1. Astronaut aging. (45 points)

In addition to the time dilation due to an object moving at a finite speed that we have learned about in special relativity, there is an effect in general relativity, termed “gravitational redshift,” caused by gravity itself. To understand this latter effect, consider a photon escaping from the Earth’s surface to infinity. It loses energy as it climbs out of the Earth’s gravitational well. As its energy $E$ is related to its frequency $\nu$ by Einstein’s formula $E = h\nu$, its frequency must therefore also be reduced, so observers at a great distance $r = \infty$ must see clocks on the surface ticking at a lower frequency as well.

Therefore an astronaut orbiting the Earth ages differently from an astronomer sitting still far from the Earth for two reasons; the effect of gravity, and the time dilation due to motion. In this problem, you will calculate both these effects, and determine their relative importance. In parts a through c we will derive analytical expressions for each effect, simplify them in part d, and plug in numbers in part e.

a. The escape speed from an object of mass $M$ if you are a distance $r$ from it is given by

$$v_{\text{escape}} = \sqrt{\frac{2GM}{r}}.$$ 

That is, if you are moving this fast, you will not fall back to the object, but will escape its gravitational field entirely.

Karl Schwarzschild’s solution to Einstein’s field equations of general relativity shows that a stationary, non-moving clock at a radius $r \geq r_\oplus$ from the Earth will tick at a rate that is

$$\sqrt{1 - \frac{1}{c^2} \frac{2GM_\oplus}{r}} = \sqrt{1 - \frac{v_{\text{escape}}^2}{c^2}}$$

times as fast as one located far away from the Earth (i.e. at $r = \infty$). Note how much this expression looks like the equivalent expression from special relativity for time
dilation. Here $r_\oplus$ is the radius of the Earth, and $M_\oplus$ is its mass. Using the formula above, calculate the rate at which a stationary clock at a radius $r$ (for $r > r_\oplus$) will tick relative to one at the surface of the Earth. Is your rate greater or less than 1? If greater than 1, this means the high altitude clock at $r > r_\oplus$ ticks faster than one on the surface; if less than one, this means the high altitude clock ticks slower than one on the surface. (10 points)

**Answer:** The equation above gives the rate at which a clock at radius $r$ ticks relative to one infinitely far away. Here we are asked to compare the rate of a clock at radius $r$ relative to one at the radius of the Earth (i.e., at the Earth's surface). We can think about this by considering the rate of each of these clocks relative to a distant clock. That is, the clock on the Earth's surface has a rate a factor

$$\sqrt{1 - \frac{1}{c^2} \frac{2GM_\oplus}{r_\oplus}}$$

slower than the distant clock, while the clock at radius $r$ has a rate:

$$\sqrt{1 - \frac{1}{c^2} \frac{2GM_\oplus}{r}}.$$ 

Note that both these expressions are less than one, but because $r > r_\oplus$, the stationary clock at radius $r$ ticks faster than that at the Earth's surface. Indeed, the relative rate of the two is just the ratio of these two expressions:

$$\frac{\sqrt{1 - \frac{1}{c^2} \frac{2GM_\oplus}{r}}}{\sqrt{1 - \frac{1}{c^2} \frac{2GM_\oplus}{r_\oplus}}}.$$ 

Again, this expression is greater than 1.

7 points for the algebra, and 3 points for the determination if the resulting expression is greater than or less than one. Full credit for any algebraic expression consistent with the above, including expressions in which the above is simplified using the approximations in part (d), as long as the logic is properly explained. Four points off for writing down the correct expression for the time dilation at both the Earth’s surface and at radius $r$, but never taking the ratio. Three points off for writing down the inverse of what was asked for. Seven points off for stating that the ratio is simply $\sqrt{1 - \frac{1}{c^2} \frac{2GM_\oplus}{r}}$, i.e., ignoring the time dilation on the surface of the Earth. If numbers are plugged in immediately (and correctly), take 3 points off, but no further points off in further parts of the problem.

