Midterm Exam Solutions, AST 203, Spring 2009

Thursday, March 12, 3:00-4:20 PM

General grading rules for calculational problems: 4 points off for each arithmetic or algebraic error. Two points off per problem for excess significant figures in a final result (i.e., more than 2 significant figures). In this part of the exam, there is no requirement to write in full sentences. Explanations should be clear, but if the context is unambiguous, no points off for undefined symbols. However, take off two points per problem if a student gives absolutely no context for their calculation. Not giving units or not giving results in the units asked for, is 2 points off. Giving the correct number with incorrect units is 4 points off. No points off for not indicating units in intermediate calculations. In no case do the results of one part depend on the answer to another, so errors should not propagate from one part to another.

In many of the problems, there is an easy way and a hard way to do it; full credit for doing problems the hard way (and getting the right answer).

The total exam is worth 200 points.

Calculational Problems

1. Comparing stars (60 points total)
Consider two stars, radiating like blackbodies. One, a main sequence A star with a mass of twice the mass of the Sun, lies at a distance of 100 light years, and has a surface temperature of 9000 K and a radius of 10^6 km. The second, a red giant with mass equal to the mass of the Sun, has a surface temperature of 3000 K and a radius 100 times larger than that of the A star.

a. *(20 points)* Calculate the ratio of the luminosity of the red giant to that of the A star.

Answer: These two stars both radiate like black bodies, so their luminosities are related to their radii and temperatures as:

$$L_A = 4\,\pi R_A^2 \sigma T_A^4$$

$$L_{RG} = 4\,\pi R_{RG}^2 \sigma T_{RG}^4$$

We divide the second equation by the first:

$$\frac{L_{RG}}{L_A} = \left(\frac{R_{RG}}{R_A}\right)^2 \left(\frac{T_{RG}}{T_A}\right)^4$$

Note that the factors of $4\,\pi$ and σ dropped out, making our lives easier (and is the reason it is always a good idea to do algebra before plugging in numbers). We know the ratio of radii and temperatures:

$$\frac{L_{RG}}{L_A} = 100^2 \left(\frac{3000\,\mathrm{K}}{9000\,\mathrm{K}}\right)^4 = \frac{10^4}{81} \approx 120,$$

or 100 to one significant figure.

A result to 2 significant figures is fine; 3 or more significant figures is not. Five points for *recognizing that $L \propto R^2T^4$, and going no further. Five points off for calculating each luminosity separately, but not determining the ratio. Four points off for calculating the reciprocal of the requested ratio.*

b. *(15 points)* What is the ratio of the mean density (i.e., mass per unit volume) of the red giant to that of the A star?
Answer: As stated, the density is the mass divided by the volume:

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

As above, we have to calculate the *ratio* of this for the two stars, so we set up the ratio, noticing that lots of stuff drops out:

$$\frac{\rho_{RG}}{\rho_A} = \frac{M_{RG}}{M_A}\left(\frac{R_A}{R_{RG}}\right)^3.$$

We know the ratio of masses and radii, so this is straightforward:

$$\frac{\rho_{RG}}{\rho_A} = \frac{1}{2} \times \left(\frac{1}{100}\right)^3 = 5 \times 10^{-7}.$$

Three points for writing down the expression for density, and going no further. Seven points for calculating the two densities, but not taking the ratio. Three points off for calculating the reciprocal of the requested ratio.

c. *(10 points)* Given its surface temperature, what is the peak wavelength at which the A star emits? Express your answer in microns (1 micron $= 10^{-6}$ meters).
Answer: This is a direct application of the relation between the peak wavelength and temperature of a blackbody:

$$\lambda_{peak} = \frac{2.9\,\mathrm{mm}}{T}.$$

The surface temperature of the A star is 9000 K. Converting to microns:

$$\lambda_{peak} = \frac{2.9\,\mathrm{mm}}{9000} \times 10^3 \frac{\mathrm{microns}}{\mathrm{mm}} \approx 0.3\,\mathrm{micron}.$$

Up to 2 significant figures are fine. 3 points for writing down the relationship between peak wavelength and temperature, and going no further. 4 points off for calculating this temperature for the red giant instead of the A star. Common errors include errors in the conversion between millimeters and microns; treat such as arithmetic errors (4 points off).

d. *(15 points)* With a sufficiently powerful space-based telescope, one could resolve the two stars (i.e., measure the angle that the diameter of each subtends). Suppose that the two stars subtend the same angle. Calculate the distance of the red giant; express your answer in light years.

