim 1\,\mathrm{fm} \sim 10^{-2} R_E$. While this can be done exactly for $U(r) = Z_1 Z_2 e^2/r$ in terms of special functions, a good approximation and a much more enlightening result can be found by WKB. We set (the incoming or outgoing part of) $\psi_0(r)$ equal to $\exp[\chi(r)]/r$, which would satisfy Schoedinger's equation,

$$\frac{\hbar^2}{2\mu r^2}\frac{d}{dr}\left(r^2\frac{d\psi_\ell}{dr}\right) + \left[E - U(r) - \frac{\hbar^2\ell(\ell+1)}{2\mu r^2}\right]\psi_\ell(r) = 0, \tag{6}$$

for a partial wave with angular momentum quantum number, ℓ, exactly if

$$\frac{d^2\chi}{dr^2} + \left(\frac{d\chi}{dr}\right)^2 = \frac{2\mu}{\hbar^2}[U(r) - E]. \tag{7}$$

The WKB approximation assumes that $d\chi/dr$ is large but slowly varying (at large r, $d\chi/dr \to \pm ik$, a constant), so that $|d^2\chi/dr^2| \ll |d\chi/dr|^2$. Then to leading order,

$$\chi(r) \approx \pm\frac{\sqrt{2\mu}}{\hbar}\int^{r} d\bar{r}\,\sqrt{U(\bar{r}) - E}\,. \tag{8}$$

The lower limit has been deliberately left unspecified, which is equivalent to allowing an arbitrary constant of integration. Plugging (8) into the previously neglected second-derivative term of (7), one can obtain a more accurate approximation for χ, though we will not need it here. The two choices for the sign in (8) yield two independent approximate solutions for $\psi_0(r)$. In the permitted region where the integrand of (8) is imaginary, the solution whose phase decreases (increases) with increasing r can be interpreted as the incoming (outgoing) wave.

The WKB approximation breaks down near the **turning point** $r = R_E$ because the integrand in (8) is not smooth there (its derivative is singular). There are standard prescriptions for matching the WKB solutions across the turning point. Since their derivation would take up too much space, we will just quote results.[1] Neglecting the small possibility of fusion, the outgoing and ingoing waves must have equal magnitudes. In this situation, the matching conditions say that we must use the upper sign of (8) at $r < R_E$, so that the solution decays inward into the forbidden zone. Let $R_0 \ll R_E$ be the range of the strong nuclear force. Then

$$\chi(R_E) - \chi(R_0) \approx \frac{\sqrt{2\mu}}{\hbar} \int_{R_0}^{R_E} d\bar{r} \sqrt{\frac{Z_1 Z_2 e^2}{\bar{r}} - E} \;=\; \frac{\sqrt{2\mu E}}{\hbar} R_E \int_{R_0/R_E}^{1} dx \sqrt{\frac{1}{x} - 1}$$
$$\approx \pi \sqrt{\frac{Z_1^2 Z_2^2 e^4 \mu}{2E\hbar^2}} = \pi Z_1 Z_2 \alpha \frac{c}{v_\infty}\,, \qquad (9)$$

where $\alpha \equiv e^2/\hbar c \approx 1/137$ is the fine-structure constant. To obtain the second line, we have replaced the lower limit of the x integral by 0: this makes only a small error because the singularity of the integrand at $x = 0$ is integrable.

The probability that the two nuclei come within R_0 is during a collision is of order

$$P_B \equiv \frac{R_0^2}{R_E^2} \frac{|\psi_0(R_0)|^2}{|\psi_0(R_E)|^2} \approx \exp\left[-2\pi \sqrt{\frac{Z_1^2 Z_2^2 e^4 \mu}{2E\hbar^2}}\right]. \qquad (10)$$

The additional factor of two in the exponential relative to (9) occurs because the wavefunction is squared. You might think that the factor R_0/R_E should be cubed rather than squared, to reflect the relative volumes, but the barrier penetration probability was originally defined by the physicist Gamow for radioactive *decay* by fission, and in that case it is the probability flux rather than the probability density that comes in. The most sensitive dependence on energy is due to the exponential factor in any case; discrepancies in the prefactor are absorbed into the nuclear factor $S(E)$ defined below. Even for two colliding protons in the solar core, where $Z_1 = Z_2 = 2$, $\mu = m_p/2$, and $(\overline{v_\infty^2})^{1/2} = \sqrt{6k_B T_c/m_p} \approx c/340$, the argument of the exponential is moderately large, ≈ -16, so $P_B \ll 1$, but this is not nearly small enough to explain the low probability of fusion per collision as estimated at the beginning of these notes ($\sim 10^{-31}$).

