HAYASHI LIMIT

Let us consider a simple "model atmosphere" of a star. The equation of hydrostatic equilibrium and the definition of optical depth are

$$\frac{dP}{dr} = -g\rho, \qquad \frac{d\tau}{dr} = -\kappa\rho, \tag{s2.24}$$

and may be combined to write

$$\frac{dP}{d\tau} = \frac{g}{\kappa}. \tag{s2.25}$$

Assuming $\kappa = const$ we may integrate this equation to obtain

$$P_{\tau=2/3} = \frac{2}{3}\frac{g}{\kappa_{\tau=2/3}}, \qquad g \equiv \frac{GM}{R^2}, \tag{s2.26}$$

where the subscript $\tau = 2/3$ indicates that we evaluate the particular quantity at the photosphere. Let us consider a cool star with the negative hydrogen ion H^- dominating opacity in the atmosphere. When temperature is low we may neglect radiation pressure in the atmosphere. Adopting

$$P = \frac{k}{\mu H}\rho T, \qquad \kappa = \kappa_0 \rho^{0.5} T^{7.7}, \qquad \kappa_0 = 10^{-25} Z^{0.5}, \tag{s2.27}$$

we may write the equation (s2.26) as

$$\frac{k}{\mu H}\rho T = \frac{2}{3}\frac{1}{\kappa_0 \rho^{0.5} T^{7.7}}\frac{GM}{R^2}, \tag{s2.28}$$

which may be rearranged to have

$$\rho^{1.5} T^{8.7} = \frac{2}{3}\frac{\mu H}{k}\frac{G}{\kappa_0}\frac{M}{R^2}. \tag{s2.29}$$

We know that a star with the H^- opacity in the atmosphere becomes convective below optical depth $\tau = 0.775$, i.e. very close to the photosphere. Let us suppose that the convection extends all the way to the stellar center, and let us ignore here all complications due to hydrogen and helium ionization. Convective star is adiabatic, and if it is made of a perfect gas with the equation of state (s2.27) then it is a polytrope with an index $n = 1.5$. Therefore, we expect a polytropic relation all the way from the photosphere down to the center, and we have

$$\frac{\rho}{T^{1.5}} = \frac{\rho_c}{T_c^{1.5}}, \qquad \rho_c = 5.99\rho_{av} = 5.99\frac{3M}{4\pi R^3}, \qquad T_c = 0.539\frac{\mu H}{k}\frac{GM}{R}. \tag{s2.30}$$

Combining equations (s2.29) and (s2.30) we obtain

$$T^{10.95} \approx \rho_c^{-1.5} T_c^{2.25}\frac{\mu HG}{k\kappa_0}\frac{M}{R^2} \approx \frac{0.10}{\kappa_0}\left(\frac{\mu HG}{k}\right)^{3.25} M^{1.75} R^{0.25}. \tag{s2.31}$$

Let us make an approximation that convection begins at the photosphere, i.e. at $T = T_{eff}$, and let us replace stellar radius with the combination of effective temperature and luminosity according to $L = 4\pi R^2 \sigma T_{eff}^4$:

$$T_{eff}^{11.45} \approx \frac{0.07}{\kappa_0 \sigma^{1/8}}\left(\frac{\mu HG}{k}\right)^{3.25} M^{1.75} L^{1/8}, \tag{s2.32a}$$

$$T_{eff} \approx 2 \times 10^3 \left(\frac{M}{M_\odot}\right)^{0.15}\left(\frac{L}{L_\odot}\right)^{0.01}\left(\frac{Z}{0.02}\right)^{-0.04}, \tag{s2.32b}$$

To the right of the Hayashi limit no stars in a hydrostatic equilibrium can exist.

Ignition Mass for Low-Mass Stars

Low mass stars are supported by gas pressure, while radiation pressure is unimportant. Electron gas may be partly degenerate. Numerical models demonstrate that very low mass stars, with $M < 0.3 M_\odot$ are fully convective, and may be very well approximated with $n = 1.5$ polytropes. We shall approximate the non.-rel. equation of state with the following formula

$$P \approx \left[\left(\frac{k}{\mu H} \rho T \right)^2 + \left(K_1 \rho^{5/3} \right)^2 \right]^{1/2}, \tag{lms.1}$$

where $k/H = 0.825 \times 10^8$ [erg g^{-1} K^{-1}], $K_1 = 0.991 \times 10^{13} \mu_e^{-5/3}$ [erg g$^{-5/3}$ cm^2]. Algebraic approximation to the stellar structure equations gives

$$\rho \approx \frac{M}{R^3}, \qquad P \approx \frac{GM^2}{R^4}. \tag{lms.2}$$

Combining the last two equations we obtain

$$\frac{G^2 M^4}{R^8} \approx \left(\frac{k}{\mu H} \rho T \right)^2 + K_1^2 \rho^{10/3} \approx \left(\frac{kT}{\mu H} \right)^2 \frac{M^2}{R^6} + K_1^2 \frac{M^{10/3}}{R^{10}}, \tag{lms.3}$$

which may be written as

$$T \approx \frac{\mu H}{k} \frac{GM}{R} \left[1 - \left(\frac{K_1}{GRM^{1/3}} \right)^2 \right]^{1/2}. \tag{lms.4}$$