Answer: The small-angle formula says that α, the angle subtended (in radians) by an object of diameter s and distance d is:

$$\alpha = s/d.$$

We can write such an expression down for the A star and the red giant. We are told that the angles subtended by the two are the same, thus:

$$\frac{s_A}{d_A} = \frac{s_{RG}}{d_{RG}}.$$

We want to know d_{RG}, the distance of the red giant. We know d_A, and we know the *ratio* of the diameters of the two stars. A little rearrangement gives us:

$$d_{RG} = d_A \left(\frac{s_{RG}}{s_A}\right) = 100\,\text{ly} \times 100 = 10^4\,\text{ly}.$$

4 points for writing down the small-angle formula and going no further. Four additional points off for getting a result of less than or equal to one light year, and not remarking that this is probably not right.

2. Orbital motion *(30 points)*

The space-shuttle, in low-Earth orbit, (i.e., in an orbit just above the Earth's surface), takes roughly 90 minutes to orbit the Earth. Now consider a space ship in orbit just above the surface of a white dwarf of the same radius as the Earth, but with a mass equal to the mass of the Sun. What is the orbital period (in seconds)? Would astronauts in such a space ship feel weightless? Explain your answer.

Hint: There are several ways to do this problem. The hard way to do it is to start with Newton's Law of Gravitation, and ignore the first sentence of the problem.

Kepler's Third Law states that for orbits around a body of mass M, the orbital period P is related to the radius of the orbit a by

$$P^2 \propto \frac{a^3}{M}.$$

Therefore the ratio of the orbital period around the white dwarf to that of the space shuttle is

$$\begin{aligned}\frac{P_{WD}^2}{P_{Earth}^2} &= \left(\frac{a_{WD}^3}{M_{WD}}\right) \Big/ \left(\frac{a_{Earth}^3}{M_{Earth}}\right)\\ &= \frac{M_{Earth}}{M_{WD}},\end{aligned}$$

where we use the fact that the radii of the two orbits are the same.

Solving for the period of the orbit around the white dwarf, we find:

$$\begin{aligned}P_{WD} &= P_{Earth}\sqrt{\frac{M_{Earth}}{M_{WD}}}\\ &= 90\,\text{min}\sqrt{\frac{6 \times 10^{24}\,\text{kg}}{2 \times 10^{30}\,\text{kg}}}\\ &= 90\,\text{min} \times \sqrt{3 \times 10^{-6}}\\ &= 90\,\text{min} \times \frac{60\,\text{sec}}{1\,\text{min}} \times 1.7 \times 10^{-3}\\ &= 9\,\text{sec},\end{aligned}$$

where we used the useful fact that $60 \times 1.7 \approx 100$.

The hard way to do it, which we recommended against in the statement of the problem, was to use Kepler's Third Law without making reference to any proportionalities, and calculating things out in full glory. For completeness, and to show that this can also be done without a calculator, let's give this a try. Kepler's Third Law, as you derived in the homework, is

$$P_{WD} = 2\,\pi\sqrt{\frac{R_{WD}^3}{GM_{WD}}}$$

Plugging in numbers, and being careful to keep the units consistent, I get:

$$\begin{aligned}
P_{WD} &= 6 \times \sqrt{\frac{(6.4 \times 10^6\,\mathrm{m})^3}{(2/3 \times 10^{-10}\,\mathrm{m^3\,s^{-2}\,kg^{-1}})(2 \times 10^{30}\,\mathrm{kg})}} \\
&= 6 \times \sqrt{\frac{240 \times 10^{18}\,\mathrm{m^3}}{1.3 \times 10^{20}\,\mathrm{m^3\,s^{-2}}}} \\
&= 6 \times \sqrt{1.8}\,\mathrm{s} \\
&= 9\,\mathrm{sec}
\end{aligned}$$

We get the same answer, but the arithmetic (still done without a calculator!) was quite a bit more difficult.