It is conventional to write the fusion cross section as the product of three factors:

1. The cross section $\sigma_0 = \pi^2 = \pi\hbar^2/2\mu E$ for the plane wave to intercept an $\ell = 0$ state.
2. The probability of what ever nuclear transition is necessary to transform the two nuclei into one once they come into "contact."
3. The probability of barrier penetration, as approximated by (10).

[1] Airy functions are involved; see any standard QM text.

Conventionally then, the energy-dependent fusion cross section is written as

$$\sigma(E) = E^{-1}S(E)\exp\left[-2\pi\sqrt{\frac{Z_1^2 Z_2^2 e^4 \mu}{2E\hbar^2}}\right],$$
$$= E^{-1}S(E)\exp\left[-\frac{b}{\sqrt{E}}\right], \qquad (11)$$

in which the three factors E^{-1}, $S(E)$, and the exponential correspond to items 1,2,3 above. The quantity b in the exponential is

$$b = \frac{2\pi}{\sqrt{2}} Z_1 Z_2 \alpha c \mu^{1/2}$$
$$= 31.3\ Z_1 Z_2 \left(\frac{\mu}{m_p}\right)^{1/2} (\,\mathrm{keV})^{1/2} \qquad (12)$$

when the energy, E, is in keV.

By far the slowest reaction in the *pp* chain is the first one (Table 1). The strong force is unable to bind two protons, *i.e.* the isotope ^{2}He has a negligible half-life. It can bind a proton and a neutron, though not terribly strongly: the binding energy of ^{2}H (deuterium) is 2.2 MeV, or about one tenth of the binding energy per nucleon of ^{4}He. So a beta decay must occur during the brief time that the two protons are in contact. All the other important weak decays in the *pp* and *CNO* cycles occur only after a bound (though only metastable) nucleus forms, so $S(E)$ is much larger for them.[2]

The factor $S(E)$ is hard to calculate because it involves nuclear structure. It is also difficult to measure experimentally at the relevent low energies ($E \lesssim 10\,\mathrm{keV}$), precisely because $\sigma(E)$ is terribly small. However, there is reason to believe that this factor should vary slowly with energy in the case of “nonresonant” reactions such as the first one in Table 1, so that it can be estimated by extrapolation. A reasonably recent estimate (Adelberger et al. 1998) is $S(E) \approx 4.00 \pm 0.03 \times 10^{-22}\,\mathrm{keV\,barn}$, where $1\,\mathrm{barn} = 10^{-24}\,\mathrm{cm}^2$. For application to stars, one averages the rate coefficient $v_\infty \sigma(E)$ over a thermal distribution of kinetic energies,

$$\overline{\sigma v}(T) = \left(\frac{8}{\pi\mu(k_\mathrm{B}T)^3}\right)^{1/2} \int_0^\infty \sigma(E) E e^{-E/k_\mathrm{B}T}\, dE,$$
$$= \left(\frac{8}{\pi\mu(k_\mathrm{B}T)^3}\right)^{1/2} \int_0^\infty S(E)\exp\left[-\frac{b}{\sqrt{E}} - \frac{E}{k_\mathrm{B}T}\right] dE\,. \qquad (13)$$

[2]An exception is $^3\mathrm{He} + p \rightarrow {}^4\mathrm{He} + e^+ + \nu_e$, but this is responsible for only $\sim 10^{-7}$ of ^{4}He production in the Sun.