b. Now consider an astronaut orbiting at $r > r_\oplus$. What is her orbital velocity as a function of $r$? Because she is moving with respect to a stationary observer at radius $r$, special relativity says that her clock is ticking slower. Calculate the ratio of the rate her clock ticks to that of a stationary observer at radius $r$. (10 points)
**Answer:** The velocity of an astronaut in a circular orbit is something we have worked out many times before. Circular motion at speed $v$ at a radius $r$ gives rise to an acceleration $v^2/r$, which we know is due to gravity. Thus if the astronaut has a mass $m$, Newton’s second law tells us:

$$F = ma$$

$$\frac{GM_\oplus m}{r^2} = m\frac{v^2}{r}.$$  

Solving for $v$ gives us:

$$v = \sqrt{\frac{GM_\oplus}{r}}.$$  

No need to re-derive this; it is fine if this expression is quoted directly from lecture or textbook.

The time dilation due to Special Relativity is the now-familiar $\sqrt{1 - v^2/c^2}$, which gives in this case:

$$\sqrt{1 - \frac{GM_\oplus}{rc^2}}.$$  

Note again how similar this looks to the expression above for time dilation due to gravity. Again, this is the rate that an orbiting clock at radius $r$ ticks relative to a stationary clock at the same radius.

4 points for the calculation of the orbital speed of the astronomer, and 6 points for the calculation of the time dilation.

c. Now multiply the results of parts (a) and (b) to determine an expression for the ratio of the rate at which the orbiting astronaut’s clock ticks to a stationary clock on the surface of the Earth, as a function of the radius $r$ at which she orbits. You may ignore the small velocity of the clock on the surface of the Earth due to the Earth’s rotation. (5 points) *Hint: the answer is an ugly expression, but have no fear – we will use approximations next!*

**Answer:** In part (a), we calculated the ratio of rates of stationary clocks at radius $r$ and $r_\oplus$ (due to General Relativity), while in part (b), we calculated the ratio of the rates of an orbiting clock at radius $r$ to a stationary clock at the same radius. This the ratio of the rate of an orbiting clock at radius $r$ to a stationary one on the ground is simply the product of these two results, namely:

$$\sqrt{\frac{1 - \frac{1}{c^2}\frac{2GM_\oplus}{r}}{1 - \frac{1}{c^2}\frac{2GM_\oplus}{r_\oplus}}} \times \left(1 - \frac{GM_\oplus}{rc^2}\right).$$

This is a big expression, and there is no obvious way to simplify it without making some approximations, as we do in the next part. *Incorrect attempts to simplify this should be treated as algebraic errors. Full credit for results consistent with parts (a) and (b).*
d. We’ll now simplify this expression using tricks similar to those we found in Homework 5 (problem # 1). There we found that if \( x \ll 1 \), then \( \sqrt{1 - x} \approx 1 - \frac{1}{2}x \). Demonstrate that, if \( x \ll 1 \) and \( y \ll 1 \):
\[
\frac{1}{1 - x} \approx 1 + x
\]
*Hint: Multiply both sides by \( 1 - x \)...*
and
\[
(1 - x)(1 - y) \approx 1 - (x + y).
\]
(5 points)
**Answer:** Let’s do the first of these. Following the hint, let us write
\[
\frac{1}{1 - x} \approx 1 + x
\]
and do algebraic manipulations on this to get to a result we know is true. If we are successful in doing so, we can be confident that the original result was correct. Multiplying both sides by \( 1 - x \) gives:
\[
1 \approx (1 + x)(1 - x) = 1 - x^2.
\]
But we know \( x \ll 1 \), so \( x^2 \) is really small and can be neglected relative to one. So the above gives us the clearly true statement, \( 1 \approx 1 \). That establishes the first relation.
The second relation follows similarly:
\[
(1 - x)(1 - y) = 1 - (x + y) + xy.
\]
But if both \( x \) and \( y \) are much less than one, their product is again really small and can be neglected. Thus
\[
(1 - x)(1 - y) \approx 1 - (x + y),
\]
as we were to show.
3 points for demonstrating only one of the two approximations. Anything reasonable, in which the student is explicit about showing the term that gets dropped in the approximation is very small, gets full credit.