Full credit for creatively using Kepler's third law in the original form and converting the radius of the Earth to AUs, and then the period from years to seconds.
3 points off for missing units in final answer.
3 points off for wrong units in final answer.
5 points off for simple arithmetic errors.
10 points credit for stating the relation between orbital period, radius and the mass of the central body.
5 points for saying that one would expect the period to be shorter.
No points off for incorrect periods, as long as the ratio is correct. 2 points off if severe intermediate rounding off results in a ratio more than 10% off from the exact value.

Astronauts in such a space ship would indeed feel weightless. The astronauts are in orbit around the white dwarf just as the spaceship is. The astronauts and the space ship are in free fall together, at the same rate, around the white dwarf (since they are falling towards the white dwarf at the same rate as the white dwarf curves away from them). There is no difference in the acceleration of the space ship, the astronauts, and everything inside the space ship; therefore there is no net force forces pushing the astronauts towards the space ship's walls; they experience weightlessness.

This question is worth 5 points. 3 points credit for saying that the astronauts feel weightless.
2 points for correct definition of orbit.
2 points credit for mentioning free fall, without further explanation.

3. Planetary systems *40 total points*

Consider a main sequence M star, called Sauron, with mass one quarter that of the Sun, a radius equal to that of the Sun, and a surface temperature of 3000K. It has two planets in circular orbits around it: the first, Gandalf, has an orbital radius of 2 AU, while the other, Frodo, has an orbital radius of 1/2 AU.

a. *(10 points)* What is the ratio of orbital periods of Gandalf and Frodo?

Answer: Newton's form of Kepler's Third Law relates the period P, the radius of the orbit a, and the mass of the central star M:

$$P^2 = C\frac{a^3}{M},$$

where C is a constant. The ratio of the orbital periods of Gandalf and Frodo cancels the dependence on the mass of the star, since they are orbiting the same star:

$$\left(\frac{P_G}{P_F}\right)^2 = \left(\frac{a_G}{a_F}\right)^3$$

or

$$\left(\frac{P_G}{P_F}\right) = \left(\frac{a_G}{a_F}\right)^{3/2} = \left(\frac{2\ AU}{0.5\ AU}\right)^{3/2} = 4^{3/2} = 8.$$

So, the period of Gandalf is 8 times longer than Frodo.

Writing down Kepler's 3rd law and going no further gets 3 points.

b. *(15 points)* The planets have no greenhouse effect and zero albedo. Find the equilibrium temperature of Frodo (in Kelvin).

Answer: Here, we will scale from what we know about the Earth. As the formulas sheet states, the quantity $T_{star}\left(\frac{R_{star}}{2\,d}\right)^{1/2}$ is equal to 300 K using parameters for the Sun and the distance from the Sun to the Earth (it is just a coincidence that this is equal to the value you get when you include both albedo and the greenhouse effect). For Frodo, we have half the distance from the parent star to the planet (i.e., the d is 0.5 AU), and we are told that the radius of the star is identical to that of the Sun. The other difference is the temperature of the star, 3000 K, rather than the 6000 K of the Sun. So for Frodo, we can simply say that:

$$T_{Frodo} = T_{Earth} \times \frac{T_{M\,star}}{T_{Sun}} \times 2^{1/2} = 300K \times \frac{1}{2} \times 1.4 = 210\,K.$$

Five points for writing down $T_{star}\left(\frac{R_{star}}{2\,d}\right)^{1/2}$, or its equivalent $(S/4\,\sigma)^{1/4}$, and getting no further. Full credit for doing this the hard way, i.e., calculating the expression directly without scaling from Earth's value. In reality, main-sequence M stars have radii of only 1/10 that of the Sun; four extra-credit points to anyone who points this out. Three points off for absurd answers ($<$ 10 K or $>$ 10^4 K), without any statement that this doesn't quite look right.

c. *(15 points)* Astronomers on Gandalf measure the change in the angular position of our Sun with respect to distant stars due to the orbital motion of Gandalf. They find that over half of the period of Gandalf, the angular position of the Sun shifts by 2 arcseconds. How far is the Sun from Sauron (in parsecs)?