Since $S(E)$ is presumed to be slowly varying, we may estimate this by the method of steepest descent: Writing the argument of the exponential as $-f(E)$, and noting that $f \gg 1$, we expect the integral to be dominated by the neighborhood of the energy E_0 that minimizes $f(E)$. Solving $f'(E_0) = 0$ yields

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3} \approx 1.22 \left(Z_1^2 Z_2^2 \mu/m_p T_6^2\right)^{1/3} \text{ keV}, \tag{14}$$

where $T_6 \equiv T/(10^6\,\text{K})$. Note $Z_1^2 Z_2^2(\mu/m_p) = 1/2$ for pp collisions. E_0 is called the **"Gamow peak" energy** and is a fundamental quantity in thermonuclear theory. It is the effective energy at which a reaction proceeds and is the energy near which experimenters want to design their measurements. The Gamow peak energy for the pp reaction in the Sun is about $4.2 k_\text{B} T_c$, so it is out on the tail of the thermal distribution, but not terribly far. For the $^{12}\text{C}(p,\gamma)^{13}\text{N}$ reaction, $E_0 \approx 3.9 T_6^{2/3}$ keV, far out on the thermal tail. Expanding $f(E) = \frac{b}{\sqrt{E}} + \frac{E}{k_\text{B}T} \approx f(E_0) + \frac{1}{2} f''(E_0)(E - E_0)^2$ around the minimum and replacing $S(E) \to S(E_0)$, one finds that

$$\overline{\sigma v}(T) \approx \left(\frac{8}{\pi\mu(k_\text{B}T)^3}\right)^{1/2} \left[\frac{2\pi}{f''(E_0)}\right]^{1/2} e^{-f(E_0)}\, S(E_0).$$

The quantity $f(E_0) = 3E_0/k_\text{B}T$, so the temperature dependence of non-resonant thermonuclear reactions is dominated by an exponential of the form $\exp[-(T_d/T)^{1/3}]$ as a result of a compromise between the barrier-penetration probability (which increases with energy) and the thermal distribution (which decreases). The constant $T_d \propto Z_1^2 Z_2^2 \mu$, where $\mu/m_p = A_1 A_2/(A_1 + A_2)$ and $A_{1,2}$ are the atomic weights of the nuclei. Note that μ is larger for the CNO reactions than for the pp ones. This is one reason the former are more temperature sensitive. On the other hand, they have much larger $S(E_0)$, so they dominate at higher temperatures (more massive or more evolved stars).

$f(E_0)$ is frequently written as $\tau = 3\left(\frac{b^2}{4k_\mathrm{B}T}\right)^{1/3}$. Therefore, the thermonuclear rate is proportional to

$$\tau^2 e^{-\tau}\,, \tag{15}$$

which has the form

$$\frac{1}{T^{2/3}}e^{-\frac{K}{T^{1/3}}}\,, \tag{16}$$

where $K \propto \left(Z_1^2 Z_2^2 \mu\right)^{1/3}$. This is the canonical temperature dependence of non-resonant thermonuclear rates. The Gamow peak has roughly a Gaussian shape

$$e^{-\left(\frac{E-E_0}{\Delta/2}\right)^2}\,, \tag{17}$$

where the width Δ is equal to $\frac{4}{\sqrt{3}}\left(E_0 k_\mathrm{B} T\right)^{1/2} \approx 0.75\left(Z_1^2 Z_2^2 \mu/m_p T_6^5\right)^{1/6}$.

If we want to write $\overline{\sigma v}(T)$ as something proportional to $\left(\frac{T}{T_0}\right)^n$ near a temperature T_0, using either eq. (15) or (16) we find that

$$\begin{aligned} n &= \frac{\partial \ln(\overline{\sigma v})}{\partial \ln(T)} \\ &= \frac{\tau - 2}{3}\,. \end{aligned} \tag{18}$$

For the *pp* chain where it applies ($T \sim 10^{6.5-7.5}$ K), $n \approx 4$. However, for the CNO cycle around $T \sim 10^{7.2-7.6}$ K, $n \approx 15-20$, a much higher power. These differences in sensitivity translate into interesting differences in the stellar context.