e. Use the above approximations to derive a final expression, of the form \( 1 - \alpha \), for the relative rate of a clicking clock on the surface of the Earth and the orbiting astronaut. Demonstrate that your quantity \( \alpha \) is indeed much smaller than 1. (8 points)
**Answer:** Now we will simplify the expression we derived in part (c), using the various tricks we’ve assembled in part (d). We will show at the end that the approximation that the various relevant pieces are much smaller than one is correct. We will do this in pieces, starting with the computation from part (a); we can write:
\[
\sqrt{\frac{1 - \frac{1}{c^2} \frac{2GM_{\oplus}}{r}}{1 - \frac{1}{c^2} \frac{2GM_{\oplus}}{r_{\oplus}}}} \approx \sqrt{\left(1 - \frac{1}{c^2} \frac{2GM_{\oplus}}{r}\right) \times \left(1 + \frac{1}{c^2} \frac{2GM_{\oplus}}{r_{\oplus}}\right)},
\]
using the first expression we derived in part (d). We then use the second expression we derived in part (d) to get:

\[ \sqrt{1 - \frac{GM_{\oplus}}{c^2} \left( \frac{2}{r} - \frac{2}{r_{\oplus}} \right)}. \]

This then gets multiplied by the expression in part (b); again we use the second expression of part (d) to simplify this to:

\[ \sqrt{1 - \frac{GM_{\oplus}}{c^2} \left( \frac{3}{r} - \frac{2}{r_{\oplus}} \right)}. \]

We’re almost done! From Problem 1 of Homework 5, we know how to simplify expressions that look like \( \sqrt{1 - x} \) where \( x \) is small; the above becomes:

\[ 1 - \frac{GM_{\oplus}}{2c^2} \left( \frac{3}{r} - \frac{2}{r_{\oplus}} \right). \]

That is our final answer.

5 points for this part of the problem. Treat incomplete use of the necessary approximations (e.g., not fully calculating the square root term) as algebraic errors.

However, we do need to justify the use of the various approximations we’ve made. We dealt with a variety of expressions of the form \( 1 - x \); in every case, \( x \) is of the form: \( \frac{GM_{\oplus}}{rc^2} \). The smallest \( r \) we considered (and therefore the largest the expression \( \frac{GM_{\oplus}}{rc^2} \) is) is \( r = r_{\oplus} \). So let’s plug in numbers at \( r = r_{\oplus} \), and see what we get:

\[ \frac{GM_{\oplus}}{r_{\oplus}c^2} = \frac{2/3 \times 10^{-10} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1} \times 6 \times 10^{24} \text{ kg}}{6.4 \times 10^6 \text{ m} \times (3 \times 10^8 \text{ m/s})^2}, \]

where we were careful to put all numbers into MKS units. To simplify this, I will approximate the radius of the Earth as \( 6 \times 10^6 \text{ m} \), which means that lots of stuff cancels, and we get a final value of \( 7 \times 10^{-9} \). This is indeed a number much much smaller than 1!

3 points for a calculation of any reasonable version of \( GM/rc^2 \). Take off points for arithmetic mistakes, as usual.

f. Using your result from (e), calculate the radius \( r \) at which the clock of the orbiting astronaut ticks at the same rate as a stationary one on the surface of the Earth; express your result in Earth radii and kilometers. Will an astronaut orbiting at a smaller radius age more or less than one who stayed home? Therefore, do astronauts on the Space Shuttle (orbiting 300 km above the Earth’s surface) age more or less than one staying home? (7 points)

**Answer:** Here we ask for the radius at which the expression we derived in part (e) is equal to unity. This clearly holds when:

\[ \frac{3}{r} - \frac{2}{r_{\oplus}} = 0 \]
or

\( r = 1.5r_\oplus. \)

Given the radius of the Earth, 6400 km, this is at a distance of 9600 km from the Earth’s center, or 3200 km above the Earth’s surface. The expression in part (e) is less than 1 at smaller radii, so astronauts on the space shuttle age less than those staying home.