Answer: Gandalf has an orbital radius of 2 AU, and the parallax angle p (*half* the angular shift observed over half the orbit) is 1 arcsecond. Therefore, the distance in parsecs from Sauron is d(parsecs) = 2/p(arcsecs). Note that 2 in the numerator comes because the orbital radius of Gandalf, and hence the baseline for the parallax formula, is 2AU. For Earth, this formula has 1 in the numerator. The Sun is then 2 parsecs from Sauron.

Five points for stating the small-angle formula, or a formula for parallax, but going no further. A correct diagram gets 3 bonus points even if the answer is wrong. Four points off for factors of two errors due to confusion over parallax referring to the half-angle of the positional change of the star. 4 points off for using baseline for the Earth (1 AU). Most common error is setting the baseline to 1 AU and parallax angle to 2 arcsec, so d=0.5 pc. This answer gets 7 points (8 points off).

Essay Questions

Please answer the following two questions, using coherent English sentences. Feel free to use equations, calculations, and/or diagrams as part of your explanation. To answer each of these questions in full will take at least three or four paragraphs, but feel free to write more if you need to. This is your opportunity to show us what you have learned, so aim for complete explanations.

Five points off for egregiously incorrect statements

4. *(35 points)* Describe the Aristotelian worldview, emphasizing its description of the physical nature of, and laws that govern the motions of, heavenly bodies relative to those on Earth. Explain how this differs from our current scientific understanding of the Universe, and explain the importance of the discoveries of Galileo Galilei, Nicolas Copernicus, Johannes Kepler, Isaac Newton and Cecilia Payne-Gaposchkin in overturning the Aristotelian worldview.

Answer: The Aristotelian worldview had several characteristics:

- The Earth lay at the center of the Universe (*3 points*)
- Earthbound objects were made of four basic elements: Earth, Water, Air, and Fire (*3 points*)
- The natural motion of of the elements was towards their "proper place": for Earth and Water, it was towards the center of the earth, while the natural motion of Air and Fire was away from the center of the Earth. (*3 points*)
- The material of the heavens, or Celestial Spheres, was made of a fifth element, different from that found on Earth, namely quintessence. (*3 points*)
- The natural motion of celestial objects was in circles; indeed, the Celestial Spheres rotated around the Earth. (*3 points*)

Indeed, for this class, the most important aspect of the Aristotelian worldview is that the substance that objects in the heavens were made of, and the physical laws that governed those objects, were different from those on Earth. *Up to 7 points for making this general statement, in the absence of the details described above. Give up to five extra points for further discussion of either Aristotelian philosophy or the Ptolemaic model, perhaps including diagrams, and explaining the nature of epicycles, etc.*

Perhaps the most profoundly revolutionary aspect of the Scientific Revolution, as exemplified by the work of Copernicus, Newton, Galileo, and Payne-Gaposchkin, was the understanding that the substance of heavenly bodies, and the laws that govern their motions, are the *same* as those on Earth. (*8 points*) Let's take the work of each of these great scientists in turn:

Nicolas Copernicus showed that motions in the Solar System are much more easily explained if instead of placing the Earth in the center, the Sun lies in the center of the Solar System. (*5 points*) This contradicts one of the Aristotelian notions, i.e., that all heavenly objects move in circles around the Earth. *3 more points for more detailed descriptions of the Copernican picture and its advantages.*

Galileo Galilei was the first to use a telescope to study the sky. His discoveries were in strong support of Copernicus' ideas, and served to further undermine the Aristotelian notions of the heavenly objects as perfect:

- He saw mountains on the Moon, very much like those on Earth. (*3 points*)
- He saw spots on the Sun, demonstrating that it was not unblemished. (*3 points*)
- He saw that Venus went through phases, easily explained (and predicted!) in the heliocentric view, and not expected in the geocentric view. (*3 points*)
- He saw that Jupiter had a set of four moons orbiting around it, showing directly that heavenly bodies can exhibit circular motions around objects other than the Earth.(*3 points*)