Resonant Reactions

A resonant reaction may be of the form:

$$A + a \to C \to B + b\,, \tag{19}$$

where state C is a compound nucleus with quantized energy levels that is an intermediate state. If the effective S-factor ($S(E)$), is not a slowly varying function of E, but a narrow resonance, then the cross section is in a **Breit-Wigner** form, basically a Lorentzian around the resonant energy:

$$\sigma_{res}(\ell) = (2\ell + 1)\,\frac{(2J_c + 1)}{(2J_a + 1)(2J_b + 1)}\,\frac{\Gamma_a \Gamma_b}{(E - E_r)^2 + \Gamma^2/4}\,, \tag{20}$$

where the factor with the $(2J_i + 1)$s accounts for the statistical weights, ℓ is the angular momentum quantum number of the incident particle, which is related to the classical impact parameter, E_r is the resonant energy, and Γ is the overall width for the decay of the compound nucleus, C. The Γ_a and Γ_b are related to the probabilities of going into and/or out of these respective entrance and exit channels and are related to the square of the associated matrix elements. The rate formula

$$\overline{\sigma v}(T) = \left(\frac{8}{\pi\mu(k_{\rm B}T)^3}\right)^{1/2} \int\limits_0^\infty \sigma(E) E e^{-E/k_{\rm B}T}\, dE \tag{21}$$

is still germane. Substituting eq. (20) into eq. (21), while at the same time assuming that in the context of eq. (21), eq. (20) behaves like a **delta function** around E_r, we arrive at a reaction rate for each resonance of the form:

$$\overline{\sigma v}(T) \propto \frac{E_r}{(kT)^{3/2}} e^{-\frac{E_r}{kT}} \int\limits_0^\infty \sigma_{res}\, dE\,. \tag{22}$$

The integral over σ_{res} is analogous to the "oscillator strength" sum-rule integral seen in atomic and molecular spectroscopy and yields

$$\int_0^\infty \sigma_{res}\, dE = 2\pi^{22}_{r}\,(2\ell+1)\,\frac{\Gamma_a\Gamma_b}{\Gamma}\,. \tag{23}$$

The final expression for the rate of a resonance reaction is then

$$\overline{\sigma v}(T) \propto \frac{1}{T^{3/2}}\frac{\Gamma_a\Gamma_b}{\Gamma}e^{-11.61 E_r/kT} \quad = 8.1\times 10^{-12}(2\ell+1)\left(\frac{m_p}{\mu\, T_6}\right)^{3/2}\frac{\Gamma_a\Gamma_b}{\Gamma}e^{-11.61 E_r/kT}\ \mathrm{cm^3 s^{-1}}\,, \tag{24}$$

where in this expression E_r is in keV. The dependence upon temperature of a resonant reaction (eq. 21) is distinctly different from that of a non-resonant reaction (eq. 13). Note that ${}^2_r$ is the reduced deBroglie wavelength squared, which is inversely proportional to E_r, and that it cancels the E in eq. (22). In general, one sums over all such resonances and channels to obtain the total rate. Usually, only one resonance dominates. Note also that for a resonant reaction the effective power index, n, is equal to $11.61 E_r(\,\mathrm{keV})/T_6 - 3/2$ and that this power index is generally large.

A classic example of a resonant reaction is the **triple-α** process, but there are many more.

Helium Burning

Helium burning occurs when there is a predominance of helium and the temperatures are higher ($T \sim 10^8$ K) than for hydrogen burning. In a star, it occurs in the core after core hydrogen burning has depleted the hydrogen into helium. It also occurs in a shell during the AGB phase, if reached. As first suggested by E. Salpeter, helium burning is a two step process:

$$^4He + ^4He \rightarrow ^8Be^* \qquad ; \quad ^4He + ^8Be^* \rightarrow ^{12}C + 2\gamma\,, \tag{25}$$

followed as carbon accumulates by

$$^4He + ^{12}C \rightarrow ^{16}O + \gamma\,. \tag{26}$$

The former reaction is the "triple-α" process and is the origin of carbon. The latter is the origin of oxygen. If helium burning is ignited under degenerate conditions it leads to the so-called **"helium flash."** The first reaction of eq. (25) produces the metastable state ^{8}Be*, which decays back to 2 alphas within $\sim$2$\times$10^{-16} seconds. This might seem like a short time, but it is long enough to supply in a "Saha" equilibrium ample ^{8}Be* to make the second reaction in eq. (25) viable at high temperatures and densities. The Saha equilibrium equation for ^{4}He and $^8Be^*$ is

$$\frac{n_\alpha^2}{n_{^8Be}} = \left(\frac{2\pi\mu kT}{h^2}\right)^{3/2} e^{-\frac{Q}{kT}}\,, \tag{27}$$

where the Q value of this endothermic reaction is a scant -91.78 keV and $\mu = m_\alpha^2/m_{^8Be} = m_\alpha/2$. At $T = 10^8$ K, $n(^8Be)/n_\alpha \sim 10^{-8}$. Note that if the Gamow peak energy is set to 92 keV, the associated temperature is $\sim$10^8 K. The Q value of the entire reaction (25) is $\sim$5.9 $\times$ 10^{17} erg g^{-1}$\sim$0.6 MeV/baryon.

