

# AST 403      Problem Set #1      Due April 5, 2017

1. The dynamical timescale of a star,  $t_{dyn}$ , the Kelvin-Helmholtz timescale,  $t_K$ , and the nuclear timescale,  $t_{nuc}$ , are given very approximately by the expressions:

$$t_{dyn} \sim \sqrt{\frac{R^3}{GM}},$$

$$t_K \sim \frac{GM^2}{R}/L,$$

and

$$t_{nuc} \sim 0.1 \times \frac{0.007Mc^2}{L},$$

where  $R$ ,  $M$ , and  $L$  have their standard meanings. (a) If the luminosity on the main sequence scales with mass as  $M^3$  and the radius of a star on the main sequence scales as  $R \propto M^{0.8}$ , compare these three timescales for main-sequence stars with masses of 0.5, 1.0, 2.0, 4.0, 6.0, and 10.0  $M_\odot$ . (b) What general conclusions (plural) can you draw from what you find?

2. The virial theorem for a star in hydrostatic equilibrium states that:

$$-3(\gamma - 1)E_{th} = \Omega,$$

where  $E_{th}$  and  $\Omega$  are the total thermal and total gravitational energies, respectively. For a polytrope we will find that:

$$\Omega = -\frac{3}{5-n} \frac{GM^2}{R}.$$

Assuming that:

$$E_{th} = \int \frac{3}{2} N k_B T dV,$$

where  $N$  is the number density of the gas particles in the star,  $T$  is the local temperature,  $k_B$  is Boltzmann's constant, and  $V$  is the volume, derive an approximate formula for the average temperature for such a star. Note that we are using the ideal gas law for the matter equation of state. Assuming that the average temperature and central temperature ( $T_c$ ) are always proportional (not always a good assumption, but ...), that a star's radius on the main sequence is proportional to  $M^{0.8}$ , that  $\gamma = 5/3$ , and that  $n = 3.0$ , derive the central temperature of a main sequence star as a function of total mass. Assume that  $T_c$  for the Sun is  $1.5 \times 10^7$  Kelvin (K). (b) Find  $T_c$  for stars of mass 0.5, 1.5, 2.0, 4.0, 6.0, and 8.0  $M_\odot$ .

3. **a.** Assume that a star is in hydrostatic equilibrium. Show that the free-fall time and the sound travel time across the star are similar. Use only order-of-magnitude arguments.

**b.** Show that the orbit time of a satellite skimming a star's surface is comparable to these times.

**c.** Calculate the dynamic times for the Sun, a white dwarf, and a neutron star. Comment.

4. The pressure ( $P$ )/mass-density ( $\rho$ )/temperature ( $T$ ) relation for an ideal gas depends upon the number density ( $N_{i,e}$ ) of free particles in the gas through the mean molecular weight,  $\mu_{i,e}$ , via the equation:

$$P_{i,e} = N_{i,e} k_B T = \frac{k_B \rho T}{\mu_{i,e} m_p},$$

where  $m_p$  is approximately the proton mass,  $i, e$  stand for the ions or the electrons, respectively, and  $N_A$  is Avogadro's number. (Note that  $k_B N_A = R$ , the gas constant, and  $N_A \sim \frac{1}{m_p}$ ). The total

gas pressure is  $P_i + P_e$  and the total pressure is this pressure, plus  $P_{rad}$ , where  $P_{rad} = \frac{aT^4}{3}$ . (a) For a completely ionized gas, derive  $\mu_i$  and  $\mu_e$  as a function of the hydrogen, helium, and heavy mass fractions,  $X$ ,  $Y$ , and  $Z$ , respectively. Note that the total  $\mu$  is naturally given by the expression:

$$\frac{1}{\mu} = \frac{1}{\mu_i} + \frac{1}{\mu_e}.$$

(b) Calculate and compare all these various pressures at  $T = 10^7$  K and  $\rho = 100 \text{ g cm}^{-3}$  (roughly values at the center of the Sun) for  $X = 0.73$ ,  $Y = 0.25$  and  $Z = 0.02$  and assemble them in order. Which dominates?

5. The Rosseland average opacity of matter to light is approximately given by the Kramers formula (everything in CGS units):

$$\kappa_{ross} = 4.0 \times 10^{25} (1 + X)(Z + 0.001) \frac{\rho}{T^{3.5}} \text{ cm}^2 \text{ g}^{-1}.$$

(a) Calculate the value of this opacity at  $T = 10^6$  K and  $\rho = 1 \text{ g cm}^{-3}$  for  $X = 0.73$  and  $Z = 0.02$ , and then calculate the corresponding mean-free-path in centimeters. Comment. (b) Assuming that this opacity is representative, calculate a very approximate diffusion time in years out of the Sun. (c) Interpret your result and comment on what it might mean.

6. For degenerate electrons at high densities and “low” temperatures, the pressure is given by the formula:

$$P_{deg} = 0.99 \times 10^{13} \left( \frac{\rho}{\mu_e} \right)^{5/3} \text{ dynes cm}^{-2},$$

where  $\rho$  is in CGS. At what temperature,  $T$ , would this pressure be equal to the ideal gas pressure for a given mass density,  $\rho$ ? This is approximately the temperature below which  $P_{deg}$  would dominate and the material can be considered degenerate.

