5-6 TWO-STREAM INSTABILITY; LINEAR ANALYSIS

The model consists of two opposing streams of charged particles as sketched in Figure 5-6a. Models with relative motion between two sets or streams of charged particles have been studied in great detail since papers by *Haeff* (1949) and *Pierce* (1948). Detailed knowledge of the nonlinear behavior of opposing streams came much later, from the simulations done by *Dawson* (1962). The fluid analog was given much earlier, as by H. Hertz in the 1880's; see comprehensive books on hydrodynamics and acoustics, such as *Lamb* (1945) or *Rayleigh* (1945).

One can readily see that an opposing stream system is unstable. When two streams move through each other one wavelength in one cycle of the plasma frequency, a density perturbation on one stream is reinforced by the forces due to bunching of particles in the other stream and vice versa; hence

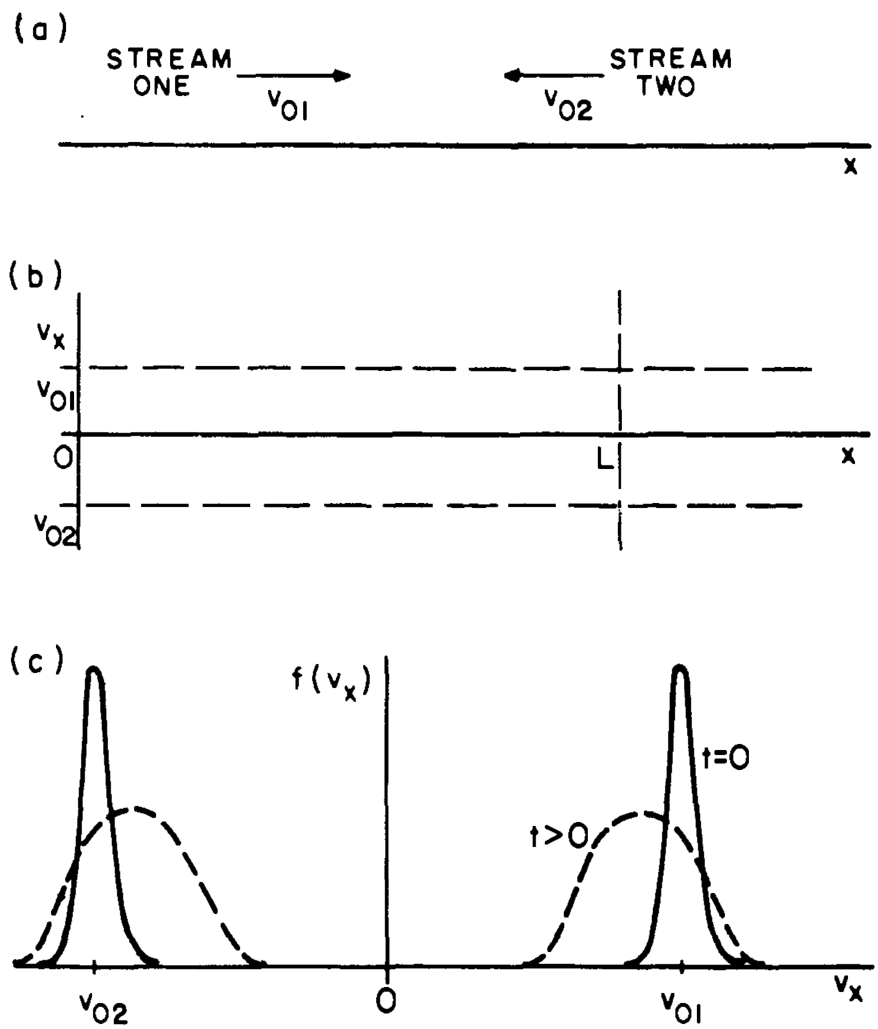

Figure 5-6a (a) Two opposing streams as seen in the laboratory. (b) The streams in phase space at the start of the problem, $t = 0$. (c) The streams in velocity space at $t = 0$ and $t > 0$.

$\Delta n_1 \propto n_1$, so that the perturbation *grows exponentially in time.* This simple relation was put forth in 1948 by Professor M. Chodorow of Stanford [and buried in Birdsall's dissertation (*Birdsall,* 1951)] for two streams moving in the same direction (*Chodorow and Susskind,* 1964). The phase relation for reinforcement is written as

$$(v_{\text{relative}})\left(\frac{2\pi}{\omega_p}\right) = \frac{2\pi}{k} \tag{1}$$

which for $v_{\text{relative}} = v_0 - (-v_0) = 2v_0$ is

$$k = \frac{\omega_p}{2v_0} \tag{2}$$

This k is very close to that found from analysis for maximum growth rate.