The rate for the reaction $^8Be^*(\alpha,\gamma)^{12}C$ depends on the product of the ^{4}He and ^{8}Be abundances. Since the Saha provides the abundance of ^{8}Be in terms of X_α^2, the rate will be proportional to X_α^3. It turns out, as F. Hoyle posited, that this reaction is resonant - the compound nucleus of ^{4}He and ^{8}Be has an energy level very close ($\sim$0.29 MeV) to the 0^+ level of carbon at 7.654 MeV. That carbon had such a level was not known when Salpeter suggested his triple-α process, but Hoyle concluded that reaction (25) could not proceed at a reasonable enough rate to explain the existence of copious carbon unless the resonance was there. Cook et al. (1957) looked for this level in carbon and found it.

Since the reaction is resonant and the $^8Be^*$ abundance is temperature-dependent through the Saha, we find that for the triple-α process

$$\varepsilon_{3\alpha} \approx 5.1 \times 10^{11} \rho^2 \frac{X_\alpha^3}{T_8^3}\, e^{-\frac{44.027}{T_8}} \text{ erg g}^{-1} \text{ s}^{-1}\,, \tag{28}$$

where $\varepsilon_{3\alpha}$ is the energy generation rate, obtained from $Qn_1n_2\overline{\sigma v}/\rho$. Note the prefactor of $\frac{1}{T_8^3}$. This comes from the product of the temperature factor for a resonant reaction and the Saha temperature term. Near $T = 2 \times 10^8$ K, this rate is approximately 23.1 $\rho^2 \left(\frac{T_8}{2}\right)^{18.5}$ erg g$^{-1}s^{-1}$, where the power index, n, is $\frac{44}{T_8} - 3$ and is roughly equal to 40 at 10^8 K. In the equations above, ρ is in cgs.

The $^{12}C(\alpha,\gamma)^{16}O$ rate is not well known and involves close resonances, one of which is $\sim$0.066 eV wide, putting the reaction out on its tail. Moreover, the associated cross sections in the keV range are nanobarns and very difficult to measure. As a result, the rate of this crucial reaction that determines the ratio of carbon to oxygen in Nature, is not known to better than a factor of two. In any case, this rate is approximately

$$\varepsilon_{12,\alpha} \approx 1.3 \times 10^{27} X_{12} X_\alpha \frac{\rho}{T_8^{2/3}} e^{-\frac{69.2}{T_8^{1/3}}} \text{ erg g}^{-1} \text{ s}^{-1} . \qquad (29)$$

The Q value of this reaction is 7.162 MeV. Note that the "large" charge results in the large 69.1 in the exponent.

The processes (25) and (26) are the final core burning processes in most stars, but for more massive stars the cores achieve temperatures sufficient to ignite **carbon burning** ($T \sim 5 - 10 \times 10^8$ K):

$$\begin{aligned} ^{12}C + ^{12}C \rightarrow & \ ^{24}Mg + \gamma \\ \rightarrow & \ ^{20}Ne + \alpha \\ \rightarrow & \ ^{23}Na + p \end{aligned} \qquad (30)$$

and **oxygen burning** ($T \geq 1 \times 10^9$) K):

$$\begin{aligned} ^{16}O + ^{16}O \rightarrow & \ ^{32}S + \gamma \\ \rightarrow & \ ^{28}Si + \alpha \\ \rightarrow & \ ^{31}P + p , \end{aligned} \qquad (31)$$

where the above reactions depict only a subset of the possible final states. **Silicon and neon burning** are more complicated and occur at still higher temperatures, mostly during the pre- and postsupernova stages of massive-star evolution. Near the onset of core carbon burning, core **thermal neutrino losses** by the pair annihilation, plasmon decay, photoneutrino rates, and (at a lower level) electron-nucleus bremmstrahlung start to exceed surface photon losses, but this is another topic.