7. If Thomson scattering off electrons (assumed free) were the dominant opacity source in the Earth’s atmosphere, what would be its optical depth? Does this seem reasonable?

8. For a radiative zone, the temperature gradient is given by a formula involving  $L(r)$  and  $\kappa_R$ . For a convective zone, the temperature gradient is given by a formula that expresses isentropy. Use these separate relations for  $\frac{dT}{dr}$  (and any other relations needed) to determine the simplest-looking condition on  $L(r)$ ,  $M(r)$ , and  $\kappa_R$  for a radiative zone to be tripped into convection, i.e. the critical local luminosity. How is this relation related to the Eddington limit and why? [Hint: Please define  $1-\beta = P_{Rad}/P$  and write your result in terms of  $\beta$ .]

## AST 403      Problem Set #3      Due April 19, 2017

9. (a) Derive in the context of mixing-length theory the relationship between the energy flux,  $F_{conv}$ , in a convective zone and the mean convective speed,  $V_{conv}$ .

(b) Calculate this speed for the convection zone of the Sun, making reasonable assumptions concerning the quantities in the resulting formula.

(c) What is the approximate relationship between the convective Mach number ( $M$ ) and the quantity  $\nabla - \nabla_{ad}$ ? Are there any stellar contexts in which we can expect  $M$  to be near one and what would this mean for the value of  $\nabla - \nabla_{ad}$ ?

10. Using the generic scaling relations for approximate  $T$  and  $\rho$  as a function of mass ( $M$ ) and radius ( $R$ ), assuming Kramers opacity, and using the ideal gas law, derive a very approximate  $R(M)$  for main-sequence stars from  $\sim 1.5 M_\odot$  to  $4.0 M_\odot$ . [Hint: You will need to know whether hydrogen burning is proceeding by the PP chain or the CNO cycle.] Retain the  $\mu$  dependence of your result. What power-law indices for  $\mu$  and  $M$  emerge?

11. (a) The Earth can be represented roughly as a polytrope of Emden index  $n = 0$ , *i.e.* constant density. Assuming this, and given that  $M_E = 6 \times 10^{27}$  g and  $R_E = 6.4 \times 10^8$  cm, calculate its central density, central pressure, and gravitational potential energy. Convert the pressure to atmospheres ( $1 \text{ atm} = 1.01 \times 10^6 \text{ dyne cm}^{-2}$ ).

(b) Calculate the corresponding quantities for Jupiter, which is composed mostly of degenerate molecular hydrogen rather than rock and is better approximated by  $n = 1$ ,  $M_J = 2 \times 10^{30}$  g =  $10^{-3} M_\odot$ ,  $R_J = 7 \times 10^9$  cm =  $0.1 R_\odot$ .

12. (a) Calculate the  $Q$  values for all the reactions in the PPI, PPII, and PPIII chains and the energy derived per helium nucleus produced (in ergs  $\text{g}^{-1}$ ). [Hint: Use a table of mass excesses.]

(b) Which reaction dominates the PPI chain?

13. Calculate the  $Q$  values for the  $3\text{-}\alpha$ ,  $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$ , and  $^{16}\text{O}(\alpha,\gamma)^{20}\text{Ne}$  reactions. [Hint: Use a table of mass excesses.]

## AST 403      Problem Set #4      Due April 26, 2017

14. (a) Calculate the positions of the Gamow peak (in keV) for the rate-limiting reaction of hydrogen burning for hydrogen-burning cores in 0.25, 1, 3, 5, and 25  $M_\odot$  stars.

(b) What are the approximate “power-law” indices for the temperature-dependence of the nuclear rates in each of these cores (both hydrogen- and helium-burning)?

15. Derive the general formula for a nuclear reaction rate:

$$\begin{aligned} \overline{\sigma v}(T) &= \left( \frac{8}{\pi \mu (k_B T)^3} \right)^{1/2} \int_0^\infty \sigma(E) E e^{-E/k_B T} dE, \\ &= \left( \frac{8}{\pi \mu (k_B T)^3} \right)^{1/2} \int_0^\infty S(E) \exp \left[ -\frac{b}{\sqrt{E}} - \frac{E}{k_B T} \right] dE. \end{aligned} \quad (1)$$

16. Identify the general stellar (or elsewhere) site and burning process (*i.e.*, H-burning, He-burning, big bang, carbon-burning, etc.) where each of the elements in the Sun or interstellar medium up to an atomic number ( $Z$ ) of 26 are created.

17. In our discussion of the “ignition mass” for a given stage of thermonuclear burning, we derived a formula that seemed to work reasonably well for the main-sequence edge. Determine using the same formula what the corresponding mass to ignite core helium burning would be. Note that the mean molecular weight and “ignition temperature” will not be the same as for hydrogen burning.

18. (a) Assuming a non-relativistic equation of state, estimate the radius, central density, and central pressure of a cold, solar-mass white dwarf consisting of pure carbon ( $^{12}\text{C}$ ). From your results, check whether the non-relativistic assumption is self-consistent. (b) Assuming that the relativistic degenerate electron equation of state obtains, derive the Chandrasekhar mass in solar mass units.