The longitudinal linear dielectric function for two independent cold streams may be obtained as was done in Section 5-3 by applying the equations of motion and continuity separately for each stream and adding the currents of each in the field equation. The result is

$$\frac{1}{\epsilon_0}\epsilon(\omega,k) = 1 - \frac{\omega_{p1}^2}{(\omega - \mathbf{k}\cdot\mathbf{v}_{01})^2} - \frac{\omega_{p2}^2}{(\omega - \mathbf{k}\cdot\mathbf{v}_{02})^2} \tag{3}$$

for two streams with drift velocities $\mathbf{v}_{01}$ and $\mathbf{v}_{02}$. This result is also obtainable directly from the usual Vlasov-Poisson set by letting the velocity distribution be two delta functions,

$$f_0(\mathbf{v}) = A\delta(\mathbf{v} - \mathbf{v}_{01}) + B\delta(\mathbf{v} - \mathbf{v}_{02}) \tag{4}$$

A system of N independent cold streams produces a sum over streams or species s:

$$\frac{1}{\epsilon_0}\epsilon(\omega,k) = 1 - \sum_{s=1}^{N}\frac{\omega_{ps}^2}{(\omega - \mathbf{k}\cdot\mathbf{v}_{0s})^2} \tag{5}$$

[Extension of the sum to an integral, for $N \rightarrow \infty$, must be done carefully, both analytically as shown by *Dawson* (1960), and also in simulation when a discrete set of beams is used to approximate a smooth distribution $f(v)$ as shown by *Byers* (1970), and *Gitomer and Adam* (1976), and discussed in Chapter 16.]

The solutions for complex ω, assuming real k (*i.e.*, an absolute instability, growth in time only, no convection in space), opposing streams of equal strength, $\omega_{p1} = \omega_{p2} \equiv \omega_p$, $v_{01} = -v_{02} \equiv v_0$, is found from $\epsilon(\omega,k) = 0$ which is quartic in ω with four independent solutions. These are

$$\omega = \pm\,[k^2v_0^2 + \omega_p^2 \pm \omega_p(4k^2v_0^2 + \omega_p^2)^{1/2}]^{1/2} \tag{6}$$

for which

$$0 < \frac{kv_0}{\omega_p} < \sqrt{2} \quad \begin{cases} \text{two roots are real} \\ \text{two roots are imaginary} \end{cases} \tag{7}$$

$$\sqrt{2} < \frac{kv_0}{\omega_p} \quad \text{all four roots are real} \tag{8}$$

$$\frac{kv_0}{\omega_p} = \frac{\sqrt{3}}{2}, \quad \omega_{\text{imaginary}} = \frac{\omega_p}{2}, \quad \text{maximum growth rate} \tag{9}$$

This behavior is sketched in Figure 5-6b; the growth ($\omega_{\text{imaginary}}$) is given in more detail in Figure 5-6c.

In this model, where there is growth ($\omega_{\text{imaginary}} > 0$), we find that $\omega_{\text{real}} = 0$; that is, there is no oscillatory part associated with the growth, a situation which is not generally true.

A point of Figure 5-6c is to make clear the existence of a *minimum unstable length L of the system;* in this model (normalized)

$$\frac{\omega_p L}{v_0} > \frac{2\pi}{\sqrt{2}} \quad \text{(unstable)} \tag{10}$$

in order to obtain growth. This is the same as (7) using $L = 2\pi / k_0$, where k_0 is the smallest wavenumber in the system.

Growth which begins at small amplitude continues until the streaming is destroyed; indeed, the distribution becomes nearly Maxwellian. Hence, we say that "the colliding streams have thermalized," although not by collisions.