4 points for calculating the radius, 1 point off for not expressing it in kilometers. Three points for answering the question about the aging of the shuttle astronauts.

2. A Hitchhiker’s Challenge (25 points)

“A full set of rules [of Brockian Ultra Cricket, as played in the higher dimensions] is so massively complicated that the only time they were all bound together in a single volume they underwent gravitational collapse and became a Black Hole.”

Chapter 17 of Life, the Universe and Everything, the third volume of the Hitchhiker’s Guide to the Galaxy series (1982, Douglas Adams)

A quote like that above is crying out for a calculation. In this problem, we will answer Adams’ challenge, and determine just how complicated these rules actually are.

An object will collapse into a black hole when its radius is equal to the radius of a black hole of the same mass; under these conditions, the escape speed at its surface is the speed of light (which is in fact the defining characteristic of a black hole!). We can rephrase the above to say that an object will collapse into a black hole when its density is equal to the density of a black hole of the same mass.

a. Derive an expression for the density of a black hole of mass \( M \). Treat the volume of the black hole as the volume of a sphere of radius given by the Schwarzschild radius. As the mass of a black hole gets larger, does the density grow or shrink? (5 points)

**Answer:** The Schwarzschild Radius of a black hole of mass \( M \) is

\[ R_{Sch} = \frac{2GM}{c^2}. \]

The volume of a sphere of this radius is just the familiar \( \frac{4}{3} \pi R_{Sch}^3 \). The density is the mass divided by the volume, giving:

\[ \text{Density} = \frac{M}{\frac{4}{3} \pi \left( \frac{2GM}{c^2} \right)^3} = \frac{3c^6}{32 \pi G^3 M^2}. \]

A messy expression! The more massive the black hole, the smaller the density. Thus there is a mass at which the black hole has the density of paper, which is what we are trying to figure out.

2 points for pointing out that the density shrinks with a larger black hole mass. Full credit for stating \( \pi \approx 3 \), and canceling the two, or similar approximations.

It is worth pointing out that the actual material which makes up the black hole is much smaller than this, at much higher density. The material is, as best we understand, crushed to a point, of extent of order the Planck radius...
b. Determine the density of the paper making up the Cricket rule book, in units of kilograms per cubic meter. Standard paper has a surface density of 75 grams per square meter, and a thickness of 0.1 millimeters. (5 points)

**Answer:** The density is the mass per unit volume. If we can figure out the volume of a square meter of paper (whose mass we know, 75 grams), we can calculate its density. The volume of a piece of paper is its area times its thickness. The thickness is 0.1 millimeter, or $10^{-4}$ meters, and so the volume of a square meter of paper is $1\text{ meter}^2 \times 10^{-4}\text{ meter} = 10^{-4}\text{ meter}^3$. Therefore, the density of paper is:

$$\rho = \frac{7.5 \times 10^{-2}\text{ kg}}{10^{-4}\text{ meter}^3} = 7.5 \times 10^2\text{ kg/m}^3,$$

similar to (but slightly less than) the density of water (remember, paper is made of wood, and wood floats in water!).

c. Calculate the mass (in solar masses), and radius (in AU) of the black hole with density equal to that of paper. (10 points)