Fully degenerate, the stellar radius satisfies a polytropic ($n = 1.5$) mass-radius relation:

$$R_{min} = \frac{K_1}{0.4242\ GM^{1/3}}, \qquad K_1 \rho^{5/3} \gg \frac{k}{\mu H} \rho T, \tag{lms.5}$$

where R_{min} is the minimum radius that a star with a mass M may have. This relation is recovered from the equation (lms.4) when we replace the square bracket with $[1 - (R_{min}/R)^2]$. In the limit when degeneracy is negligible we should have

$$T_c = 0.539 \frac{\mu H}{k} \frac{GM}{R}, \qquad K_1 \rho^{5/3} \ll \frac{k}{\mu H} \rho T, \tag{lms.6}$$

where T_c is the central temperature of an $n = 1.5$ polytrope with pressure provided by non-degenerate gas. Combining the two limiting cases with the equation (lms.4) we may write it as

$$T_c = 0.539 \frac{\mu H}{k} \frac{GM}{R} \left[1 - \left(\frac{R_{min}}{R} \right)^2 \right]^{1/2}, \qquad R_{min} = \frac{K_1}{0.4242\ GM^{1/3}}. \tag{lms.7}$$

The central temperature reaches its maximum where $dT_c/dR = 0$ for $R = R_{T_{max}}$, i.e.

$$R_{T_{max}} = 2^{1/2} R_{min}, \tag{lms.8}$$

and the corresponding maximum central temperature is

$$T_{c,max} = \frac{0.539}{2^{1/2}} \frac{\mu H}{k} \frac{GM}{R_{T_{max}}} = \frac{0.539}{2} \frac{\mu H}{k} \frac{GM}{R_{min}} = \tag{lms.9}$$

$$0.1143 \frac{\mu H G^2}{k K_1} M^{4/3} = 1.56 \times 10^8\ \mu \mu_e^{5/3} \left(\frac{M}{M_\odot} \right)^{4/3} \approx 6 \times 10^6 \left(\frac{M}{0.1\ M_\odot} \right)^{4/3} \qquad [\mathrm{K}],$$

Schönberg-Chandrasekhar Limit

After core hydrogen burning, the **quasi-isothermal helium core** grows by the accumulation of helium ashes from the hydrogen burning shell surrounding it. However, the core can maintain itself in hydrostatic equilbrium only if the mass of the helium core is below the so-called **"Schönberg-Chandrasekhar"** limit. If it exceeds this limit, the core starts to contract on **Kelvin-Helmholtz** timescales, inaugurating the next phase of stellar evolution. We can derive the SC critical mass by using the Virial theorem with the core boundary pressure retained and assuming an ideal gas, finding the maximum surface pressure the core can sustain (calculated in terms of core quantities), calculating the surface pressure due to the weight of the overlying envelope, and then setting these two pressures equal. A rearrangement of the resulting equation yields the ratio of the critical core mass to the total stellar mass (as long as this ratio is small) in terms of the mean molecular weights in the envelope and core. The derivation proceeds as follows:
We state the **Virial theorem**:

$$4\pi R_c^3 P_c - 2U_c = \Omega_c\,, \tag{sc.1}$$

where $U_c = \frac{3}{2}\left(\frac{M_c}{\mu_c m_p}\right) kT_c$ and the subscript c stands for "core." We will assume that $\Omega_c \sim \frac{3GM_c^2}{3R_c}$. Then, we solve for P_c, and take dP_c/dM_c, and set it to zero. The result, written for R_c, is

$$R_c = \left(\frac{2\mu_c m_p G M_c}{5kT_c}\right)\,.$$

Plugging this radius into eq. (sc.1) and solving for P_c, we have

$$P_c(crit) = \frac{375}{64\pi G^3 M_c^2}\left(\frac{kT_c}{\mu_c m_p}\right)^4\,.$$

The pressure at the surface of the core due to the envelope weight is

$$P_{c,env} = -\int_{\mathrm{M_c}}^{M} \frac{GM}{4\pi r^4} dM \sim \frac{G\left(M^2 - M_c^2\right)}{8\pi < r^4 >}\,. \tag{sc.2}$$

We assume that $< r^4 > \sim R^4/2$, where R is the outer radius of the star and M is the total mass of the star. The pressure on the core due to the envelope, $P_{c,env}$, can be written in terms of the mass density ($\rho_{c,env}$), temperature (T_c), and envelope mean molecular mass (μ_{env}), and if we assume that $\rho_{c,env} \sim \frac{3M}{4\pi R^3}$, we find (solving for R)

$$R \sim \frac{\mu_{env} m_p G M}{3kT_c} \, .$$

Plugging this into the eq. (sc.2) for the pressure, we find

$$P_c(crit) = \frac{81}{4\pi G^3 M^2} \left(\frac{kT_c}{\mu_{env} m_p} \right)^4 \, .$$

Setting the two pressures equal (note that T_c is in both expressions), we find

$$\frac{M_c}{M} \sim 0.54 \left(\frac{\mu_{env}}{\mu_c} \right)^2 .$$

Doing this a bit more rigorously yields a coefficient of **0.37**, not much different. If $\mu_{env} \sim 0.63$ and $\mu_c \sim 1.34$, we find that $\frac{M_c}{M} \sim 0.08$ (~8%). This is the Schönberg-Chandrasekhar ratio. Note that the subsequent KH collapse of the core leads to a **self-bound object** with central pressures much larger than the pressure in the hydrogen-burning shell. As we will state again, “once an isothermal core becomes self-gravitating, it remains self-gravitating forever, even if a new nuclear fuel ignites in the core, and the core is no longer isothermal."