Isaac Newton developed his laws of motion, and his laws of gravity, showing explicitly that the laws that govern motions on Earth and in the heavens are the same *Up to 5 points.* In particular, he showed that the acceleration that pulls an apple to the ground, is due to the same force of gravity that keeps the Moon "falling" in its orbit around the Earth. (*4 points*). *Up to 5 more points for a coherent description of Newton's Laws, and/or its explanation of Kepler's Laws of planetary motion.*

Finally, Cecilia Payne-Gaposchkin studied the spectra of stars, and used the (then newfangled) notions of quantum mechanics to infer the chemical composition of stars. (*4 points*) The important point here was that stars are made up of the same elements that are found on Earth, going against one of the basic Aristotelian tenets. (*4 points*) In particular, the Sun, and stars in general, are composed of 73% Hydrogen and 25% Helium, leaving 2% for all the rest of the elements in the periodic table. (*4 points*). *Up to 4 more points for further discussion of spectral line formation in stars, how the lines which appear in a star depend on the surface temperature of a star, and that Payne-Gaposchkin sorted out what the OBAFGKM sequence actually means.*

5. *(35 points)* Describe how we infer the properties of stars. Make sure to discuss the measurements of distance, temperature, radius, mass and chemical composition of stars.

Give typical numbers and ranges where appropriate. When using the names of techniques or other terminology, try to briefly explain what they mean, to show your understanding.

Answer The most reliable way to measure distance to nearby stars is using parallax (*4 points*). This technique involves measuring the change in angular position of the star due to the motion of the Earth around its orbit (*2 points*). The distance to the star in parsecs is then d(parsecs) = 1/p(arcsecs) where p is the parallax angle in arcseconds (*3 points*). Using this method, we know distances of stars within 500 parsecs from the Sun (*2 points*). A good diagram of the parallax technique is worth 3 points.

The spectrum of stars is mostly black body (*3 points*) with superimposed absorption lines. From the wavelength of the peak of the black-body spectrum we can infer the temperature of the surface of the star, using Wein's law: $\lambda_{\text{peak}} = 2.9\text{mm}/\text{T(K)}$ (*4 points*). Typical temperatures range from $50,000\ K$ for hottest stars to $2000\ K$ for coldest ones (*4 points*).

Knowing the distance to a star, we can infer its luminosity (power output) by measuring star's brightness (amount of energy per second per unit area on the Earth). The formula that relates brightness B to luminosity L is $B = L/(4\pi d^2)$, where d is the distance to a star. (*4 points*) Typical luminosities range from 10^6 solar luminosities for massive stars to 10^{-4} solar for low mass stars. (*3 points*). The luminosity of the star is obtained from the Stefan-Boltzmann law $L = \sigma T^4 \times$ Area (*4 points*). This relation and the formula for the area of a sphere Area $= 4\pi R^2$ allows us to find the radius of the star, because we know the luminosity and temperature. (*5 points*). Typical radii are near the solar radius on the main sequence, but can range from 1000 solar radii for giants, to less than the radius of the Earth for white dwarfs (even less for neutron stars, but we have not covered these yet) (*5 points*). For very few stars, like Betelguese (or the Sun), we can resolve the radius of the star by direct observations (*2 points*).

Masses of stars are determined from binary stars (*4 points*). The technique involves measuring the radial velocity of stars using the Doppler shift (*3 points*), period of the binary (from Doppler shift or visual eclipses, or changes in position on the sky due to orbital motion) (*2 points*), and then using Newton's version of the Kepler's third law (*3 points*) to relate velocity and period to mass. *Extra 5 points for showing the equations.* Typical masses range from 0.08 solar masses to 50 solar masses (*3 points*).

Chemical composition is determined from the strength of the absorption lines in the atmospheres of stars (*4 points*). Knowing the temperature of the surface and identifying the absorption lines with lines of a particular element, we can relate the strength of the line with the abundance of this element in the photosphere of the star (*5 points*). We find that stars are 73% Hydrogen and 25% Helium, leaving 2% for all the rest of the elements in the periodic table. (*4 points*).

Temperature and luminosity of stars correlate well on the Hertzsprung-Russell (HR) diagram, which shows the main sequence (hydrogen burning) stars and giant and dwarf branches (*3 points*). (5 points for sketching the diagram and labeling the axes). *Extra 4 points for OBAFGKM sequence.*