**Answer:** Here we equate the expression for density we found in part (a) with the density we calculated in part (b), and solve for mass. Let’s first do it algebraically:

$$\rho = \frac{3\epsilon^6}{32\pi G^3 M^2}$$

$$M = \sqrt{\frac{3\epsilon^6}{32\pi G^3 \rho}}.$$

Now let’s plug in numbers. This will be fun without a calculator:

$$M \approx \sqrt{\frac{3 (3 \times 10^8 \text{ m/s})^6}{32 \pi (2/3 \times 10^{-10} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1})^3 \times 7.5 \times 10^2 \text{ kg} / \text{m}^3}} \approx 3 \times 10^{38} \text{ kg},$$

where I made all the usual approximations of $\pi = 3$, $3^2 = 10$, and so on. We need to express this in solar masses, so we divide by $2 \times 10^{30} \text{ kg}$, i.e., one solar mass, to get $1.5 \times 10^8 M_\odot$ as our final answer. A black hole 150 million times the mass of the Sun has the same density as a piece of paper...

Surely this is science fiction! Well, yes, “The Hitchhiker’s Guide to the Galaxy” is indeed science fiction, but do such incredibly massive black holes actually exist? Indeed they do: the cores of massive galaxies (including our own Milky Way, as we shall see in the next problem) do contain such enormous black holes. Indeed, the most massive such black hole known to exist is in the core of a particularly luminous galaxy known as Messier 87, with a mass of 3 billion solar masses.

We still have to calculate the Schwarzschild radius of a black hole. We could plug into the formula for a Schwarzschild radius and calculate away, but I prefer a simpler approach. I know the Schwarzschild radius is proportional to the mass of a black hole, and I happen to remember (indeed, it is on our formulas sheet) that a one solar mass
black hole has a Schwarzschild radius of 3 kilometers. So a 150 million solar mass black hole has a Schwarzschild radius 150 million times larger, or $4.5 \times 10^8$ kilometers. We are asked to express this in terms of AU; one AU is $1.5 \times 10^8$ kilometers, so the Schwarzschild radius of such a black hole is 3 AU.

*Six points for the calculation of only the radius or the mass, but not both. Calculation of the Schwarzschild radius need not scale off the Sun, as was done here...*

d. How many pages long is the Brockian Ultra Cricket rule book? Assume the pages are standard size (8.5” × 11”). For calculational simplicity, treat the book as spherical (a common approximation in this kind of problem). What if the rule book were even longer than you have just calculated? Would it still collapse into a black hole? (5 points)

**Answer:** We know the entire mass of the black hole. If we can calculate the mass of a single piece of paper, the ratio of the two gives the total number of pages. So let’s calculate the mass of a single piece of paper. We know that a square meter of paper has a mass of 75 grams. How many square meters is a standard-size sheet? One inch is $2.5 \text{ centimeters} = 2.5 \times 10^{-2} \text{ meters}$. So $8.5 \times 11 \text{ inch}^2 \approx 100 \text{ inch}^2 \approx 6 \times 10^{-2} \text{ meter}^2$. Thus the mass is:

$$\text{Mass of a piece of paper} = 7.5 \times 10^{-2} \text{ kg/meter}^2 \times 6 \times 10^{-2} \text{ meter}^2 \approx 5 \times 10^{-3} \text{ kg}.$$ 

That is, a piece of paper weighs about 5 grams. We divide this into the mass we calculated above:

$$\text{Number of sheets of paper} = \frac{\text{Mass of rule book}}{\text{Mass per page}} = \frac{3 \times 10^{38} \text{ kg}}{5 \times 10^{-3} \text{ kg/page}} = 6 \times 10^{40} \text{ pages}.$$ 

(Strictly speaking, if the rule book is printed on both sides of the page, we should multiply the above result by a factor of two.) That is one seriously long set of rules!

Finally, note that because the density of a more massive black hole is smaller, as we saw in part (a), the above mass and number of pages of the Brockian Ultra Cricket rule book is really just a lower limit. That is, if the rule book were even larger than what we’ve just calculate, it would still collapse into a black hole.