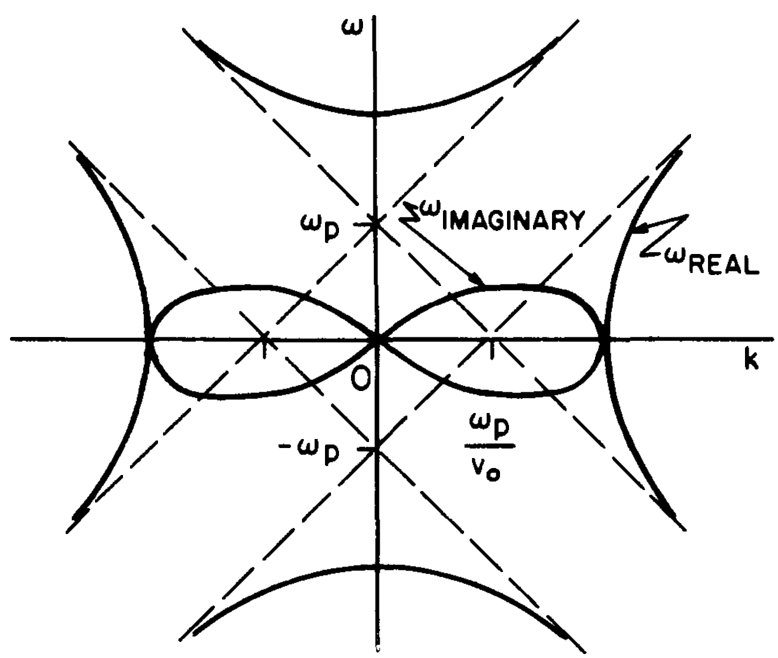

Figure 5-6b Dispersion, or ω-k, diagram for two equal opposing streams, real k, complex ω. The uncoupled space-charge waves are shown dashed. For each value of k, there are four values of ω that correspond to four linearly independent waves.

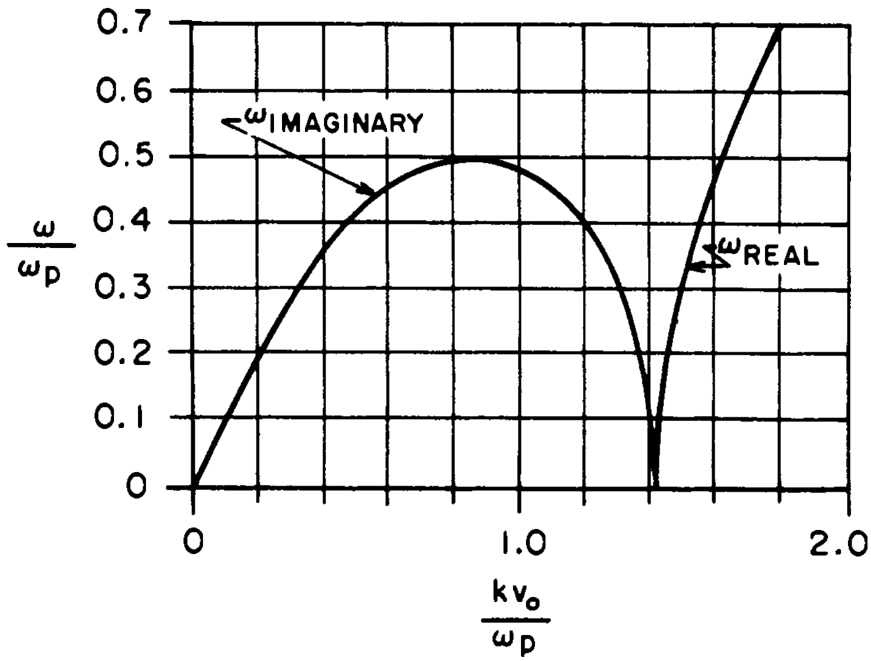

Figure 5-6c Growth rate $\omega_{\text{imaginary}}$ for two opposing streams.

Instead, collective effects build up large electric fields at long wavelengths ($\lambda >>$ particle spacing) and these *scatter the particles* in phase space.

As the instability grows, two changes are readily observed in $f(v)$ as indicated for $t > 0$ in Figure 5-6a(c). The *width* of each beam increases [measured directly on an $f(v)$ plot or by $(\overline{v^2} - \bar{v}^2)$ of one stream], which is taken as an increase in the *temperature* of each beam (but perhaps carelessly so, for if the electric field were suddenly shut off—and you should try this—the spread might decrease). The drift or mean velocity $\bar{v}$ decreases.

We might expect, as v_{thermal} increases and v_{drift} decreases, that the conditions for linear growth would cease to be met [see *Stringer* (1964), who shows the threshold for growth for electron-electron streams to be $v_{\text{drift}} \approx 1.3 v_{\text{thermal}}$] and that the exponential growth would stop. However, at this time, the conditions for linearity are largely violated, with perturbed charge densities comparable to the zero-order density; particles in one stream are about to pass their neighbors and wrap into *vortices in phase space,* that is, become *trapped*. Hence, the growth need not stop, although we might be tempted to look for a change in character of the growth (*e.g.*, away from exponential in time) at the time where v_t exceeds $\bar{v}/1.3$; keep this in mind in your project. Of course, ES1 can readily be run with warm beams; hence, look for growth with $v_0 = 2v_t$ (Section 5-9), but stability with $v_0 = v_t$.