*3 points for the calculation of the number of pages (and there is no need to include the factor of two!), and two points for answering the question about what would happen if the book were even longer.*

3. Falling into a black hole (30 points)

The tides near black holes can be so extreme that a process informally called “spaghettification” occurs in which a body falling towards a black hole is strongly stretched due to the difference in gravitational force at different locations along the body (this is called a tidal effect). In the following, imagine that you are falling into a 3 solar mass black hole.

a) What is the Schwarzschild radius of this black hole (in km)? (5 points)

**Solution:**

The Schwarzschild radius of a black hole is $R_s = 2GM/c^2 = 9 \text{ km}$ for $M = 3M_\odot$, where $c = 3 \times 10^8 \text{ m/s}$ the speed of light. If you remember that for 1 Solar mass the Schwarzschild radius is 3 km, you can scale from there.
b) You are 1.5 meters tall and 70 kg in mass and are falling feet first. At what distance from the black hole would the gravitational force on your feet exceed the gravitational force on your head by 10kN (kN stands for kilo-Newton; Newton is an MKS unit of force)? Express this distance in km and in Schwarzschild radii of the black hole. (10 points)

*Hint: Remember that Newton’s law of gravitation tells that the force between bodies of mass $m_1$ and $m_2$ a distance $r$ apart is given by $F = Gm_1m_2/r^2$, where $G$ is Newton’s gravitational constant. You may also find useful the following mathematical simplification:

$$\frac{1}{r^2} - \frac{1}{(r + \Delta)^2} \approx \frac{2\Delta}{r^3},$$

which is valid when $\Delta \ll r$. Proving that this approximation is correct will get you extra 5 points – this is extra credit, but will not lead to your total exceeding 100 points for this homework.*

**Solution**

Using the Newton’s law of gravitation, we can write the difference in gravitational forces acting on two bodies of mass $m$ which are located at distances $r_1$ and $r_2$ from the massive body of mass $M$.

$$\delta F \equiv F_1 - F_2 = \frac{GmM}{r_1^2} - \frac{GmM}{r_2^2} = GmM \left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right).$$

We are interested in the difference in gravitational forces in two locations that are close to each other, since the height of the person falling into the black hole is small compared to the Schwarzschild radius. Let’s take $r_2 = r_1 + \Delta$, where $\Delta \ll r_1$. Then, we can simplify the previous expression (dropping the subscript 1):

$$\delta F = GmM \left(\frac{1}{r^2} - \frac{1}{(r + \Delta)^2}\right) = GmM \frac{(r + \Delta)^2 - r^2}{r^2(r + \Delta)^2} = GmM \frac{r^2 + 2r\Delta + \Delta^2 - r^2}{r^2(r + \Delta)^2} \approx GmM \frac{2r\Delta}{r^4} = GmM \frac{2\Delta}{r^3}.$$

In obtaining this expression, we used the approximations $r + \Delta = r(1 + \Delta/r) \approx r$ and $2r\Delta + \Delta^2 = 2r\Delta(1 + \Delta/(2r)) \approx 2r\Delta$, since $\Delta \ll r$.

Now let’s use this formula to find the distance $r_{crit}$ from the black hole where the relative stretching force between your head and your legs is equal to some critical force $\delta F_{crit}$.

$$\delta F_{crit} = GmM \frac{2\Delta}{r_{crit}^3};$$

$$r_{crit} = \left(\frac{2GmM\Delta}{\delta F_{crit}}\right)^{1/3}.$$

Now we can plug in the numbers. The mass of the black hole is $M = 3M_\odot = 6 \times 10^{30}$ kg. The mass of the body is $m = 70$ kg, $\delta F_{crit} = 10$ kN. The critical radius is then, $r_{crit} \approx 2000$ km, or $\sim 200R_S$, remembering from a) that the Schwarzschild radius for 3 $M_\odot$
black hole is \( R_s \sim 10 \text{km} \). Note, that in this case a significant amount of stretching occurs already relatively far from the black hole.

5 extra points for the derivation of the expansion that was given as a hint. These points can be added to the homework total up to 100 points only. 1 point off for using \( m = 1/2 \times 70 \text{kg} \).

c) To appreciate if this amount of force is enough to “spaghettify” and kill you, imagine that you are suspended from a ceiling of your room (on Earth) with a steel plate tied to your feet. Calculate the mass of the plate (in kg) that will give you a nice tug of 10kN (you can ignore the weight of your body here). Do you think this pull will kill you? (5 points)

**Solution**

The force with which the metal plate is pulling on you is given by the Newton’s second law, \( F = m_p g \), where \( m_p \) is the mass of the plate, and \( g \sim 10 \text{m/s}^2 \) is the gravitational acceleration on the Earth. Thus \( m_p = 10 \text{kN}/(10 \text{m/s}^2) = 1000 \text{kg} \), or 1 ton. If you still have hard time imagining how much weight 10 kN is, this is the weight of a typical car. So, imagine attaching a car to your feet – not pleasant! Most likely this is enough to kill or at the very least severely disable a person!

*Full credit for any reasonable conclusion about whether this is enough to kill a human. This course is not about enhanced interrogation techniques...*

d) Now consider a trip toward the supermassive black hole at the center of our Galaxy, which has an estimated mass of 4 million solar masses. How does this change the distance at which you will be “spaghettified” by the differential gravity force of 10kN? Express your answer in km and in Schwarzschild radii of the black hole. (5 points)

**Solution**

We can apply the formula for \( r_{crit} \) from part b). Now the mass of the black hole is \( 1.3 \times 10^6 \) times larger, so the radius increases by \( (1.3 \times 10^6)^{1/3} \sim 100 \) times. The answer is then \( r_{crit} = 2 \times 10^5 \text{ km} \). In terms of the Schwarzschild radii, remember that \( R_s \) is linearly proportional to the mass. For 4 million solar mass black hole, the Schwarzschild radius is then \( 1.3 \times 10^6 \times 10 \text{km} \approx 10^7 \text{km} \). Since \( r_{crit} < R_s \) (\( r_{crit} = 2 \times 10^{-2} R_s \)), the “spaghettification” happens inside the Schwarzschild radius!

2 points off for calculating using the Schwarzschild radius of the 3 \( M_\odot \) black hole.

e) Find the smallest mass of the black hole for which you would not die by “spaghettification” before falling within its Schwarzschild radius. (5 points)

**Solution**

As we saw in d), the radius at which the tidal force reaches 10kN grows as the third root of the mass of the black hole, but the Schwarzschild radius grows linearly with the mass of the hole. In part c) for the 3 \( M_\odot \) hole the critical radius was outside \( R_s \), while in part d) for \( 4 \times 10^6 M_\odot \) hole the critical radius was inside. Thus, there should be a minimum mass of the hole, at which \( r_{crit} = R_s \), i.e., we can just barely pass through the horizon before getting fatally stretched. Let’s find this mass:

\[
 r_{crit} = R_s
\]
\[
\left( \frac{2GmM_{\text{min}}\Delta}{\delta F_{\text{crit}}} \right)^{1/3} = \frac{2GM_{\text{min}}}{c^2}
\]

We can solve this equation for \( M_{\text{min}} \) (divide both sides by \((2GM_{\text{min}})^{1/3}\), and take both sides to the power of \(3/2\)).

\[
M_{\text{min}} = \left( \frac{c^3}{2G} \right) \left( \frac{m\Delta}{\delta F_{\text{crit}}} \right)^{1/2} = 2 \times 10^{34} \text{kg} = 10^4 M_\odot.
\]

So, if you fall into a 10000 solar mass black hole, you will be killed right as you go through the horizon. If the black hole is more massive, then you can go through the horizon while still alive, and enjoy the sights! Sadly, you will not have much time to enjoy the view anyways, because you will be crushed by the singularity in 0.01 seconds for this \(10^4 M_\odot\) black hole. This time is proportional to mass of the hole. You can view the animations of what you will see here:

http://jilawww.colorado.edu/~ajsh/insidebh/schw.html